TWO NEARLY EQUAL DISTANCES IN R^d

PAUL ERDŐS, ENDRE MAKAI, JR., JÁNOS PACH

Alfréd Rényi Institute of Mathematics,
HUN-REN, 1364 Budapest, POB 127, Hungary*

makai.endre@renyi.hu, http://www.renyi.hu/~makai
ORCID ID: https://orcid.org/0000-0002-1423-8613
pach@renyi.hu, http://www.renyi.hu/~pach
ORCID ID: https://orcid.org/0000-0002-2389-2035

2020 Mathematics Subject Classification: Primary:52C10, Secondary: 05C35

Keywords: Erdős-type problems, separated point sets, nearly equal distances, k-distance sets, Turán's theorem, Ramsey's theorem

ABSTRACT. A point set $P \subset \mathbb{R}^d$ is *separated* if the minimum distance between any two points in P is at least 1. For $d \neq 4, 5$, we determine, for every $t_1, t_2 \geq 1$, and for n at least a suitable n_d, the maximum number of point pairs in a separated n-element point set in $\mathbb{R}^d$, with distances in the set $[t_1, t_1 + 1] \cup [t_2, t_2 + 1]$. For $d = 4, 5$ we establish a weaker, similar asymptotic estimate. Recently N. Frankl and A. Kupavskii have generalized this result to unions of $k \geq 2$ intervals. We also determine the maximum number of point pairs in an n-element point set in $\mathbb{R}^d$, whose distances belong to the union of $k \geq 2$ intervals of the form $[t_i, t_i(1+\varepsilon)]$, where $t_i > 0$ and $\varepsilon > 0$ is small.

§1. INTRODUCTION

Around 1945, Paul Erdős found two interesting applications of extremal combinatorics. One is related to an algebraic question of Littlewood and Offord [19], and the other one is in geometry. In [6], he applied Sperner's lemma to give a tight upper bound on the number of subsets of a set of n real numbers, whose absolute values are at least 1, and whose sums fall into a given interval of length 1. In [7], Erdős addressed the following question: At most how many times can the same distance occur among n points in $\mathbb{R}^d$? More precisely, what is the maximum number of unordered point pairs that determine the same distance?

Erdős modified the second question in the spirit of the first one, cf. [12]. At most how many unordered pairs $\{p, q\}$ of distinct points can be selected from an n-element point set $P \subset \mathbb{R}^d$ so that all distances $d(p, q)$ are *nearly the same*, in the sense that they fall into the same unit interval? To avoid the degenerate situation where all points are very close to each other and, hence, all distances are nearly 0, we consider only *separated* point sets P. That is, we assume that the distance between any two points of P is at least 1. To give an answer to the last question, we recall Turán's theorem [26, 1]. For $n, k \geq 1$ integers, define the *Turán number* $T(k, n)$, as the maximum number of edges that a graph on n vertices can have without containing a complete subgraph K_k on k vertices. According to *Turán's theorem*, for a fixed k, we have

$$T(k,n) = \frac{n^2}{2}\left(1 - \frac{1}{k-1}\right) + O_k(1) \le \frac{n^2}{2}\left(1 - \frac{1}{k-1}\right).$$

Moreover, the only K_k-free graph for which this maximum is attained is the so-called Turán graph. This is a complete $(k-1)$-partite graph whose classes are as equal as possible, i.e., each class consists of $\lfloor n/(k-1)\rfloor$ or $\lceil n/(k-1)\rceil$ points.

Theorem A. ([12], Theorem 5) *For any $d \ge 2$, there exist positive constants c_d, n_d such that for every $t \ge 1$, every separated set $P \subset \mathbb{R}^d$ with $n \ge n_d$ elements the following holds. The set P has at most $T(d+1,n)$ unordered point pairs whose distances belong to the interval $[t, t + c_d n^{1/d}]$. This bound is best possible for every d and every $n \ge n_d$.*

To see that the bound $T(d+1,n)$ can be attained, we write $\mathbb{R}^d$ as $\mathbb{R}^{d-1} \times \mathbb{R}$, and let $q_1, \dots, q_d \in \mathbb{R}^{d-1}$ be the vertices of a regular $(d-1)$-simplex of edge length t. Write n as a sum, $n = n_1 + \ldots + n_d$, where $n_i = \lfloor n/d\rfloor$ or $\lceil n/d\rceil$ for every i. Let $P := \{q_i + je_d : 1 \le i \le d,\ 1 \le j \le n_i\}$, where $e_d = (0, \dots, 0, 1)$. If t is large enough (depending on n), then all distances between two points in distinct sets $P_{i(1)}$ and $P_{i(2)}$ belong to the interval $[t, t+1]$, and the number of such pairs is $T(d+1,n)$.

Originally, Theorem A was stated for unit intervals $[t, t+1]$, but its proof easily extends to this case. (See the paragraph after Lemma 3.1 in [12].)

We say that a set *determines a distance* $t > 0$ if it has two points at distance t from each other. It is our goal to extend Theorem A and obtain an upper bound for the number of pairs whose distances fall into the union of $k \ge 2$ unit, or short, intervals. In [9], we made the first step in this direction by providing an asymptotically tight bound in the plane.

Theorem B. ([9], Theorem 2) *For any $k \ge 2$ and $\varepsilon > 0$, there exist positive constants $c_{k,\varepsilon}$ and $n_{k,\varepsilon}$ such that for every $t_1, \dots, t_k \ge 1$, for every separated set $P \subset \mathbb{R}^2$ with $|P| = n \ge n_{k,\varepsilon}$, the following holds.*

The number of unordered point pairs from P that determine a distance belonging to the set $\cup_{i=1}^k [t_i, t_i + c_{k,\varepsilon} n^{1/2}]$, is at most

$$\frac{n^2}{2}\left(1 - \frac{1}{k+1} + \varepsilon\right).$$

This statement is asymptotically tight: it does not remain true if we replace the last expression by $T(k+2,n) - 1$.

An example of an n-element point set with $T(k+2,n)$ pairs whose distances are nearly equal to one of k numbers, $t_1, \dots, t_k$ is the following. Let $t_i := it$, for $1 \le i \le k$, and let $n = n_1 + \ldots + n_{k+1}$, where the n_h's, for $1 \le h \le k+1$, are as equal as possible. Let $P_h = \{((h-1)t, j) : 1 \le j \le n_h\}$ and $P = \cup_{h=1}^{k+1} P_h$. If, for a given n, t is large enough, then every distance between two points belonging to distinct P_h's lies in $\cup_{i=1}^k [t_i, t_i + 1]$.

To generalize Theorem B to higher dimensions, we need a definition.

Definition 1. For any positive integers d and k, we call a finite subset of $\mathbb{R}^d$ a k-*distance set* if it determines at most k distinct (positive) distances. Let $m(d,k)$ denote the maximum cardinality of a k-distance set in $\mathbb{R}^d$. (This exists by Ramsey's theorem.) If $k=2$, we write $m(d) := m(d,2)$, for simplicity.

Estimating the value of $m(d,k)$ is equivalent to Erdős's *distinct distances problem* [6, 7] and has a huge literature. In particular, it is known [2, 3] that

$$\binom{d+1}{k} \le m(d,k) \le \binom{d+k}{k}. \tag{1.1}$$

This implies that for a fixed k and $d \to \infty$, we have $m(d,k) = (d^k/k!)\,(1+o_k(1))$, while for a fixed d and $k \to \infty$, we have $m(d,k) \le (k^d/d!)\,(1+o_d(1))$. The asymptotically best upper bounds for $m(2,k)$ and $m(d,k)$ for $d \ge 3$ have been established by Guth and Katz [16] and by Solymosi and Vu [25], resp.

For our purposes, the case $k=2$ will be relevant. For the maximum cardinality $m(d) = m(d,2)$ of a 2-distance set in $\mathbb{R}^d$, it is known that

$$\begin{cases} m(1)=3,\ m(2)=5\ [8],\ m(3)=6\ [5],\ m(4)=10, \\ m(5)=16,\ m(6)=27,\ m(7)=29,\ m(8)=45\ [18]. \end{cases} \tag{1.2}$$

In particular,

$$\text{for all } d \ge 2, \text{ we have } m(d-1) > d\,. \tag{1.3}$$

Our main result is the following generalization of the special case $k=2$ of Theorem B to higher dimensions.

Theorem 1. *For any integer $d \ge 2$, $d \ne 4,5$, there exist positive constants c_d, n_d such that for any $t_1, t_2 \ge 1$, for every separated point set $P \subset \mathbb{R}^d$ with $n \ge n_d$ elements, the following holds. The number of unordered point pairs in P that determine a distance belonging to the set $[t_1, t_1 + c_d n^{1/d}] \cup [t_2, t_2 + c_d n^{1/d}]$, is at most*

$$T\left(m(d-1)+1, n\right) = \frac{n^2}{2}\left(1 - \frac{1}{m(d-1)}\right) + O_d(1).$$

For $d=4$ or 5, for any $\varepsilon > 0$, there exist positive constants $c_{d,\varepsilon}, n_{d,\varepsilon}$ such that for any $t_1, t_2 \ge 1$, for every separated point set $P \subset \mathbb{R}^d$ with $n \ge n_{d,\varepsilon}$ elements, the following holds. The number of unordered point pairs in P that determine a distance belonging to the set $[t_1, t_1 + c_{d,\varepsilon}(\log n)^{1/d}] \cup [t_2, t_2 + c_{d,\varepsilon}(\log n)^{1/d}]$ is at most

$$\frac{n^2}{2}\left(1 - \frac{1}{m(d-1)} + \varepsilon\right).$$

These upper bounds cannot be reduced to $T\left(m(d-1)+1, n\right) - 1$, for any d and n.

We also study a closely related problem, where two distances are considered nearly equal if they fall into an interval $[t, t(1+\varepsilon)]$, for some small $\varepsilon > 0$. To formulate our result we need to extend Definition 1, as follows.

Definition 2. For any $\varepsilon \ge 0$ and integers $d, k \ge 1$, we call a finite subset of $\mathbb{R}^d$ a (k,ε)-*distance set* if all distances determined by it lie in the union of k intervals of the form $[t_1, t_1(1+\varepsilon)], \ldots, [t_k, t_k(1+\varepsilon)]$, for some $t_1, \ldots, t_k > 0$. Let $m(d, k, \varepsilon)$ denote the maximal cardinality of a (k, ε)-distance set in $\mathbb{R}^d$. (This is finite for every $\varepsilon > 0$. In fact, by applying Ramsey's theorem, it is enough to see that $m(d, 1, \varepsilon)$ is finite, and this follows from a volume argument.)

Obviously, a $(k, 0)$-distance set is a k-distance set and $m(d, k, 0) = m(d, k)$.

Theorem 2. *For any fixed integers $d, k \geq 1$ there exists $\varepsilon_{d,k} > 0$ such that for $0 < \varepsilon < \varepsilon_{d,k}$ the following two statements hold.*

(A) *For the maximum cardinality of a (k, ε)-distance set in $\mathbb{R}^d$, we have*

$$m(d, k, \varepsilon) = (d+1)^k.$$

(B) *For any set $P \subset \mathbb{R}^d$ of $n \geq 1$ points, and for any $t_1, \ldots, t_k > 0$, the following holds. The number of unordered pairs in P that determine a distance belonging to the set $\cup_{j=1}^k [t_j, t_j(1+\varepsilon)]$, is at most the Turán number $T\left((d+1)^k + 1, n\right)$. This upper bound cannot be reduced to $T\left((d+1)^k + 1, n\right) - 1$, for any d, k and n, and any $\varepsilon > 0$.*

It follows from Theorem 2 (A) and (1.1) that, for k fixed and $d \to \infty$,

$$1 \geq \frac{m(d,k)}{\lim\limits_{\varepsilon \searrow 0} m(d,k,\varepsilon)} = \frac{m(d,k)}{(d+1)^k} = \frac{1}{k!} + o_k(1).$$

Observe that in Definition 2 and Theorem 2, the assumption that P is separated is not required. (Actually, the concept of a (k, ε)-distance set is similarity invariant, so we could have required this property as well.)

