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FURTHER REMARKS ON δ- AND θ-MODIFICATIONS*

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Abstract. In generalizing constructions of N.V. Veličko, the paper starts from two generalized topologies μ and μ' on a set X and introduces two more generalized topologies $\delta(\mu,\mu')$ and $\theta(\mu,\mu')$ with the examination of their properties.

0. Introduction

In the paper [5], the first author has generalized the construction of δ- and θ-modifications of topologies, introduced in [7], by replacing the topology used in [7] by a generalized topology in the sense of [1]; we recall that a subset μ of the power set $\exp X$ of a set $X \neq \emptyset$ is said to be a *generalized topology* (briefly GT) iff $\emptyset \in \mu$ and every union of elements of μ belongs to μ. In the paper [5], given a GT μ on X, two other GT's $\delta(\mu)$ and $\theta(\mu)$ were constructed and their properties examined.

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In the present paper, we generalize the above constructions in the following way. We use two GT's μ and μ' on X and define two more GT's $\delta(\mu,\mu')$ and $\theta(\mu,\mu')$ in such a way that, in the case $\mu=\mu'$, we have $\delta(\mu,\mu)=\delta(\mu)$ and $\theta(\mu,\mu)=\theta(\mu)$. Our purpose is to discuss the properties of $\delta(\mu,\mu')$ and $\theta(\mu,\mu')$ (involving the results in [5] in the case $\mu=\mu'$).

1. Preliminaries

Let μ be a GT on X. We call *μ-open* the elements of μ, their complements *μ-closed.* If $A\subset X$ then $i_\mu A$ denotes the largest μ-open set contained in A and $c_\mu A$ the smallest μ-closed set containing A (see e.g. [3]). Then both i_μ and c_μ are *operations*, i.e. mappings from $\exp X$ to $\exp X$, monotonic (where $\beta:\exp X\to\exp X$ is said to be *monotonic* when $A\subset B$ implies $\beta A\subset\beta B$; for an operation β we write βA for $\beta(A)$), idempotent (where β is *idempotent* iff $\beta\beta A=\beta A$) and $c_\mu(X-A)=X-i_\mu A$ (see [3]). If β and β' are operations, we write $\beta\beta'$ for $\beta\circ\beta'$. According to [5],

$(*)$ if $A\subset X$ and $x\in X$, then $x\in c_\mu A$ iff $x\in M\in\mu$ implies $M\cap A\neq\emptyset$.

We say that the GT μ is *strong* iff $X\in\mu$ (see [2]) and a *quasi-topology* (briefly QT) iff $A\in\mu$ and $B\in\mu$ implies $A\cap B\in\mu$ (see [4]). Clearly the GT μ is a topology on X iff it is a strong quasi-topology.

If μ is a GT on X and ν is a GT on Y then the mapping $f:X\to Y$ is said to be *(μ,ν)-continuous* iff $N\in\nu$ implies $f^{-1}(N)\in\mu$ (see [1]).

2. $\delta(\mu,\mu')$ and $\theta(\mu,\mu')$

We consider two (not necessarily distinct) GT's μ and μ' on X. For the sake of brevity, in what follows we write i, c, i', c' for i_μ, c_μ, $i_{\mu'}$, $c_{\mu'}$ respectively.

Let $\delta(\mu,\mu')$ be composed of the sets $A\subset X$ such that $x\in A$ implies the existence of a μ'-closed set Q satisfying $x\in iQ\subset A$.

Similarly, $\theta(\mu,\mu')$ is composed of all sets $A\subset X$ such that $x\in A$ implies the existence of a set $M\in\mu$ satisfying $x\in M\subset c'M\subset A$.

Then clearly $\delta(\mu,\mu)=\delta(\mu)$ and $\theta(\mu,\mu)=\theta(\mu)$ (see [5]). If possible, we shall simply write δ and θ instead of $\delta(\mu,\mu')$ and $\theta(\mu,\mu')$, respectively.

