Periodica Mathematica Hungarica Vol. 5 (4), (1974), pp. 353–354.

RESEARCH PROBLEMS

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In this column Periodica Mathematica Hungarica publishes current research problems whose proposers believe them to be within reach of existing methods. Manuscripts should preferably contain the background of the problem and all references known to the author. The length of the manuscript should not exceed two doublespaced typewritten pages.

11. Let $D_1, \dots, D_r$ be convex bodies in R^n. If $\lambda_1, \dots, \lambda_r$ are real numbers then $\lambda_1 D_1 + \dots + \lambda_r D_r$ and $-D_i$ denote the point sets described by the vectors $\{\lambda_1 x_1 + \dots + \lambda_r x_r;\ x_1 \in D_1, \dots, x_r \in D_r\}$ and $\{-x, x \in D_i\}$, respectively, while $V(D_i)$ is the volume of D_i.

If $\lambda_1, \dots, \lambda_r$ are non-negative numbers, the volume $V(\lambda_1 D_1 + \dots + \lambda_r D_r)$ is a homogeneous polynomial in $\lambda_1, \dots, \lambda_r$ of the form $\Sigma \lambda_{i_1} \dots \lambda_{i_n} V(D_{i_1}, \dots, D_{i_n})$, where the summation runs over all n-tuples of integers $1 \leqq i_1, \dots, i_n \leqq r$. The coefficients $V(D_{i_1}, \dots, D_{i_n})$ of this polynomial, called "the mixed volumes", are symmetric in their arguments. Moreover, they are monotonous in each argument (cf. [2], 38—44).

Let $r = 2$, $D_1 = D$, $D_2 = -D$, $0 \leqq k \leqq n$. If $V_{(k)}(D, -D)$ is the abbreviation for the coefficient $V(D, \dots, D, -D, \dots, -D)$, where D and $-D$ occur $(n-k)$ times and k times, respectively, then by [5], 50—51 we have $V_{(k)}(D, -D) \geq V(D)$ with equality iff D is centrosymmetric, or $k(n-k) = 0$ (and D is arbitrary). Thus $\dfrac{V_{(k)}(D, -D)}{V(D)}$ $(0 < k < n)$ can serve as a measure of asymmetry of D. On the other hand we conjecture that $V_{(k)}(D, -D) \leqq V(D)\binom{n}{k}$. Here equality stands e.g. for a simplex (this is seen by [3] 230—231). Thus in the above sense the simplex would be among the most asymmetric convex bodies.

Using the method of [1], in [2] 105—106 it has been shown that $V_{(k)}(D, -D) \leqq V(D)\, n^{\min(k, n-k)}$. This establishes our conjecture for $k \leqq 1$ and $k \geqq n-1$. With the method of [1] and using a result from [4], it can be shown that $V_{(k)}(D, -D) < V(D)2^n$. By [4] Theorem 3, there is a centrosymmetric convex body C, for which $D \subset C$ and $V(C) < V(D)\ 2^n$. Then, like in [1], by $D \subset C$ and $-D \subset C$ we have $V_{(k)}(D, -D) = V(D, \dots, D, -D, \dots, -D) \leqq V(C, \dots, C, C, \dots, C) = V(C) < V(D)\, 2^n$.

Two consequences of our conjecture are shown to be true in [3], [4]. Namely, $V(D+(-D)) \leq V(D)\binom{2n}{n}$ and $\int_0^1 V[tD+(1-t)(-D)]\,dt \leq \leq V(D)\frac{2^n}{n+1}$ ([3] Theorem 1 and [4] Theorem 2). They are equivalent to

$$\sum_{k=0}^{n} V_{(k)}(D,-D)\binom{n}{k} \leq V(D)\binom{2n}{n} = V(D)\sum_{k=0}^{n}\binom{n}{k}^2$$

and

$$\sum_{k=0}^{n} V_{(k)}(D,-D) \leq V(D)\,2^n = V(D)\sum_{k=0}^{n}\binom{n}{k}.$$

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