The rest of this paper is organized as follows. In §2, we describe several constructions showing the tightness of Theorems 1 and 2. §3 and §4 contain the proofs of Theorems 1 and 2, resp. In §5, we make some concluding remarks.

The present paper is a minimally edited version of a manuscript written in the early 1990s. We posted it on arXiv in January 2019 [10]. A somewhat weaker version of Theorem 1 was announced in [21] in 2002. Our proofs use simple Turán-type results and elementary geometric observations. The first inequality of Theorem 1 has been generalized by Nóra Frankl and Andrey Kupavskii to unions of $k \geq 2$ intervals, for any $d \geq d(k)$ for some $d(k)$ ([14] Theorem 1.2 and [15] Theorem 13).

Moreover, they proved in [15] Theorem 12 the following. Let us fix any $d, k \geq 2$. Then there exists a natural number $N_k(d)$, such that the following holds. For any $\varepsilon > 0$, there exists a natural number $n(d, k, \varepsilon)$, such that for all $n \geq n(d, k, \varepsilon)$ the following is valid. The maximum number of unordered pairs of points, from any n points in $\mathbb{R}^d$, whose distances lie in the union of k intervals, lies in $[T(N_k(d), n), T(N_k(d), n) + \varepsilon n^2]$.

§2. Constructions

The aim of this section is to describe the constructions showing the tightness of Theorems 1 and 2.

Construction 1. We regard $\mathbb{R}^{d-1}$ as the hyperplane of $\mathbb{R}^d$ spanned by the first $d-1$ usual basic unit vectors. Let $Q \subset \mathbb{R}^{d-1}$ be a finite point set, with all distances sufficiently large, and let $m := |Q|$. Suppose that the distances determined by

Q all lie in the union of k intervals of length ε each, where $0 \le \varepsilon < 1$. Let $Q = \{q_1, \dots, q_m\}$. Let $n = n_1 + \ldots + n_m$, where each n_i is $\lfloor n/m \rfloor$ or $\lceil n/m \rceil$. We construct a point system $P = P(Q)$ of n points in $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ as follows. We let

$$P(Q) := \{q_i + je_d : 1 \le i \le m,\ 1 \le j \le n_i\},$$

where e_d is the d-th usual unit basic vector in $\mathbb{R}^d$. If, for given n, all distances determined by Q are large enough, then the following holds. The distances of all pairs of points $q_{i(1)} + j(1)e_d$, $q_{i(2)} + j(2)e_d \in P(Q)$ with $i(1) \ne i(2)$ lie in the union of k unit intervals (or of k arbitrarily small intervals, provided ε can be made arbitrarily small). The number of these pairs of points is $(n^2/2)(1-1/m) + O_{d,k}(1) \le (n^2/2)(1-1/m)$, for $n \to \infty$.

We present two particular cases of Construction 1.

Construction 1$'$. The case $k = 2$ of this construction will show the tightness of Theorem 1.

Let k be fixed and $d \to \infty$. In Construction 1, we choose $Q \subset \mathbb{R}^{d-1}$ as a k-distance subset of maximum cardinality $m(d-1,k)$, with all distances sufficiently large. By (1.1),

$$|Q| = m(d-1,k) = \frac{d^k}{k!}\left(1 + o_k(1)\right).$$

Then the set $P(Q)$ determines

$$\begin{aligned}\frac{n^2}{2}\left(1 - \frac{1}{|Q|}\right) + O_{d,k}(1) &= \frac{n^2}{2}\left(1 - \frac{1}{m(d-1,k)}\right) + O_{d,k}(1)\\ &\le \frac{n^2}{2}\left(1 - \frac{1}{m(d-1,k)}\right)\end{aligned}$$

distances, taken with multiplicity, that lie in the union of k intervals of arbitrarily small length.

Construction 1$''$. Let d be fixed and $k \to \infty$. We construct a set $Q \subset \mathbb{R}^{d-1}$ as follows. Let $k = k_1 + \ldots + k_{d-1}$, where each k_i is $\lfloor k/(d-1) \rfloor$ or $\lceil k/(d-1) \rceil$. We write $\{e_1, \dots, e_d\}$ for the usual basic unit vectors in $\mathbb{R}^d$. Let $n \ll \lambda_1 \ll \lambda_2 \ll \ldots \ll \lambda_{d-1}$ and let

$$Q := \Big\{\sum_{i=1}^{d-1} j_i \lambda_i e_i : j_i \in \{0, 1, \dots, k_i\}\Big\}.$$

Then the distance between any two distinct points, $\sum_{i=1}^{d-1} j_{i(1)} \lambda_i e_i$, $\sum_{i=1}^{d-1} j_{i(2)} \lambda_i e_i \in Q$, is very close to one of the distances $\lambda_i, 2\lambda_i, \dots, k_i\lambda_i$, where i is the largest index $\ell \in \{1, \dots, d-1\}$ such that $j_{\ell(1)} \ne j_{\ell(2)}$. The total number of these distances is $k_1 + \ldots + k_{d-1} = k$, and we have $|Q| = \prod_{i=1}^{d-1}(k_i + 1)$. Hence, for a fixed d and $k \to \infty$, we have

$$|Q| = \frac{k^{d-1}}{(d-1)^{d-1}}\left(1 + o_d(1)\right) \le \frac{(k+d-1)^{d-1}}{(d-1)^{d-1}}.$$

Using that $n \ll \lambda_1$, the number of distances determined by $P(Q)$ that lie in the union of k intervals of arbitrarily small length is

$$\frac{n^2}{2}\left(1 - \frac{1}{|Q|}\right) + O_{d,k}(1) \le \frac{n^2}{2}\left(1 - \frac{(d-1)^{d-1}}{(k+d-1)^{d-1}}\right).$$

It is somewhat surprising that for a fixed $d \geq 3$ and any $\varepsilon > 0$, a point set in $\mathbb{R}^{d-1}$ in which all distances are at least 1 and belong to k intervals of length ε, can be much larger than the conjectured maximum size of a point set in $\mathbb{R}^{d-1}$ in which every point pair determines one of k specific distances. (The conjectures are $m(2,k) = \Theta\left(k(\log k)^{1/2}\right)$, and $d-1 \geq 3 \Longrightarrow m(d-1,k) = \Theta_d(k^{(d-1)/2})$, see [7].) This is in sharp contrast with Theorem 1.1 in [14], and Theorem 1 in [15], stating that if k is fixed, $d \geq d_k$, and $\varepsilon \in (0, \varepsilon_{d,k})$, then these two quantities coincide.

Construction 2. We construct, for any $d, k \geq 1$ and any $\varepsilon \in (0,1)$, a (k,ε)-distance set in $\mathbb{R}^d$, of cardinality $(d+1)^k$. This will show the tightness of Theorem 2, (A).

Let us choose, for some small $\varepsilon_1 \in (0,1)$, positive numbers $s_1, \ldots, s_k$, satisfying $s_i/s_{i+1} \leq \varepsilon_1$ for every $i \in \{1, \ldots, k-1\}$. Fix k regular simplices centred at 0, with circumradii $s_1, \ldots, s_k$, and with vertices

$$\{v_{1,i} : 1 \leq i \leq d+1\}, \ldots, \{v_{k,i} : 1 \leq i \leq d+1\}.$$

Define the set of $(d+1)^k$ vectors,

$$S := \{v_{1,i(1)} + \ldots + v_{k,i(k)} : 1 \leq i(1) \leq d+1, \ldots, 1 \leq i(k) \leq d+1\}.$$

For any different $v_{1,i(1)} + \ldots + v_{k,i(k)},\ v_{1,j(1)} + \ldots + v_{k,j(k)} \in S$, let h be the largest index $\ell \in \{1, \ldots, k\}$ such that $i(\ell) \neq j(\ell)$. Then their distance equals

$$d(v_{1,i(1)} + \ldots + v_{h,i(h)}, v_{1,j(1)} + \ldots + v_{h,j(h)}) \in$$

$$[d(v_{h,i(h)}, v_{h,j(h)}) - 2s_{h-1} - \ldots - 2s_1, d(v_{h,i(h)}, v_{h,j(h)}) + 2s_{h-1} + \ldots + 2s_1] =$$

$$[(2(1+1/d))^{1/2} s_h - 2s_{h-1} - \ldots - 2s_1, (2(1+1/d))^{1/2} s_h + 2s_{h-1} + \ldots + 2s_1].$$

If ε_1 is sufficiently small, then for any $h \in \{1, \ldots, k\}$ the quotient of the maximum and the minimum of the last interval lies in $[1, 1+\varepsilon]$. Hence, S is a (k,ε)-distance set, with

$$t_h := (2(1+1/d))^{1/2} s_h - 2s_{h-1} - \ldots - 2s_1 \text{ for any } h \in \{1, \ldots, k\}.$$

Construction 3. We construct, for any $d, k \geq 1$, any $\varepsilon > 0$ and any n, a set $\{p_1, \ldots, p_n\}$ of n points in $\mathbb{R}^d$ with the following property. The number of point pairs determining a distance that belongs to $\cup_{j=1}^k [t_j, t_j(1+\varepsilon)]$, for some $t_1, \ldots, t_k > 0$, is equal to $T\left((d+1)^k + 1, n\right)$. This will show the tightness of Theorem 2, (B).

The points $p_1, \ldots, p_n$ are divided into $(d+1)^k$ classes, with $\lfloor n/(d+1)^k \rfloor$ or $\lceil n/(d+1)^k \rceil$ points in each class, so that the distance between any two points in different classes belongs to $\cup_{j=1}^k [t_j, t_j(1+\varepsilon)]$. Each of the $(d+1)^k$ classes of points is chosen in the 1-neighbourhood of one of the $(d+1)^k$ points of the set S as in Construction 2, where we also assume that $1/s_1 \leq \varepsilon_1$. Like in Construction 2, the distance between any two points in different classes belongs to the interval

$$[(2(1+1/d))^{1/2} s_h - 2s_{h-1} - \ldots - 2s_1 - 2, (2(1+1/d))^{1/2} s_h + 2s_{h-1} + \ldots + 2s_1 + 2].$$

Here, h is the largest index $\ell \in \{1, \ldots, k\}$ such that $i(\ell) \neq j(\ell)$, with $v_{1,i(1)} + \ldots + v_{k,i(k)}$ and $v_{1,j(1)} + \ldots + v_{k,j(k)}$ being the elements of S in Construction 2, associated with the classes of the two points. If ε_1 is sufficiently small, then for any $h \in \{1, \ldots, k\}$, the quotient of the maximum and the minimum elements of the last (displayed) interval lies in $[1, 1+\varepsilon]$. Thus we can choose

$$t_h := (2(1+1/d))^{1/2} s_h - 2s_{h-1} - \ldots - 2s_1 - 2, \text{ for any } h \in \{1, \ldots, k\}.$$

§3. Proof of Theorem 1

We first agree on some notation and terminology. We denote the vertex set of a graph G by $V(G)$. Throughout this paper, the term *subgraph* will always stand for *induced* or *spanned subgraph.* Let $d(p,q)$ denote the *distance between two points* $p,q \in \mathbb{R}^d$. The *norm of* $p \in \mathbb{R}^d$ is denoted by $\|p\|$. We write S^{d-1} for the unit sphere in $\mathbb{R}^d$. For any set $P \subset \mathbb{R}^d$, we write $\operatorname{diam} P$, $\operatorname{aff} P$ and $\operatorname{lin} P$ for the *diameter, affine hull,* and *linear hull of* P, resp. The *volume* (Lebesgue measure) *of a set in* $\mathbb{R}^d$ is denoted by $V(\cdot)$, while the $(d-1)$-*volume* is denoted by $V_{d-1}(\cdot)$. For $x_1,\dots,x_d \in \mathbb{R}^d$, denote by $\det(x_1,\dots,x_d)$ the *determinant with columns* $x_1,\dots,x_d$. For any $1 \le \ell \le d+1$ and any affinely independent vectors $x_1,\dots,x_\ell \in \mathbb{R}^d$, let $S(x_1,\dots,x_\ell)$ stand for the $(\ell-1)$-*dimensional simplex spanned by these vertices.*

Throughout, we suppose that $t_1 \le \ldots \le t_k$. The interval $[t_\kappa, t_\kappa + c_d n^{1/d}]$ will be referred to as the κ*-th interval.* The symbols $\mathrm{const}_d, C_d, D_d, c_d$ will denote positive constants depending on d (or on other parameters in the subscript). At different places, const_d may stand for different constants. We always assume that n is sufficiently large in terms of all fixed parameters.