Proposition 2.1. *δ is a GT on X.*

Proof. $\emptyset\in\delta$ is evident. If $A=\bigcup_{i\in I}A_i\subset X$ and $A_i\in\delta$ for every $i\in I$ then $x\in A$ implies $x\in A_i$ for some $i\in I$ and then there is a μ'-closed set Q such that $x\in iQ\subset A_i\subset A$. Hence $A\in\delta$. □

PROPOSITION 2.2. *θ is a GT on X.*

PROOF. Clearly $\emptyset \in \theta$. If $A = \bigcup_{i \in I} A_i \subset X$ and $A_i \in \theta$ for $i \in I$ then $x \in A$ implies $x \in A_i$ for some $i \in I$ and then there is $M \in \mu$ such that $x \in M \subset c'M \subset A_i \subset A$. Hence $A \in \theta$. □

PROPOSITION 2.3. *If μ is a strong GT then the same is true for δ.*

PROOF. $X \in \delta$ since $x \in X$ implies that X is μ'-closed (as $\emptyset \in \mu'$), moreover $X \in \mu$ by hypothesis so that $x \in iX = X \subset X$. □

PROPOSITION 2.4. *If μ is a strong GT then the same holds for θ.*

PROOF. $X \in \theta$ since $x \in X$ implies $x \in X = c'X \subset X$, and $X \in \mu$ by hypothesis. □

LEMMA 2.5. *If μ is a QT and $P, Q \subset X$ then*

$$i(P \cap Q) = iP \cap iQ.$$

PROOF. As i is monotonic, $i(P \cap Q) \subset iP \cap iQ$. On the other hand, $iP \cap iQ \in \mu$ by hypothesis and $iP \cap iQ \subset P \cap Q$ so that $iP \cap iQ \subset i(P \cap Q)$. □

PROPOSITION 2.6. *If μ is a QT then δ is a QT as well.*

PROOF. Assume $A, B \in \delta$. Then $x \in A \cap B$ implies the existence of μ'-closed sets P and Q such that $x \in iP \subset A$ and $x \in iQ \subset B$. By 2.5 $i(P \cap Q) = iP \cap iQ$ so that $x \in i(P \cap Q) = iP \cap iQ \subset A \cap B$, moreover $P \cap Q$ is μ'-closed. Hence $A \cap B \in \delta$. □

PROPOSITION 2.7. *If μ is a QT then the same holds for θ.*

PROOF. Assume $A, B \in \theta$. If $x \in A \cap B$ then there are $M, N \in \mu$ satisfying $x \in M \subset c'M \subset A$ and $x \in N \subset c'N \subset B$. By hypothesis $M \cap N \in \mu$. Hence $x \in M \cap N \subset c'(M \cap N) \subset c'M \cap c'N \subset A \cap B$. □

THEOREM 2.8. *If μ is a topology then the same is true for $\delta(\mu, \mu')$ and $\theta(\mu, \mu')$.*

PROOF. By 2.3, 2.4, 2.6, 2.7 both δ and θ are strong QT's. □

THEOREM 2.9. *$\theta \subset \delta \subset \mu$.*

PROOF. Assume $A \in \theta$ and $x \in A$. Then there is $M \in \mu$ such that $x \in M \subset c'M \subset A$. Choose $Q = c'M$. Then Q is μ'-closed and $M \subset Q$ implies $M \subset iQ$. Hence $x \in M \subset iQ \subset Q \subset A$.

Now let $A \in \delta$, $x \in A$. Then there is a μ'-closed set Q_x such that $x \in iQ_x \subset A$. As $iQ_x \in \mu$ and clearly A is the union of these sets iQ_x, we have $A \in \mu$. □

In general, $\theta \neq \delta$, even if $\mu = \mu'$ is a topology. In fact, let $X = \{a, b, c\}$, $\mu = \mu' = \big\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\big\}$ (see [5], 4.6). Then $\theta = \{\emptyset, X\}$ by [5], 4.6. However, $A = \{a\} \in \delta$ as $Q = \{a, c\}$ is μ-closed and $a \in iQ = \{a\} \subset A$.