In the rest of this section, we present the proof of Theorem 1. The proof falls into eight simple steps marked as **Step 1**, **Step 2**, etc. For $d \ne 4,5$, we give the proof in full detail. The treatment of the cases d=4,5 requires only minor modifications which are described in **Step 7** below.

Proof of Theorem 1. **Step 1**. The tightness of Theorem 1 was shown by Construction $1'$. Therefore, we only have to prove the upper bounds. Let $P = \{p_1,\dots,p_n\}$.

Lemma 1. *It is sufficient to prove Theorem 1 under the following assumptions.*

(1) *The intervals* $[t_1, t_1+c_d n^{1/d}]$ *and* $[t_2, t_2+c_d n^{1/d}]$ *are disjoint, and both contain at least one distance between two points of* P.

(2) *We have* $t_2 > t_1 \ge C_d n^{1/d}$, *where* $C_d > 1$ *can be chosen arbitrarily large.*

(3) *The ratio of any two distances that belong to the* κ*-th interval* ($\kappa = 1,2$) *lies in* $[(1+c_d)^{-1}, 1+c_d]$. *Hence, it lies in an arbitrarily small neighbourhood of 1, provided that we choose* $c_d > 0$ *sufficiently small.*

Proof. (1) If $[t_1, t_1+c_d n^{1/d}] \cap [t_2, t_2+c_d n^{1/d}] \ne \emptyset$, then the length of the union of the two intervals is at most $2c_d n^{1/d}$. Hence, if $c_d > 0$ is sufficiently small, Theorem A yields the following. The number of pairs $\{p_{i(1)}, p_{i(2)}\}$ whose distances belong to the union of the two intervals is at most $T(d+1,n)$. By (1.3), we have $T(d+1,n) \le T(m(d-1)+1,n)$, and Theorem 1 follows.

The same argument applies if one of the intervals does not contain any distance $d(p_{i(1)}, p_{i(2)})$.

(2) Suppose that $t_1 \le C_d n^{1/d}$ for an arbitrarily large constant C_d. By our assumptions, the open balls of radius $1/2$ centred at the points p_i are disjoint. Thus, by volume considerations, for any fixed $p_{i(1)}$, the number of $p_{i(2)}$'s with $d(p_{i(1)}, p_{i(2)}) \in [t_1, t_1+c_d n^{1/d}]$ is at most $\mathrm{const}_d \cdot \left[(t_1+c_d n^{1/d}+1/2)^d - (t_1-1/2)^d\right]$. Hence, the number of all pairs $\{p_{i(1)}, p_{i(2)}\}$, where $d(p_{i(1)}, p_{i(2)})$ belongs to the first interval, is at most

$$\mathrm{const}_d \cdot \left[n\left((t_1+c_d n^{1/d}+1/2)^d - (t_1-1/2)^d\right)\right] \le n \cdot \mathrm{const}_d \cdot (t_1+c_d n^{1/d})^{d-1} \cdot c_d n^{1/d}$$

$$\leq n \cdot \mathrm{const}_d \cdot (C_d n^{1/d} + c_d n^{1/d})^{d-1} \cdot c_d n^{1/d} = n^2 \cdot \mathrm{const}_d \cdot (C_d + c_d)^{d-1} c_d \leq \delta n^2,$$

provided that we choose $c_d > 0$ so small compared to const_d and C_d that $\mathrm{const}_d \cdot (C_d + c_d)^{d-1} c_d \leq \delta$ holds.

By Theorem A, the number of pairs $\{p_{i(1)}, p_{i(2)}\}$ with $d(p_{i(1)}, p_{i(2)}) \in [t_2, t_2 + c_d n^{1/d}]$ is at most $T(d+1, n) = (n^2/2)(1 - 1/d) + O_d(1)$. Hence, the number of pairs for which $d(p_{i(1)}, p_{i(2)})$ belongs to the union of the two intervals in question is at most $(n^2/2)(1 - 1/d + 2\delta) + O_d(1)$. In view of (1.3), the last expression is bounded from above by

$$T(m(d-1)+1, n) = \frac{n^2}{2}\left(1 - \frac{1}{m(d-1)}\right) + O_d(1),$$

provided that $2\delta < 1/d - 1/(d+1) \leq 1/d - 1/m(d-1)$ and n is sufficiently large. Thus, in the case $t_1 \leq C_d n^{1/d}$, Theorem 1 is true.

(3) It follows from part (2) that

$$\frac{t_2 + c_d n^{1/d}}{t_2} \leq \frac{t_1 + c_d n^{1/d}}{t_1} \leq \frac{C_d n^{1/d} + c_d n^{1/d}}{C_d n^{1/d}} = 1 + \frac{c_d}{C_d} \leq 1 + c_d,$$

which proves (3). □

Proof of Theorem 1, continuation. **Step 2**. In the rest of the proof, we assume that conditions (1), (2), and (3) of Lemma 1 are satisfied. Consider the *graph G with vertex set $\{p_1, \dots, p_n\}$, where $p_{i(1)}$ and $p_{i(2)}$ are connected by an edge if and only if $d(p_{i(1)}, p_{i(2)})$ belongs to one of the two intervals in question.*

Suppose, in order to obtain a contradiction, that the number of edges of G is greater than $T(m(d-1)+1, n)$. By [1], Ch. 6, G contains a subgraph $G_1 = K(1, 1, \dots, 1, \lfloor \mathrm{const}_d \cdot n \rfloor)$, that is, a complete $(m(d-1)+1)$-partite graph with $1, 1, \dots, 1, \lfloor \mathrm{const}_d \cdot n \rfloor$ points in its parts called *primary colour classes.* (So here we consider the vertices coloured.) Obviously, it makes sense to speak about the j-th primary colour class of any (spanned) subgraph of G_1, for $1 \leq j \leq m(d-1)+1$. This is the intersection of the j-th primary colour class of G_1 with the vertex set of the subgraph. For the subgraphs considered later in this proof, these primary colour classes are always non-empty.

Define the *secondary colouring* of the *edges* of G_1, as follows. Assign to each edge the symbol L and R, according to whether the length of the corresponding segment lies in the first or in the second interval. Since the two intervals are disjoint (cf. Lemma 1 (1)), the secondary colouring is uniquely determined.

At least half of the points of the $(m(d-1)+1)$-st primary colour class of $G_1 = K(1, 1, \dots, 1, \lfloor \mathrm{const}_d \cdot n \rfloor)$ are joined by edges of the same secondary colour L or R to the unique point in the first primary colour class. By induction, the unique points in the 1st, 2nd, …, $m(d-1)$-st primary colour classes of G_1 and some $\lfloor \mathrm{const}_d \cdot n \rfloor$ points in the $(m(d-1)+1)$-st primary colour class of G_1 satisfy the following. The secondary colour of an edge between any two of these points only depends on the primary colour classes the endpoints of the edge belong to. We denote the subgraph induced by all these points by G_1^*.

From now on, we will consider G_1^* rather than G_1. We will show that such a graph G_1^* cannot exist, for $c_d > 0$ a sufficiently small constant. This contradiction will prove that the number of pairs $\{p_{i(1)}, p_{i(2)}\}$, whose distances lie in the union of our two intervals, is at most as large as is stated in Theorem 1.

Step 3. Let $D_d > 2$ be a sufficiently large constant. We distinguish two cases:

$$\begin{aligned} \text{Case I:} \quad & t_2/(t_1 + c_d n^{1/d}) \leq D_d, \\ \text{Case II:} \quad & t_2/(t_1 + c_d n^{1/d}) > D_d > 2\,. \end{aligned}$$

In Case I, we use two-distance sets in $\mathbb{R}^{d-1}$. The proof will be presented in **Step 4**, and will be completed using Lemma 2.

In Case II, the two types of distances, i.e., those belonging to the first interval and to the second one, can be treated separately. The segments corresponding to different types of distances will turn out to be "almost orthogonal". We will describe the structure of our edge coloured graph. The proof in this case will be carried out in **Step 5** and completed by Lemma 9.

Step 4. First, we analyze Case I. The proof of the following lemma consists of six easy parts (enumerated as **A**, **B**, ..., **F**).

Lemma 2. *The upper estimate of Theorem 1 holds in Case I.*

Proof. **A.** In Case I, we have by Lemma 1 (3), for $c_d > 0$ sufficiently small

$$\frac{t_2 + c_d n^{1/d}}{t_1} = \frac{t_2 + c_d n^{1/d}}{t_2} \cdot \frac{t_2}{t_1 + c_d n^{1/d}} \cdot \frac{t_1 + c_d n^{1/d}}{t_1} \leq (1 + c_d)^2 D_d \leq \text{const}_d \cdot D_d\,.$$

Thus, in Case I, the quotient of any two distances lying in the union of our two intervals is at most $\text{const}_d \cdot D_d$. Therefore, these quotients lie between two positive bounds, namely $(\text{const}_d \cdot D_d)^{-1}$ and $\text{const}_d \cdot D_d$. In particular, this holds for the distances between the endpoints of the edges of the graph G_1^*.

B. Definition 3. Let $m > d$ be any integer, and let $x_1, \ldots, x_m, x_{m+1} \in \mathbb{R}^d$ be any distinct points in $\mathbb{R}^d$. Let $\Delta(x_1, \ldots, x_m, x_{m+1})$ be the maximum absolute value of all determinants whose columns are any d vectors from the set

$$\{(x_1 - x_{m+1})/d(x_1, x_{m+1}), \ldots, (x_m - x_{m+1})/d(x_m, x_{m+1})\} \subset S^{d-1}.$$

Clearly, $\Delta(x_1, \ldots, x_{m+1})$ is invariant under simultaneous similarity transformations of $x_1, \ldots, x_{m+1}$. Furthermore, it is nonnegative, and equals 0 if and only if $x_1, \ldots, x_{m+1}$ lie in an (affine) hyperplane. Thus, it can be considered as a measure of "non-hyperplanarity of $x_1, \ldots, x_{m+1}$". We will apply the above definition for the case $m := m(d-1)$ (recall (1.3)).

C. Claim 1. *Let $q_1, \ldots, q_{m(d-1)+1}$ be vertices of G_1^*, one from each respective primary colour class.* (Thus, $q_1, \ldots, q_{m(d-1)}$ are fixed, but $q_{m(d-1)+1}$ can assume $\lfloor \text{const}_d \cdot n \rfloor$ values, i.e., points.) *Then $\Delta(q_1, \ldots, q_{m(d-1)+1})$ is at least some positive constant, provided that $c_d > 0$ is sufficiently small.*

Proof. Suppose, for contradiction, that $c_d > 0$ is very small, i.e., we have $c_d^N < 1/N$, say, for a large integer N, but $\Delta(q_1, \ldots, q_{m(d-1)+1})$ can get arbitrarily close to 0. That is, $\Delta(q_1, \ldots, q_{m(d-1)+1}) < 1/N$, say, for some choice $q_1^N, \ldots, q_{m(d-1)+1}^N$ of the points $q_1, \dots , q_{m(d-1)+1}$. (Actually, only the last point can vary.) We apply to each of $q_1^N, \dots , q_{m(d-1)+1}^N$, simultaneously, a similarity transformation Φ_λ with ratio $\lambda > 0$ such that the following holds. We have $\operatorname{diam}\{\Phi_\lambda q_1^N, \dots , \Phi_\lambda q_{m(d-1)+1}^N\} = 1$ and $\{\Phi_\lambda q_1^N, \dots , \Phi_\lambda q_{m(d-1)+1}^N\}$ lies in the unit ball of $\mathbb{R}^d$. Then, by **A**, the minimal distance in $\{\Phi_\lambda q_1^N, \dots , \Phi_\lambda q_{m(d-1)+1}^N\}$ is at least $1/(\mathrm{const}_d \cdot D_d) > 0$.