Let us consider another GT μ_0 on X and write i_0 and c_0 for i_{μ_0} and c_{μ_0}, respectively.

Proposition 2.10. *If* $\mu' \subset \mu_0$ *then* $\delta(\mu,\mu') \subset \delta(\mu,\mu_0)$.

Proof. Let $A \in \delta(\mu,\mu')$ and $x \in A$. Then there is a μ'-closed set Q such that $x \in iQ \subset A$. By hypothesis, Q is μ_0-closed so that $A \in \delta(\mu,\mu_0)$. □

Now consider another set Y and two GT's ν and ν' on Y. Consider also a mapping $f : X \to Y$. The following proposition shows the functoriality of θ.

Proposition 2.11. *If* f *is* (μ,ν)*-continuous and* (μ',ν')*-continuous then it is* $\big(\theta(\mu,\mu'),\theta(\nu,\nu')\big)$*-continuous as well.*

Proof. Let $B \in \theta(\nu,\nu')$ and $y \in B$. Then there exist sets $G \in \nu$ and $Y - F \in \nu'$ such that $y \in G \subset F \subset B$. For the set $A = f^{-1}(B) \subset X$, consider $x \in A$. Then $y = f(x) \in B$ and there are $G \in \nu$ and $Y - F \in \nu'$ as above. Therefore $x \in f^{-1}(G) \subset f^{-1}(F) \subset A$ and $G \in \nu$ implies $f^{-1}(G) \in \mu$, $Y - F \in \nu'$ implies $X - f^{-1}(F) = f^{-1}(Y - F) \in \mu'$ so that $A \in \theta(\mu,\mu')$. □

Corollary 2.12. *If* ν *and* ν' *are two GT's on* X *and* $\nu \subset \mu$, $\nu' \subset \mu'$ *then* $\theta(\nu,\nu') \subset \theta(\mu,\mu')$.

Proof. Consider $Y = X$, $f = \mathrm{id}$. □

3. Properties of $\delta(\mu,\mu')$

Let us say that a set $R \subset X$ is $r(\mu,\mu')$*-open* iff $R = ic'R$. If there is no confusion, we simply call these sets r-open.

Lemma 3.1. *If* Q *is* μ'*-closed then* iQ *is* r*-open.*

Proof. Define $R = iQ$. Then $R \subset c'R \subset Q$ and, as $R \in \mu$, $R \subset ic'R \subset iQ = R$. Thus $R = ic'R$. □

Theorem 3.2. δ *coincides with the collection of all unions of* r*-open sets.*

Proof. Let R be r-open, i.e. $R = ic'R$. Then $Q = c'R$ is μ'-closed and $x \in R$ implies $x \in iQ = R$ so that $R \in \delta$.

By 2.1 each union of r-open sets belongs to δ.

Now let $D \in \delta$ and $x \in D$. Then there is a μ'-closed set Q_x such that $x \in iQ_x \subset D$. By 3.1 iQ_x is r-open and clearly D is the union of all these sets. □

Theorem 3.3. *If* $A \subset X$ *and* $x \in X$, *we have* $x \in c_\delta A$ *iff every* r*-open set* R *such that* $x \in R$ *fulfils* $R \cap A \neq \emptyset$.

Proof. $x \in c_\delta A$ iff $x \in D \in \delta$ implies $D \cap A \neq \emptyset$ iff $x \in R$, $R = ic'R$ implies $R \cap A \neq \emptyset$ by 3.2. □

4. Properties of $\theta(\mu, \mu')$

Let us introduce an operation $\gamma(\mu,\mu') = \gamma$ by $x \in \gamma A$ iff $x \in M \in \mu$ implies $c'M \cap A \neq \emptyset$ $(A \subset X)$.

LEMMA 4.1. *The operation* $\gamma : \exp X \to \exp X$ *is monotonic.*

PROOF. If $A \subset B$ and $x \in \gamma A$ then $x \in M \in \mu$ implies $c'M \cap A \neq \emptyset$, hence $c'M \cap B \neq \emptyset$ and $x \in \gamma B$. $\square$

LEMMA 4.2. $A \subset X$ *implies* $A \subset cA \subset \gamma A$.

PROOF. A is contained in cA (see Section 1). Moreover, $x \in cA$ and $x \in M \in \mu$ imply $M \cap A \neq \emptyset$ (cf. $(*)$) and a fortiori $c'M \cap A \neq \emptyset$. $\square$

THEOREM 4.3. $A \subset X$ *is* θ*-closed iff* $A = \gamma A$.

PROOF. A is θ-closed iff $X - A \in \theta$ iff $x \in X - A$ implies the existence of $M \in \mu$ satisfying $x \in M \subset c'M \subset X - A$ iff $x \in X - A$ implies the existence of $M \in \mu$ satisfying $x \in M$ and $c'M \cap A = \emptyset$ iff $x \in X - A$ implies $x \in X - \gamma A$ iff $X - A \subset X - \gamma A$ iff $\gamma A \subset A$ iff $\gamma A = A$ by 4.2. $\square$

For the relation of γA and $c_\theta A$, we can say:

PROPOSITION 4.4. $A \subset X$ *implies* $\gamma A \subset c_\theta A$.

PROOF. $x \in X - c_\theta A$ implies the existence of $T \in \theta$ satisfying $x \in T$ and $T \cap A = \emptyset$ and then of $M \in \mu$ such that $x \in M$ and $c'M \subset T \subset X - A$, implying $x \notin \gamma A$. $\square$

However, in general $\gamma A \neq c_\theta A$ since γ is not idempotent, not even if $\mu = \mu'$ is a topology (see [5], 4.6 and 4.7 where, for $A = \{a\}$, we have $c_\theta A = X$ but $b \notin \gamma A$ as $M = \{b\}$ implies $cM \cap A = \emptyset$).

In order to obtain a more precise connection, let us define an operation γ^α for each ordinal number α and $A \subset X$. Define $\gamma^0 A = A$ and $\gamma^{\alpha+1}A = \gamma\gamma^\alpha A$, $\gamma^\alpha A = \cup_{\beta<\alpha}\gamma^\beta A$ for a limit ordinal α. (Clearly $\gamma^\beta A$ stabilizes for a sufficiently large ordinal α, see [6].) Then define $\bar\gamma A = \cup_\alpha \gamma^\alpha A$ for all ordinals α. Clearly

$$(**) \qquad A \subset \bar\gamma A \subset X.$$

THEOREM 4.5. $A \subset X$ *implies* $c_\theta A = \bar\gamma A$.

PROOF. $c_\theta A$ is the minimal θ-closed set B containing A, i.e. by 4.3 the minimal set B such that

$$A \subset B = \gamma B.$$

Thus

$$(***) \qquad A \subset c_\theta A = \gamma c_\theta A$$

(cf. 4.3) while

$$(****) \qquad B \subset X, \quad A \subset B = \gamma B \Rightarrow c_\theta A \subset B.$$

From $(***)$ and 4.1 it follows by transfinite induction that $\gamma^\alpha A \subset c_\theta A$ for each ordinal α. Hence

$$(*****) \qquad \bar{\gamma} A \subset c_\theta A.$$

However $\gamma\bar{\gamma}A = \bar{\gamma}A$ and thus by $(**)$ and $(****)$ we get $c_\theta A \subset \bar{\gamma}A$. Consequently by $(*****)$ $c_\theta A = \bar{\gamma}A$. □

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