Now let $N \to \infty$. Then, for a certain subsequence $N(\nu)$ of the N's, the following four statements are true:
(i) for each $1 \le j \le m(d-1)+1$, we have that $\lim_{\nu\to\infty} \Phi_\lambda(q_j^{N(\nu)})$ exists;
(ii) these limit points have pairwise distances at least $1/(\mathrm{const}_d \cdot D_d)$;
(iii) for any $j(1) \neq j(2)$, the distance $d(\Phi_\lambda(q_{j(1)}^{N(\nu)}), \Phi_\lambda(q_{j(2)}^{N(\nu)}))$ lies in $[\lambda t_\kappa, \lambda(t_\kappa + c_d n^{1/d})]$ for some $\kappa \in \{1, 2\}$.
(iv) $\lim_{\nu\to\infty} \Delta\left(q_1^{N(\nu)}, \dots , q_{m(d-1)+1}^{N(\nu)}\right) = 0$.

By (ii), $\left(\lim_{\nu\to\infty} \Phi_\lambda(q_1^{N(\nu)}), \dots , \lim_{\nu\to\infty} \Phi_\lambda(q_{m(d-1)+1}^{N(\nu)})\right)$ belongs to the domain of definition of the function $\Delta(x_1, \dots , x_{m(d-1)+1})$.

By (iii), Lemma 1 (3) and $c_d^{N(\nu)} < 1/N(\nu)$, we have that any two numbers that belong to the same new interval $[\lambda t_\kappa, \lambda(t_\kappa + c_d n^{1/d})]$, for $\kappa \in \{1, 2\}$, have a ratio in $[(1 + c_d^{N(\nu)})^{-1}, 1 + c_d^{N(\nu)}] \subset [(1 + (1/N(\nu)))^{-1}, 1 + (1/N(\nu))]$. Thus, this ratio lies in an as small neighbourhood of 1, as we want. Therefore, for $\nu \to \infty$, both our new κ-th intervals converge to degenerate intervals, i.e., to points. In particular, the second new interval converges to $\{1\}$.

By the similarity invariance of $\Delta(\cdot)$ and (iv), we have

$$\Delta\left(\lim_{\nu\to\infty} \Phi_\lambda q_1^{N(\nu)}, \dots , \lim_{\nu\to\infty} \Phi_\lambda q_{m(d-1)+1}^{N(\nu)}\right)$$

$$= \lim_{\nu\to\infty} \Delta\left(\Phi_\lambda q_1^{N(\nu)}, \dots , \Phi_\lambda q_{m(d-1)+1}^{N(\nu)}\right) = 0.$$

That is, the points $\lim_{\nu\to\infty} \Phi_\lambda q_1^{N(\nu)}, \dots , \lim_{\nu\to\infty} \Phi_\lambda q_{m(d-1)+1}^{N(\nu)}$ lie in some hyperplane of $\mathbb{R}^d$, their number is $m(d-1)+1$, and they determine only two distinct distances. This contradiction ends the proof of Claim 1. $\square$

D. Let us fix some $q_{m(d-1)+1}$ in the $(m(d-1)+1)$-st primary colour class of G_1^*. By Claim 1, among the $m(d-1) > d$ unit vectors

$$u_1(q_{m(d-1)+1}) := (q_1 - q_{m(d-1)+1})/d(q_1, q_{m(d-1)+1}), \dots ,$$

$$u_{m(d-1)}(q_{m(d-1)+1}) := (q_{m(d-1)} - q_{m(d-1)+1})/d(q_1, q_{m(d-1)+1}),$$

there are $u_{j(1)}(q_{m(d-1)+1}), \dots , u_{j(d)}(q_{m(d-1)+1})$ such that

$$|\det\left(u_{j(1)}(q_{m(d-1)+1}), \dots , u_{j(d)}(q_{m(d-1)+1})\right)| \ge \mathrm{const}_d > 0.$$

Since there are only const_d choices for these d-tuples, still for $\lfloor \mathrm{const}_d \cdot n\rfloor$ many choices of $q_{m(d-1)+1}$ this d-tuple is the same, $\{u_{j(1)}, \dots , u_{j(d)}\}$, say. We will write $C_{m(d-1)+1}$ for the set of these $\lfloor \mathrm{const}_d \cdot n\rfloor$ points $q_{m(d-1)+1}$ in the $(m(d-1)+1)$-st primary colour class of G_1^*. Thus,

$$|\det\left(u_{j(1)}, \dots , u_{j(d)}\right)| \ge \mathrm{const}_d > 0\,.$$

Hence, we have a *d-dimensional simplex* $S(q_{j(1)}, \dots , q_{j(d)}, q_{m(d-1)+1})$, and a $(d-1)$-*dimensional simplex* $S(q_{j(1)}, \dots , q_{j(d)})$.

From the set of unit vectors $u_1, \dots , u_{m(d-1)}$, we will consider only $u_{j(1)}, \dots , u_{j(d)}$. Further, from among all $\mathrm{const}_d \cdot n$ points $q_{m(d-1)+1}$ in the $(m(d-1)+1)$-st primary colour class of G_1^*, we will restrict our attention to the subset $C_{m(d-1)+1}$. *We write* ${G_1^*}'$ *for the induced subgraph of* G_1^**, containing all (single) vertices of* G_1^* *in its first* $m(d-1)$ *primary colour classes, and* $C_{m(d-1)+1}$ *from its last primary colour class.*

E. Recall from **Step 2** of the proof of Theorem 1 the following. For any $h = 1, \dots , d$, either
(1) for any choice of the vertex $q_{m(d-1)+1,i}$ of the $(m(d-1)+1)$-st primary colour class of G_1^*, the distance $d(q_{j(h)}, q_{m(d-1)+1,i})$ lies in the first interval, or
(2) for any choice of the vertex $q_{m(d-1)+1,i}$ of the $(m(d-1)+1)$-st primary colour class of G_1^*, the distance $d(q_{j(h)}, q_{m(d-1)+1,i})$ lies in the second interval. In particular, this holds for the subgraph ${G_1^*}'$ of G_1^*. This means that

$$C_{m(d-1)+1} \subset \cap_{h=1}^d S_{j(h)}, \tag{3.1}$$

where $S_{j(1)}, \dots , S_{j(d)}$ are spherical shells with centres $q_{j(1)}, \dots , q_{j(d)}$, inner radii either t_1 or t_2, and outer radii either $t_1 + c_d n^{1/d}$ or $t_2 + c_d n^{1/d}$, resp. For each of these spherical shells, the quotient of the difference of the outer and inner radii and of the inner radius is $c_d n^{1/d}/t_\kappa \le c_d n^{1/d}/(C_d n^{1/d}) = c_d/C_d \le c_d$, by Lemma 1 (2). Hence, this quotient is in an arbitrarily small neighbourhood of 0, if $c_d > 0$ is chosen sufficiently small.

We are going to show that, for $c_d > 0$ sufficiently small, the inclusion (3.1) is impossible, yielding the desired contradiction.

Before this, we have to introduce some notations. Observe that $\mathrm{aff}\,\{q_{j(1)}, \dots , q_{j(d)}\}$ is a hyperplane of symmetry of $\cap_{h=1}^d S_{j(h)}$, which will be identified with the hyperplane $x_d = 0$. (As $S(q_{j(1)}, \dots , q_{j(d)})$ is $(d-1)$-dimensional, so is its affine hull.) Let H^+ and H^- denote the closed half-spaces $x_d \ge 0$ and $x_d \le 0$, resp. One of them contains at least half of the points of $C_{m(d-1)+1}$. We may suppose this is H^+. Thus

$$H^+ \text{ contains } \lfloor \mathrm{const}_d \cdot n \rfloor \text{ points of } C_{m(d-1)+1}.$$

Let us fix a point $q_{m(d-1)+1,1} \in C_{m(d-1)+1} \cap H^+$.

For any $h \in \{1, \dots , d\}$, define the slab $S'_{j(h)}$, as follows. Let $S'_{j(h)}$ be bounded by two hyperplanes, both orthogonal to $q_{j(h)} - q_{m(d-1)+1,1}$. Further, they intersect the half-line from $q_{j(h)}$, passing through $q_{m(d-1)+1,1}$, at points with distances $d(q_{j(h)}, q_{m(d-1)+1,1}) + c_d n^{1/d}$ and $d(q_{j(h)}, q_{m(d-1)+1,1}) - 2c_d n^{1/d}$ from $q_{j(h)}$. (By Lemma 1 (2), this difference is positive, for $c_d > 0$ sufficiently small.) We need the following

Claim 2. *If* $c_d > 0$ *is sufficiently small, then*

$$(\cap_{h=1}^d S_{j(h)}) \cap H^+ \subset \Pi := \cap_{h=1}^d S'_{j(h)}.$$

This holds both in Case I and in Case II.

This statement appears to be intuitively clear, but we have been unable to show it by a simple geometric argument. We provide a proof in the original version of our paper [10], on arXiv; see parts **8-10** of the proof of Theorem 1, pp. 9-13. It uses elements of the algebraic topology of Euclidean spaces [17].

F. Again, we handle both Cases I and II. The set Π in Claim 2 is a parallelepiped, and is circumscribed about a ball of diameter $3c_d n^{1/d}$. Its volume is $(3c_d/2)^d n$ times the volume of its homothetic copy Π^1 circumscribed about the unit ball. Moreover,

$$V(\Pi^1) = 2^d / |\det(u_{j(1),1}, \dots, u_{j(d),1})|,$$

with the denominator at least $\mathrm{const}_d > 0$, by **D**. (The easiest way to see this volume formula is as follows. The polar body $(\Pi^1)^*$ of Π^1 is a cross-polytope, with $V\left((\Pi^1)^*\right) = (2^d/d!)|\det(u_{j(1),1}, \dots, u_{j(d),1})|$. Simultaneously, the product $V(\Pi^1)V\left((\Pi^1)^*\right)$ of the two volumes is invariant under linear maps. Hence, it equals $4^d/d!$, as can be calculated from the case when Π_1 is the unit cube. Cf. [20], pp. 165, 169.) All these imply that

$$V(\Pi) \leq (3c_d/2)^d n 2^d / \mathrm{const}_d .$$

Now a standard volume consideration finishes the proof of Lemma 2. Consider the open balls of unit diameter, with centres at all $\lfloor \mathrm{const}_d \cdot n \rfloor$ points $q_{m(d-1)+1,i} \in C_{m(d-1)+1} \cap H^+ \subset \Pi$. *These are pairwise disjoint open balls contained in a concentric homothetic copy* Π' *of* Π, *with inradius* $3c_d n^{1/d}/2 + 1/2$. However,

$$V(\Pi') \leq \mathrm{const}_d \cdot c_d^d n.$$

So, if $c_d > 0$ is sufficiently small, then the volume of Π' is not large enough to contain $\lfloor \mathrm{const}_d \cdot n \rfloor$ disjoint open balls of unit diameter. This contradiction completes the proof of Lemma 2 and, hence, Theorem 1 in Case I (see **Step 3**). □

Next, we turn to the proof of Theorem 1 in Case II.

Proof of Theorem 1, continuation. **Step 5**. Now we assume that $t_2/(t_1 + c_d n^{1/d}) > D_d > 2$, where D_d is a sufficiently large constant (Case II).

We investigate the secondary (edge) colourings of the graph $G_1^{*\prime}$ from **Step 2** of the proof of Lemma 2. Each edge is coloured either by L or by R. Each edge coloured by R has length at least t_2, and each edge coloured by L has length at most $t_1 + c_d n^{1/d}$. By $t_2/(t_1 + c_d n^{1/d}) > 2$, *any edge coloured by* R *is more than twice as long as any edge coloured by* L.

This implies that one can define an equivalence relation $\sim$ on the vertices of $G_1^{*\prime}$ as follows.

Definition 4. For any two vertices $q_{j(1)}$, $q_{j(2)}$ of $G_1^{*\prime}$, we write $q_{j(1)} \sim q_{j(2)}$ if either $q_{j(1)} = q_{j(2)}$, or the edge $q_{j(1)}q_{j(2)}$ is coloured by L.

Recall from **Step 2** and the proof of Lemma 2, **D**, that the colour of an edge of $G_1^{*\prime}$ between vertices of two primary colour classes does not depend on the vertices chosen from the primary colour classes. (This is equivalent to its special case when one of the primary colour classes is the $(m(d-1)+1)$-st primary colour class.)

$$\text{Therefore, we may consider the relation } \sim \text{ as defined}$$

$$\text{alternatively on the set of primary colour classes of } G_1^{*\prime}.$$

Whether we consider it on the set of vertices, or on the primary colour classes, will be clear from the context. Let ℓ denote the number of $\sim$-equivalence classes.

Let us choose for each of the ℓ $\sim$-equivalence classes of primary colour classes of $G_1^{*\prime}$ one vertex q_j from their union; let these be $r_1, r_2, \ldots, r_\ell$. By Lemma 1 (3), edges coloured the same way have a ratio in an as small neighbourhood of 1 as we want, provided $c_d > 0$ is sufficiently small. Note that any edge among $r_1, \ldots, r_\ell$ is coloured by R. Therefore the quotients of the lengths of these edges are in an as small neighbourhood of 1 as we want, for $c_d > 0$ sufficiently small. Hence, for $c_d > 0$ sufficiently small, we have $\ell \leq d+1$. Namely, for $d+2$ points in $\mathbb{R}^d$ the quotient of the maximum and the minimum distances is at least some constant strictly greater than 1. (Cf. Schütte [24], Satz 3, which gives the sharp lower bound, which is $(1+2/d)^{1/2}$, for d even and $[1+2(d+2)/\left(d(d+2)-1\right)]^{1/2}$, for d odd.) The same argument shows that $r_1, \ldots, r_\ell$ cannot lie in an affine $(\ell-2)$-plane, thus determine an $(\ell-1)$-simplex, namely $S(r_1, \ldots, r_\ell)$.

Our goal is to show that the simplex $S(r_1, \ldots, r_\ell)$ is "close" to a regular $(\ell-1)$-simplex of edge length t_2. Similarly, the vertices of $G_1^{*\prime}$ in single $\sim$-equivalence classes are "close" to the vertices of regular simplices of edge length t_1, of dimensions at most $d-\ell+1$, with affine hulls nearly orthogonal to $\operatorname{aff}\{r_1, \ldots, r_\ell\}$. The number of primary colour classes of $G_1^{*\prime}$ is maximum if all of the last simplices have dimension $d-\ell+1$.

Lemma 3. *In Case II, the number ℓ of the $\sim$-equivalence classes is at least 2, provided $c_d > 0$ is sufficiently small.*

Proof. If $\ell = 1$, then all distances between the vertices of $G_1^{*\prime}$ lie in $[t_1, t_1 + c_d n^{1/d}]$, contradicting Lemma 1 (1). This proves Lemma 3. $\square$

Lemma 4. *In Case II, let $q_j \in V(G_1^{*\prime})$ be in the $\sim$-equivalence class of $r_1 \in V(G_1^{*\prime})$ such that $q_j \neq r_1$. Further, let $r_2 \in V(G_1^{*\prime})$ be in another $\sim$-equivalence class, as r_1. Then $|\sphericalangle q_j r_1 r_2 - \pi/2|$ is as small as we want, for D_d sufficiently large and $c_d > 0$ sufficiently small.* (Here r_2 exists by Lemma 3.)

Proof. We are going to estimate from above

$$|\cos(\sphericalangle q_j r_1 r_2)| = \left|d(r_1,r_2)^2 + d(r_1,q_j)^2 - d(q_j,r_2)^2\right| / \left(2d(r_1,r_2)d(r_1,q_j)\right)$$

$$= \left|\left(d(r_1,r_2) + d(q_j,r_2)\right)\left(d(r_1,r_2) - d(q_j,r_2)\right) + d(r_1,q_j)^2\right|$$

$$/\left(2d(r_1,r_2)d(r_1,q_j)\right) \leq [2(t_2 + c_d n^{1/d})c_d n^{1/d} + (t_1 + c_d n^{1/d})^2]/(2t_1t_2)\,.$$

By Lemma 1 (3), any two numbers from the same interval $[t_\kappa, t_\kappa + c_d n^{1/d}]$ have quotients as close to 1 as we want, for $c_d > 0$ sufficiently small. Thus, we suppose $t_1 + c_d n^{1/d} \leq 2t_1$ and $t_2 + c_d n^{1/d} \leq 2t_2$, for $c_d > 0$ sufficiently small. Then

$$|\cos(\sphericalangle q_j r_1 r_2)| \leq [2 \cdot 2t_2 \cdot c_d n^{1/d} + 4t_1^2]/(2t_1t_2) =$$

$$2c_d n^{1/d}/t_1 + 2t_1/t_2 \leq 2c_d/C_d + 2/D_d < 2c_d + 2/D_d,$$

by Lemma 1 (2), and by $t_2/t_1 \geq t_2/(t_1 + c_d n^{1/d}) > D_d$ (Case II). If D_d is sufficiently large and $c_d > 0$ is sufficiently small, then this last expression, and hence also $|\cos(\sphericalangle q_j r_1 r_2)|$ is as small as we want. This proves Lemma 4. $\square$

Lemma 5. *In Case II, the number ℓ of the $\sim$-equivalence classes is at most d, for D_d sufficiently large and $c_d > 0$ sufficiently small.*

Proof. We already know that $\ell \le d+1$ (cf. **Step 5**), so we have to exclude $\ell = d+1$ only.

Suppose $\ell = d+1$. At the beginning of **Step 5**, we selected points $r_1, \dots, r_\ell = r_{d+1}$, one from the union of each $\sim$-equivalence class of the primary colour classes of $V({G_1^*}')$. By Lemma 1 (3), for $c_d > 0$ sufficiently small, we have that the quotients of any two distances among these points are in an as small neighbourhood of 1 as we want. This also implies that any three of these points determine a triangle with angles as close to $\pi/3$ as we want.

Let $v_\mu := (r_\mu - r_{d+1})/d(r_\mu, r_{d+1}) \in S^{d-1}$, for $\mu = 1, \dots, d$. Let V denote the $d \times d$ matrix with columns $v_1, \dots, v_d$. We have

$$|\det(v_1, \dots, v_d)| = |\det V| = [\det(V'V)]^{1/2} = [\det(\langle v_{\mu(1)}, v_{\mu(2)}\rangle)]^{1/2},$$

where V' is the transposed matrix of V. Moreover, $(\langle v_{\mu(1)}, v_{\mu(2)}\rangle)$ is a $d \times d$ matrix for which $\langle v_{\mu(1)}, v_{\mu(1)}\rangle = 1$, and $\mu(1) \ne \mu(2)$ implies that $\langle v_{\mu(1)}, v_{\mu(2)}\rangle$ is as close to $\cos(\pi/3) = 1/2$ as we want. Hence, $|\det(v_1, \dots, v_d)|$ is as close to $[\det((1+\delta_{\mu(1)\mu(2)})/2)]^{1/2}$ as we want, for D_d sufficiently large and $c_d > 0$ sufficiently small. Here $[\det\big((1+\delta_{\mu(1)\mu(2)})/2\big)]^{1/2}$ equals the absolute value of the determinant whose columns are the unit vectors pointing from a vertex of a regular d-simplex to all other vertices. Thus it is a non-zero constant. All this implies that $|\det(v_1, \dots, v_d)|$ *is greater than a non-zero constant.* In particular, $v_1, \dots, v_d$ *are linearly independent.*

By (1.3), $m(d-1)+1 > d+1$ for $d \ge 2$, hence some of the $\ell = d+1$ $\sim$-equivalence classes must contain at least two vertices q_j of ${G_1^*}'$. Assume without loss of generality that r_{d+1} belongs to such a class and q_j is one of its elements different from r_{d+1}. In view of Lemma 4, the scalar product of the vector $v := (q_j - r_{d+1})/d(q_j, r_{d+1}) \in S^{d-1}$ with any v_μ, $1 \le \mu \le d$, is as close to 0 as we want, provided that D_d is sufficiently large and $c_d > 0$ is sufficiently small. In other words, $\max_{1\le\mu\le d} |\langle v, v_\mu\rangle|$ *is as small as we want, for D_d sufficiently large and $c_d > 0$ sufficiently small.*

By the linear independence of $v_1, \dots, v_d$, we have $v = \sum_{\mu=1}^d \lambda_\mu v_\mu$ for some $\lambda_\mu \in \mathbb{R}$. Consider this as a system of equations for λ_μ, and note that the absolute value of any coordinate of v and any v_μ is at most 1. Then we have by Cramer's rule for λ_μ, and by $|\det(v_1, \dots, v_d)| \ge \mathrm{const}_d$, that $|\lambda_\mu| \le \mathrm{const}_d/|\det(v_\mu)| \le \mathrm{const}_d$.

Hence

$$1 = \langle v, v\rangle = \langle v, \sum_{\mu=1}^d \lambda_\mu v_\mu\rangle = \sum_{\mu=1}^d \lambda_\mu \langle v, v_\mu\rangle \le$$

$$d \cdot \max_{1\le\mu\le d} |\lambda_\mu| \cdot \max_{1\le\mu\le d} |\langle v, v_\mu\rangle| \le \mathrm{const}_d \cdot \max_{1\le\mu\le d} |\langle v, v_\mu\rangle|.$$

This contradicts the fact that $\max_{1\le\mu\le d} |\langle v, v_\mu\rangle|$ is as small as we want, for D_d sufficiently large and $c_d > 0$ sufficiently small. This completes the proof of Lemma 5. □

Lemma 6. *In Case II, for D_d sufficiently large and $c_d > 0$ sufficiently small, any $\sim$-equivalence class contains at most $d - \ell + 2$ points q_j.*

Proof. Let q_j be any vertex in the $\sim$-equivalence class of r_ℓ, say, and let $q_j \neq r_\ell$. (If such a vertex did not exist, then this $\sim$-equivalence class would have $1 < d-\ell+2$ points, by Lemma 5.) Then, by Lemma 4, $|\cos(\sphericalangle q_j r_\ell r_1)|, \ldots, |\cos(\sphericalangle q_j r_\ell r_{\ell-1})|$ are as small as we want, for D_d sufficiently large and $c_d > 0$ sufficiently small. Let $w^j := (q_j - r_\ell)/d(q_j, r_\ell) \in S^{d-1}$, and, for $\mu = 1, \ldots, \ell-1$, let $w_\mu := (r_\mu - r_\ell)/d(r_\mu, r_\ell) \in S^{d-1}$. Then we have that $|\langle w^j, w_\mu \rangle|$ *is as small as we want, for D_d sufficiently large and $c_d > 0$ sufficiently small. Suppose* $r_\ell = 0$, and let $(w^j)'$ be the orthogonal projection of w^j to the linear $(\ell-1)$-subspace $\mathrm{aff}\,\{r_1, \ldots, r_\ell\} = \mathrm{lin}\,\{r_1, \ldots, r_{\ell-1}\}$. Then $w_1, \ldots, w_{\ell-1}, (w^j)' \in \mathrm{lin}\,\{r_1, \ldots, r_{\ell-1}\}$. Moreover, $|\langle (w^j)', w_\mu \rangle| = |\langle w^j, w_\mu \rangle|$ *is as small as we want, for D_d sufficiently large and $c_d > 0$ sufficiently small.*

Now we proceed in the linear subspace $\mathrm{lin}\,\{r_1, \ldots, r_{\ell-1}\}$, as we proceeded in $\mathbb{R}^d$ in the proof of Lemma 5. We have $|\det(w_1, \ldots, w_{\ell-1})| \geq \mathrm{const}_d > 0$. (Observe that ℓ can assume only finitely many values. This is why we could write here $\mathrm{const}_d > 0$.) Moreover, $(w^j)' = \sum_{\mu=1}^{\ell-1} \lambda_\mu w_\mu$, where now by Cramer's rule $|\lambda_\mu| \leq \mathrm{const}_d \cdot \|(w^j)'\|$. Then

$$\|(w^j)'\|^2 = \langle (w^j)', (w^j)' \rangle = \langle (w^j)', \sum_{\mu=1}^{\ell-1} \lambda_\mu w_\mu \rangle = \sum_{\mu=1}^{\ell-1} \lambda_\mu \langle (w^j)', w_\mu \rangle$$

$$\leq d \cdot \max_{1 \leq \mu \leq \ell-1} |\lambda_\mu| \cdot \max_{1 \leq \mu \leq \ell-1} |\langle (w^j)', w_\mu \rangle|$$

$$\leq \mathrm{const}_d \cdot \|(w^j)'\| \cdot \max_{1 \leq \mu \leq \ell-1} |\langle (w^j)', w_\mu \rangle|.$$

Hence,

$$\|(w^j)'\| \leq \eta := \mathrm{const}_d \cdot \max_{1 \leq \mu \leq \ell-1} |\langle (w^j)', w_\mu \rangle|,$$

and here η *is as small as we want, for D_d sufficiently large and $c_d > 0$ sufficiently small.*

Suppose that the equivalence class of r_ℓ contains $d-\ell+2$ other points, $q_{\ell+1}, \ldots, q_{d+2}$, besides r_ℓ (any of which could be identical with the point denoted by q_j at the beginning of the proof of the lemma). Let $q_\ell = q_\ell^* := r_\ell = 0$. Further, let $q_{\ell+1}^*, \ldots, q_{d+2}^*$ denote the orthogonal projections of $q_{\ell+1}, \ldots, q_{d+2}$ to the linear $(d-\ell+1)$-subspace which is the orthocomplement of $\mathrm{lin}\,\{r_1, \ldots, r_{\ell-1}\}$. Then we have for distinct $j(1), j(2) \in \{\ell, \ell+1, \ldots, d+2\}$ that

$$d(q_{j(1)}^*, q_{j(2)}^*) \leq d(q_{j(1)}, q_{j(2)}) \leq t_1 + c_d n^{1/d}.$$

On the other hand, for $j(1) \in \{\ell, \ell+1, \ldots, d+2\}$, the orthogonal projection of $q_{j(1)}$ to $\mathrm{lin}\,\{r_1, \ldots, r_{\ell-1}\}$ is $q_{j(1)} - q_{j(1)}^*$. Here for $j(1) \geq \ell+1$ we have $q_{j(1)} = q_{j(1)} - r_\ell = d(q_{j(1)}, r_\ell) w^{j(1)}$, hence its orthogonal projection to $\mathrm{lin}\,\{r_1, \ldots, r_{\ell-1}\}$ is $q_{j(1)} - q_{j(1)}^* = d(q_{j(1)}, r_\ell)(w^{j(1)})'$. Therefore, we have $d(q_{j(1)}, q_{j(1)}^*) = d(q_{j(1)}, r_\ell) \cdot \|(w^{j(1)})'\| \leq d(q_{j(1)}, r_\ell)\eta$ and, analogously, $d(q_{j(2)}, q_{j(2)}^*) \leq d(q_{j(2)}, r_\ell)\eta$. For $j(1) = \ell$, we have $d(q_{j(1)}, q_{j(1)}^*) = 0 \leq d(q_{j(1)}, r_\ell)\eta = 0$ and, analogously for $j(2) = \ell$.

These imply by Lemma 1 (2), for D_d sufficiently large and $c_d > 0$ sufficiently small, that for $j(1), j(2) \in \{\ell, \ell+1, \ldots, d+2\}$ we have (both for $j(\kappa) = \ell$, and for $j(\kappa) > \ell$) that

$$d(q_{j(1)}^*, q_{j(2)}^*) \geq d(q_{j(1)}, q_{j(2)}) - d(q_{j(1)}^*, q_{j(1)}) - d(q_{j(2)}^*, q_{j(2)}) \geq$$

$$t_1 - \eta \cdot d(q_{j(1)}, r_\ell) - \eta \cdot d(q_{j(2)}, r_\ell) \geq t_1 - 2\eta(t_1 + c_d n^{1/d}) \geq t_1 - 4\eta t_1.$$

Let D_d be sufficiently large and $c_d > 0$ be sufficiently small. Then $c_d n^{1/d}/t_1$ is sufficiently small (Lemma 1 (2)), and also η is sufficiently small. Therefore, each $d(q^*_{j(1)}, q^*_{j(2)})$ lies in an interval, whose maximum and minimum have a quotient that is as close to 1 as we want. Thus, there are $d - \ell + 3$ points, $q^*_\ell, q^*_{\ell+1}, \ldots, q^*_{d+2}$, with this property in a $(d - \ell + 1)$-dimensional linear subspace of $\mathbb{R}^d$; namely in the orthocomplement of $\operatorname{lin}\{r_1, \ldots, r_{\ell-1}\}$. This is impossible by the theorem of Schütte [24], cited at the beginning of **Step 5**, for D_d sufficiently large and $c_d > 0$ sufficiently small. $\square$

Lemma 7. *In Case II, for $d \geq 6$, the upper estimate of Theorem 1 holds.*

Proof. By Lemma 6, the number of all primary colour classes of $G_1^{*\prime}$, i.e. $m(d-1)+1$, is at most $\ell \cdot (d - \ell + 2) \leq \lfloor ((d+2)/2)^2 \rfloor$. Hence, by (1.1), we have $d(d-1)/2+1 \leq m(d-1)+1 \leq \lfloor ((d+2)/2)^2 \rfloor$. Thus, $d(d-1)/2+1 \leq ((d+2)/2)^2$, implying $d \leq 6$. That is, for $d \geq 7$ we have a contradiction.

For $d = 6$, by (1.2) we have $17 = m(d-1)+1 \leq \ell(d-\ell+2) \leq \lfloor ((6+2)/2)^2 \rfloor = 16$, a contradiction.

At the beginning of **Step 2**, we assumed, in order to obtain a contradiction, that Theorem 1 was false. This led to a contradiction for every $d \geq 6$. $\square$

Lemma 8. *In Case II, for $d = 2, 3$, the upper estimate of Theorem 1 holds.*

Proof. By (1.2), for $d = 2$ we have

$$4 = m(d-1) + 1 \leq \ell(d - \ell + 2) \leq \lfloor ((d+2)/2)^2 \rfloor = 4,$$

implying $\ell = 2$, while for $d = 3$ we have

$$6 = m(d-1) + 1 \leq \ell(d - \ell + 2) \leq \lfloor ((d+2)/2)^2 \rfloor = 6,$$

implying $\ell \in \{2, 3\}$. For both $d = 2$ and 3, equality in the first inequality implies that each of the ℓ $\sim$-equivalence classes contains maximally many, i.e., $d - \ell + 2$ primary colour classes of $G_1^{*\prime}$.

First, let $d = 2$. Then $\ell = d - \ell + 2 = 2$. Let the $\sim$-equivalence classes on the primary colour classes of $G_1^{*\prime}$ be represented by the vertices $\{q_1, q_3\}$ and $\{q_2, q_4\}$. Here all q_j's belong to distinct ones among the four primary colour classes of $G_1^{*\prime}$, which have $1, 1, 1, \lfloor \text{const} \cdot n \rfloor$ vertices, resp. (Thus, these vertices form a subgraph of $G_1^{*\prime}$ which is a four-clique – in particular, each distance determined by them lies in $[t_1, t_1 + c_d n^{1/d}] \cup [t_2, t_2 + c_d n^{1/d}]$.) Further, the secondary colour of an edge only depends on the primary colour classes the edge endpoints belong to. Up to notation, we may assume that q_4 belongs to the last one of these primary colour classes (it plays the role of $q_{m(d-1)+1}$ from part **E** of the proof of Lemma 2). By Lemma 4, $\sphericalangle q_2 q_4 q_3$ is close to $\pi/2$. Fix q_1, q_2, q_3, and vary q_4 in its own primary colour class in $G_1^{*\prime}$, so that it assumes $\lfloor \text{const} \cdot n \rfloor$ values (points). Then all these points lie in the intersection of two circular shells (defined analogously as in part **E** of the proof of Lemma 2). These have centres q_2 and q_3, inner radii some t_κ's, and outer radii the respective $(t_\kappa + c_d n^{1/d})$'s. Moreover, the unit vectors pointing from q_4 to q_2 and to q_3 enclose an angle close to $\pi/2$. Then Claim 2 and the arguments in part **F** of the proof of Lemma 2 yield a contradiction, for D_d sufficiently large and $c_d > 0$ sufficiently small.

Second, let $d = 3$. We copy the proof of the case $d = 2$. Then either $\ell = 2$ and $d-\ell+2 = 3$, or $\ell = 3$ and $d-\ell+2 = 2$. Let the $\sim$-equivalence classes on the primary colour classes of $G_1^{*\prime}$ be represented either by the vertices $\{q_1, q_3, q_5\}$, $\{q_2, q_4, q_6\}$, or by the vertices $\{q_1, q_4\}$, $\{q_2, q_5\}$, $\{q_3, q_6\}$, resp. Here, all q_j-s belong to distinct primary colour classes of $G_1^{*\prime}$, which have $1, 1, 1, 1, 1, \lfloor \mathrm{const} \cdot n \rfloor$ vertices, resp. Up to notation, in both cases we may assume that q_6 belongs to the last one of these primary colour classes. By Lemma 4, for $\ell = 2$, both $\sphericalangle q_2 q_6 q_3$ and $\sphericalangle q_4 q_6 q_3$ are close to $\pi/2$, while for $\ell = 3$, both $\sphericalangle q_3 q_6 q_4$ and $\sphericalangle q_3 q_6 q_5$ are close to $\pi/2$. Moreover, by Lemma 1 (3), $\ell = 2$ implies that $\sphericalangle q_2 q_6 q_4$ is close to $\pi/3$, while $\ell = 3$ implies that $\sphericalangle q_4 q_6 q_5$ is close to $\pi/3$. Fix $q_1, \ldots, q_5$, and vary q_6 in its own primary colour class in $G_1^{*\prime}$, so that it assumes $\lfloor \mathrm{const} \cdot n \rfloor$ values (points). Then all these points lie in the intersection of three spherical shells. These have centres q_2, q_4, q_3 for $\ell = 2$, and centres q_3, q_4, q_5 for $\ell = 3$. Moreover, their inner radii are some t_κ's, and the outer radii are the respective $(t_\kappa + c_d n^{1/d})$'s. Further, for $\ell = 2$ (and 3), the angles enclosed by the unit vectors pointing from q_6 to q_2, q_3, q_4 (and to q_3, q_4, q_5, resp.,) are close to $\pi/3, \pi/2, \pi/2$. Then Claim 2 and the arguments in part **F** of the proof of Lemma 2 yield a contradiction, for D_d sufficiently large and $c_d > 0$ sufficiently small. □

Proof of Theorem 1, continuation. **Step 6**. By **Step 1** (about tightness) and Lemmas 2, 7 and 8, the proof of Theorem 1 for $d \neq 4, 5$ follows.

Step 7. Now we give the differences in the proof of Theorem 1 for the cases $d = 4, 5$. Recall that the proof for Case I already has been given in Lemma 2, so we need to investigate Case II only (cf. **Step 3**).

Analogously, as at the beginning of §3, we will have several positive constants, now depending on d and ε, like $\mathrm{const}_{d,\varepsilon}$, etc. Of these, $C_{d,\varepsilon}$, $D_{d,\varepsilon}$ will be fixed large constants, and $c_{d,\varepsilon}$ will be a sufficiently small positive constant, in terms of the already fixed values of all the other constants.

As in **Step 2** of the proof of Theorem 1, suppose, in order to obtain a contradiction, that the number of edges of G is greater than $(n^2/2)(1-1/m(d-1)+\varepsilon)$. Then by [1], Ch. 6, now G contains a subgraph $G_2 = K(\lfloor \mathrm{const}_{d,\varepsilon} \cdot \log n \rfloor, \ldots, \lfloor \mathrm{const}_{d,\varepsilon} \cdot \log n \rfloor)$, which is a complete $(m(d-1)+1)$-partite graph, with $\lfloor \mathrm{const}_{d,\varepsilon} \cdot \log n \rfloor$ points in each colour class. (For the dependence of $\mathrm{const}_{d,\varepsilon}$ on d and ε in this statement, the best known bound is given in [4].)

The primary (vertex) colouring and the secondary (edge) colouring of G_2 are defined as for G_1 in **Step 2**. (Each edge of G_2 is coloured by L or R.) Analogously to the definition of the subgraph G_1^* of G_1 in **Step 2**, for any j, where $1 \leq j \leq m(d-1)+1$, we define an induced subgraph $G_{2,j}^*$ of G_2 with the following properties. Each primary colour class of G_2, except the j-th one, has exactly one point in $G_{2,j}^*$. Further, still $\lfloor \mathrm{const}_{d,\varepsilon} \cdot \log n \rfloor$ points of the j-th primary colour class of G_2 belong to $G_{2,j}^*$. Moreover, the secondary colour L or R of an edge in $G_{2,j}^*$ depends only on the primary colour classes the edge endpoints belong to.

Analogously to how we have defined the subgraph $G_1^{*\prime}$ of G_1^* in part **D** of the proof of Lemma 2, now we define the subgraph $(G_{2,j}^*)'$ of $G_{2,j}^*$. In what follows, we will deal with the graphs $(G_{2,j}^*)'$, for $1 \leq j \leq m(d-1)+1$.

We want to show that for some $1 \leq j \leq m(d-1)+1$ such a graph $(G_{2,j}^*)'$ cannot exist, for $c_{d,\varepsilon} > 0$ a sufficiently small constant. This contradiction will show that the number of pairs $\{p_{i(1)}, p_{i(2)}\}$, whose distances lie in the union of our two intervals, is at most as large as stated in Theorem 1.

Lemma 9. *In Case II, for $d = 4, 5$, the upper estimate of Theorem 1 holds.*

Proof. Suppose that none of the ℓ $\sim$-equivalence classes of the primary colour classes of $(G^*_{2,j})'$ contains $d-\ell+2$ primary colour classes (cf. Lemma 6). Then we have $m(d-1)+1 \le \ell(d-\ell+1) \le \lfloor((d+1)/2)^2\rfloor$. Thus, by (1.2), for $d = 4$ we have $7 = m(3)+1 \le 6$, and for $d = 5$ we have $11 = m(4)+1 \le 9$, i.e., in both cases we get a contradiction. Therefore, *one of the $\sim$-equivalence classes of primary colour classes, C, say, contains maximally many, i.e., $d-\ell+2$ primary colour classes of $(G^*_{2,j})'$.*

Let us choose from C one primary colour class, the $j(0)$-th one, say, where $1 \le j(0) \le m(d-1)+1$, and consider $(G^*_{2,j(0)})'$. We use the notation $r_1, \dots, r_\ell$ introduced in **Step 5**.

Let the $d-\ell+2$ primary colour classes in C contain $d-\ell+2$ points as follows. One of them is $r_{j(0)}$, and the others are $q_{\ell+1}, \dots, q_{d+1}$. Now consider the $d-\ell+1$ vectors $(q_j - r_{j(0)})/d(q_j, r_{j(0)}) \in S^{d-1}$, for $j \in \{\ell+1, \dots, d+1\}$, and the $\ell-1$ vectors $(r_{j(1)} - r_{j(0)})/d(r_{j(1)}, r_{j(0)}) \in S^{d-1}$, for $j(1) \in \{1, \dots, \ell\} \setminus \{j(0)\}$. Let M be the $d \times d$ matrix formed by the above $(d-\ell+1)+(\ell-1) = d$ column vectors, in the above order. Then $|\det M| = [\det(M'M)]^{1/2}$, where M' is the transpose of M. The entries of $M'M$ are the cosines of the angles formed by the d column vectors of M. The diagonal entries of $M'M$ are equal to 1. Outside the diagonal, in the intersection of the first $d-\ell+1$ rows and the first $d-\ell+1$ columns, as well as in the intersection of the last $\ell-1$ rows and the last $\ell-1$ columns, by Lemma 1 (3), the entries of $M'M$ are close to $1/2$. By Lemma 4, the remaining entries of $M'M$ are close to 0.

Let N_0 denote the $d \times d$ matrix, with the exact entries $1, 1/2$ and 0 at its respective positions. Then $\det(M'M)$ is close to $\det N_0$, hence

$$|\det M| = [\det(M'M)]^{1/2} \text{ is close to } [\det(N_0)]^{1/2}\ (\in [0,\infty)). \tag{3.2}$$

Now we define a $d \times d$ matrix M_0 (it will not be unique) as follows. Its first $d-\ell+1$ column vectors are the edge vectors of a regular $(d-\ell+1)$-simplex of unit edge lengths in the coordinate subspace spanned by the first $d-\ell+1$ basic unit vectors, pointing from some of its vertices to all its other vertices. Moreover, its last $\ell-1$ column vectors are the edge vectors of a regular $(\ell-1)$-simplex of unit edge lengths in the coordinate subspace spanned by the last $\ell-1$ basic unit vectors, pointing from some of its vertices to all its other vertices. Then all these d column vectors form a base of $\mathbb{R}^d$, hence $|\det M_0|$ is some positive constant $\mathrm{const}_{d,\ell}$, independently of the choice of M_0. Since ℓ can assume only finitely many values (cf. Lemma 5), therefore $|\det M_0| \ge \mathrm{const}_d > 0$. Moreover, the entries of $M_0'M_0$ are the cosines of the angles formed by the d column vectors of M_0. Hence, we have $M_0'M_0 = N_0$, which implies

$$[\det(N_0)]^{1/2} = [\det(M_0'M_0)]^{1/2} = |\det M_0| \ge \mathrm{const}_d > 0. \tag{3.3}$$

By (3.2) and (3.3), also

$$|\det M| \ge \mathrm{const}_d > 0, \text{ provided } D_{d,\varepsilon} \text{ is}$$

$$\text{sufficiently large and } c_{d,\varepsilon} > 0 \text{ is sufficiently small.}$$

On the other hand, the $\lfloor \mathrm{const}_{d,\varepsilon} \cdot \log n \rfloor$ points of $(G^*_{2,j})'$ in its $j(0)$-th primary colour class should be contained in an intersection of d spherical shells (called $S_{j(h)}$ in part **E** of the proof of Lemma 2). These have centres $q_{\ell+1}, \ldots, q_{d+1}$ and $r_{j(1)}$ for $j(1) \in \{1, \ldots, \ell\} \setminus \{j(0)\}$, inner radii some t_κ, and outer radii (differently from part **E** of the proof of Lemma 2) the respective $t_\kappa + c_{d,\varepsilon}(\log n)^{1/d}$.

Moreover, the unit vectors pointing from $r_{j(0)}$ to the above centres, are the column vectors of a $d \times d$ matrix, having a determinant of absolute value bounded from below by a positive number. Then the slabs $S'_{j(h)}$ in Claim 2 will be replaced by new slabs. More exactly, $r_{j(0)}$ replaces $q_{m(d-1)+1,1}$, the present d centres replace $q_{j(1)}, \ldots, q_{j(d)}$ in part **D** of the proof of Lemma 2, $\log n$ replaces n, and a suitable half-space replaces H^+ (in part **E** of the proof of Lemma 2). Then Π in Claim 2 of the proof of Lemma 2 will be replaced by a parallelepiped, circumscribed about a ball of diameter $3c_{d,\varepsilon}(\log n)^{1/d}$. Moreover, Π' in part **F** of the proof of Lemma 2 will be replaced by a parallelepiped, with inradius $3c_{d,\varepsilon}(\log n)^{1/d}/2 + 1/2$, hence of volume at most

$$\mathrm{const}_{d,\varepsilon} \cdot c_{d,\varepsilon}^d \log n.$$

Thus, with these changes, the analogue of Claim 2 of the proof of Lemma 2 (with the same proof as cited after Claim 2) and the arguments in part **F** of the proof of Lemma 2 yield a contradiction. Namely, for $c_{d,\varepsilon} > 0$ sufficiently small, we have the following. The parallelepiped replacing Π' has not enough volume in order to contain $\lfloor \mathrm{const}_{d,\varepsilon} \cdot \log n \rfloor$ disjoint open balls of unit diameter. □

Proof of Theorem 1, continuation. **Step 8**. By **Step 1** (about tightness) and Lemmas 2 and 9, the proof of Theorem 1 for $d = 4, 5$ follows.

Together with **Step 6**, this completes the proof of Theorem 1. □

§4. Proof of Theorem 2

In this section, we present the proof of Theorem 2. The proof falls into five simple steps marked as **Step 1**, **Step 2**, etc.

Proof of Theorem 2. **Step 1**. Recall that the tightness of Theorem 2 (A) and (B) was shown by Constructions 2 and 3. It remains to establish that $(d+1)^k$ in (A) and $T\left((d+1)^k + 1, n\right)$ in (B) are upper bounds for the respective quantities. For (B), this follows from (A), by Turán's theorem.

Step 2. We need to show that, for $0 < \varepsilon < \varepsilon_{d,k}$, where $\varepsilon_{d,k} > 0$ is sufficiently small, any (k, ε)-distance set P in $\mathbb{R}^d$ has a cardinality at most $(d+1)^k$. We use induction on k.

For $k = 1$, this statement is valid for $1 + \varepsilon < (1 + 2/d)^{1/2}$ (for d even), or for $1 + \varepsilon < [1 + 2(d+2)/\left(d(d+2) - 1\right)]^{1/2}$ (for d odd), resp. (cf. Schütte [24], Satz 3).

Now let $k \geq 2$. We may suppose without loss of generality that $t_1 < \ldots < t_k$. (If two of these numbers are equal, then the statement follows by induction.) We may and will suppose $\varepsilon \leq 1$.

Let $D_{d,k} > 0$ be a sufficiently large constant. We distinguish two cases:

$$\text{Case I: } t_k/t_1 \leq D_{d,k},$$

$$\text{Case II: } t_k/t_1 > D_{d,k}.$$

Step 3. In Case I, we prove

Lemma 10. *If $t_k/t_1 \le D_{d,k}$, then the upper estimate stated in* **Step 2** *is valid.*

Proof. The ratio of any two distances determined by P is at most $t_k(1+\varepsilon)/t_1 \le 2D_{d,k}$. Then, for sufficiently small $\varepsilon > 0$, we get

$$|P| \le m(d,k),$$

by using the analogues of the compactness considerations from the proof of Lemma 2, Claim 1. (Actually, only (i), the analogue of (ii) with $1/(2D_{d,k})$, and (iii) from Claim 1 are needed.)

Further, by (1.1) we have

$$m(d,k) \le \binom{d+k}{k} = \frac{d+1}{1} \cdot \ldots \cdot \frac{d+k}{k} = \left(\frac{d}{1}+1\right) \cdot \ldots \cdot \left(\frac{d}{k}+1\right) \le (d+1)^k.$$

Hence,

$$|P| \le m(d,k) \le (d+1)^k,$$

as claimed in **Step 2**. □

Step 4. In Case II, we prove

Lemma 11. *If $t_k/t_1 > D_{d,k}$, then the upper estimate from* **Step 2** *holds.*

Proof. In Case II, there exists an integer $j \in \{1, \ldots, k-1\}$ such that $t_{j+1}/t_j > D_{d,k}^{1/(k-1)}$. We consider a colouring of the edges of the complete graph on the vertex set P with k colours. Namely, every edge $\{p_{i(1)}, p_{i(2)}\}$ gets a colour j with $d(p_{i(1)}, p_{i(2)}) \in [t_j, t_j(1+\varepsilon)]$. (Such a colouring is not necessarily unique, but this makes no difference.)

Let us call a distance $d(p_{i(1)}, p_{i(2)})$ *small* if its colour is at most j, and *large* if its colour is at least $j+1$. The quotient of any large and any small distance is at least $t_{j+1}/\left(t_j(1+\varepsilon)\right) > D_{d,k}^{1/(k-1)}/(1+\varepsilon) \ge D_{d,k}^{1/(k-1)}/2 =: D'_{d,k}$, where $D'_{d,k}$ is a large constant. In particular, we will assume that $D'_{d,k} > 1$, which implies that $(0, t_j(1+\varepsilon)] \cap [t_{j+1}, \infty) = \emptyset$. Thus, the length $d(p_{i(1)}, p_{i(2)})$ uniquely determines whether it is a small or a large distance. From now on, we also assume that $D'_{d,k} > 2$. This yields that *every large distance is more than twice as large as every small distance.*

This implies that we can define an equivalence relation $\sim$ on the points $p_{i(1)}$, $p_{i(2)} \in P$.

Definition 5. For $p_{i(1)}, p_{i(2)} \in P$ we write $p_{i(1)} \sim p_{i(2)}$ if either $i(1) = i(2)$, or $d(p_{i(1)}, p_{i(2)})$ is a small distance. By the last italicized text, $\sim$ is an equivalence relation on P.

In each $\sim$-equivalence class of the points $p_i \in P$, each edge has a colour at most j. Thus, each $\sim$-equivalence class is a (j,ε)-distance set. Since $j \le k-1$, by the induction hypothesis we have, for $\varepsilon > 0$ sufficiently small, that the cardinality of any $\sim$-equivalence class is at most $(d+1)^j$.

Now consider a set of representatives from each $\sim$-equivalence class. In this set, each edge has a colour at least $j+1$, so it is a $(k-j,\varepsilon)$-distance set. Since $k-j \le k-1$, by the induction hypothesis we have, for $\varepsilon > 0$ sufficiently small, that the cardinality of this set is at most $(d+1)^{k-j}$.

Using the results of the last two paragraphs, we obtain the following. For $\varepsilon > 0$ sufficiently small, $|P|$ is at most the number of $\sim$-equivalence classes times the maximum cardinality of a $\sim$-equivalence class. That is,

$$|P| \leq (d+1)^{k-j}(d+1)^j = (d+1)^k,$$

as asserted in **Step 2**. □

Step 5. Now Theorem 2 follows from **Steps 1**, **2**, and Lemmas 10 and 11. □

§5. Concluding remarks

1. Suppose that neither k nor d is much larger than the other. It seems likely that in this case one can obtain reasonably good constructions for Q in Construction 1 the following way. Suppose that $d = d(1)+\ldots+d(h)$ and $k = k(1)+\ldots+k(h)$, where all $d(g)$ and $k(g)$, for $1 \leq g \leq h$, are natural numbers. Then $\mathbb{R}^d = \mathbb{R}^{d(1)+\cdots+d(h)} = \mathbb{R}^{d(1)} \oplus \ldots \oplus \mathbb{R}^{d(h)}$. In each $\mathbb{R}^{d(g)}$, for $1 \leq g \leq h$, we take a subset Q_g. Here, each Q_g is one of the examples from Construction $1'$ or Construction $1''$, with all distances in Q_g lying in the union of $k(g)$ intervals of arbitrarily small lengths. We scale $Q_1, \ldots, Q_h$ in such a way that for each $1 \leq g \leq h-1$, the maximal distance in Q_g is much smaller than the minimal distance in Q_{g+1}. Moreover, all distances in Q_g still belong to the union of $k(g)$ intervals of arbitrarily small lengths. Let $Q := \oplus_{g=1}^h Q_g$. For any two distinct points $q(1) = \oplus_{g=1}^h q_{g(1)}$, $q(2) = \oplus_{g=1}^h q_{g(2)} \in Q$, there is a largest $g \in \{1, \ldots, h\}$ such that $q_{g(1)} \neq q_{g(2)}$. Then the distance between $q(1)$ and $q(2)$ is arbitrarily close to the distance between $q_{g(1)}$ and $q_{g(2)}$. Therefore, all distances determined by Q lie in the union of $k(1)+\ldots+k(h) = k$ intervals of arbitrarily small length.

2. A related question was studied by Pach, Radoičić and Vondrák [22], [23]. They proved that for any $d \geq 2$ and any $0 < \gamma < 1/4$, the following statement holds. Suppose that in an n-element separated point set $P \subset \mathbb{R}^d$ there are at least γn^2 point pairs whose distances differ by at most 1. Then the diameter of P is at least $\mathrm{const}_{d,\gamma} \cdot n^{2/(d-1)}$. Apart from the value of the constant, this bound is tight for all $d \geq 2$ and all $0 < \gamma < 1/4$.

3. Another related question is treated in [11]. Suppose that in a separated n-element point set P in the plane, the number of pairs that determine a distance nearly equal to one of $t_1 < \ldots < t_k$ is maximal. Does it follow that then we have, "approximately," $t_2 = 2t_1, \ldots, t_k = kt_1$ (as in the example after Theorem B)? In this direction, they proved the following. Let $\delta > 0$ and suppose that for any $1 \leq i(1) \leq i(2) < i(3) \leq k$, the inequality $|t_{i(3)}/(t_{i(1)} + t_{i(2)}) - 1| > \delta$ holds. Then, for $n \geq n_{k,\delta}$, the number of unordered pairs that determine a distance belonging to $[t_1, t_1+1] \cup \ldots \cup [t_k, t_k+1]$, is at most $n^2/4 + \mathrm{const}_{k,\delta} \cdot n$. This bound is sharp, up to the value of $\mathrm{const}_{k,\delta} > 0$. It is easy to see that if $t_{i(3)} = t_{i(1)} + t_{i(2)}$ holds for some $i(1) \leq i(2) < i(3)$, then the number of pairs with the above property can attain $\lfloor n^2/3 \rfloor$.

4. We pose the following

Question. What would be the results analogous to our Theorem 2, for unions of intervals of the form $[t_1, t_1^{1+\varepsilon}] \cup \ldots \cup [t_k, t_k^{1+\varepsilon}]$, for $\varepsilon > 0$?

Acknowledgements. This research has been partially supported over the years by several OTKA (Hungarian National Foundation for Scientific Research) and

NKFIH (National Office of Research, Development, and Innovation) grants, most recently, by grant no. K131529, and by ERC Advanced grant ”GeoScape”.

References

[1] B. Bollobás, *Extremal graph theory,* London Math. Soc. Monographs **11**, Academic Press, Inc. [Hartcourt Brace Jovanovich, Publishers], London-New York, 1978, MR **80a:**05120

[2] A. Blokhuis, A new upper bound for the cardinality of 2-distance sets in Euclidean space, in: *Convexity and Graph Theory (Jerusalem, 1981)*, North-Holland Math. Stud. **87**, *Ann. Discr. Math.* **20**, North-Holland, Amsterdam, 1984, 65–66, MR **86h:**52008

[3] A. Blokhuis, J. J. Seidel, Few-distance sets in $\mathbf{R}^{p,q}$, in: *Symposia Mathematica*, **38** (Rome, 1983), Academic Press, London - New York, 1986, 145–158, MR **88d:**52011

[4] V. Chvátal, E. Szemerédi, On the Erdős-Stone theorem, *J. London Math. Soc.* **23** (1981), 207-214, MR **82f:**05057

[5] H. T. Croft, 9-point and 7-point configurations in 3-space, *Proc. London Math. Soc. (3)* **12** (1962), 400–424, corr.: ibid. **13** (1963), 384, MR **27#** 5167

[6] P. Erdős, On a lemma of Littlewood and Offord, *Bull. Amer. Math. Soc.* **51** (1945), 898–902, MR **7,**309j

[7] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly* **53** (1946), 248-250, MR **7,**741c

[8] P. Erdős, L. M. Kelly, Elementary Problems and Solutions: Solutions: E735. Isosceles n-points, *Amer. Math. Monthly* **54** (1947), 227–229, MR **1526679**

[9] P. Erdős, E. Makai, Jr., J. Pach, Nearly equal distances in the plane, *Combinatorics, Probability and Computing* **2** (1993), 401–408, MR **95i:**52018

[10] P. Erdős, E. Makai, Jr., J. Pach, Two nearly equal distances in $\mathbb{R}^d$, arXiv: 1901.01055

[11] P. Erdős, E. Makai, Jr., J. Pach, Nearly equal distances in the plane II, arXiv:2112.08852

[12] P. Erdős, E. Makai, Jr., J. Pach, J. Spencer, Gaps in difference sets and the graph of nearly equal distances, *Applied Geometry and Discr. Math., The V. Klee Festschrift, DIMACS Series in Discr. Math. and Theoretical Computer Science* **4** (1991), 265–273, MR **92i:**52021

[13] P. Erdős, A. H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091, MR **8,**333b

[14] N. Frankl, A. Kupavskii, Nearly k-distance sets, Acta Math. Univ. Comenian. (N. S.) **88** (2019) (3), 689-693, MR **4012869**

[15] N. Frankl, A. Kupavskii, Nearly k-distance sets, Discrete Comput. Geom. **70** (2023) (3), 455-494, MR **4650014**

[16] L. Guth, N. H. Katz, On the Erdős distinct distances problem in the plane, Ann. of Math. (2) **181** (2015), 155-190, MR **3272924**

[17] J. W. Hurewicz, H. Wallman, *Dimension Theory*, Princeton Math. Series **4**, Princeton University Press, Princeton, N. J., 1941, MR **3,**312b

[18] P. Lisoněk, New maximal two-distance sets, *J. Combin. Th., Ser. A* **77** (1997), 318–338, MR **98a:**51014

[19] J. E. Littlewood, A. Offord, On the number of real roots of a random algebraic equation. III. *Rec. Math. [Mat. Sbornik] N.S.* **12** (1943), 277–286, MR **5,**179h

[20] E. Lutwak, Selected affine isoperimetric inequalities, *Handbook of Convex Geometry* (Eds. P.M. Gruber, J. M. Wills), North Holland, Amsterdam etc., 1993, Vol. A, Chapter 1.5, 151-176, MR **94h:**52014

[21] E. Makai, Jr., J. Pach, J. Spencer, New results on the distribution of distances determined by separated point sets, *Paul Erdős and His Math. II*, Bolyai Soc. Math. Studies **11**. (Ed. by G. Halász, L. Lovász, M. Simonovits, V. T. Sós.) Springer, Berlin etc., J. Bolyai Math. Soc., Budapest, 2002, 499–511, MR **2004a:**52026

[22] J. Pach, R. Radoičić, J. Vondrák, Nearly equal distances and Szemerédi's regularity lemma, *Comput. Geom.* **34** (2006), 11–19, MR **2006i:**52022

[23] J. Pach, R. Radoičić, J. Vondrák, On the diameter of separated point sets with many equal distances, *European J. Combin.* **27** (2006), 1321–1332. MR **2007f:**51026

[24] K. Schütte, Minimale Durchmesser endlicher Punktmengen mit vorgeschriebenem Mindestabstand (The minimal diameter of finite point sets with prescribed minimal distance, in German) *Math. Ann.* **150** (1963), 91–98, MR **26#**5479

[25] J. Solymosi, V. H. Vu, Near optimal bounds for the Erdős distinct distances problem in high dimensions, *Combinatorica* **28** (2008), 113–125, MR **2009f:**52042

[26] P. Turán, On an extremum problem in graph theory (in Hungarian), *Matematikai és Fizikai Lapok* **48** (1941), 436–452, MR **8,**284j