Projections of normed linear spaces with closed subspaces of finite codimension as kernels

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Abstract

It follows from [1] and [7] that any closed n-codimensional subspace ($n \geq 1$ integer) of a real Banach space X is the kernel of a projection $X \to X$, of norm less than $f(n) + \varepsilon$ ($\varepsilon > 0$ arbitrary), where

$$f(n) = \frac{2 + (n-1)\sqrt{n+2}}{n+1}.$$

We have $f(n) < \sqrt{n}$ for $n > 1$, and

$$f(n) = \sqrt{n} - \frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right).$$

(The same statement, with $\sqrt{n}$ rather than $f(n)$, has been proved in [2]. A small improvement of the statement of [2], for $n = 2$, is given in [3], pp. 61-62, Remark.) In [1] for this theorem a deeper statement is used, on approximations of finite rank projections on the dual space X^* by adjoints of finite rank projections on X. In this paper we show that the first cited result is an immediate consequence of the principle of local reflexivity, and of the result from [7].

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§ 1 Introduction

First we cite two theorems, from [7], [4], [1], or [1], respectively. We use the notation $f(n)$ from our abstract.

Theorem 1. (cf. [7], Theorem 1.1, [4], Theorem 3, and [1], p. 210) *Let $n \geq 1$ be an integer. Let X, X^n be real Banach spaces,* $\dim X^n = n$*, and let $X^n \subset X$. Then X^n is the range of a projection $P : X \to X$, of norm at most $f(n)$.*

We note that [7], Theorem 1.1 showed Theorem 1 with $f(n)+\varepsilon$ rather than $f(n)$, with $\varepsilon > 0$ arbitrary. Furthermore, [4], Theorem 3, and [1], p. 210, showed that under the conclusion of [7], Theorem 1.1, we have the same conclusion with $f(n)$ rather than $f(n) + \varepsilon$.

For a real or complex normed linear space X we denote by X^* its dual space with its usual normed linear space structure.

Theorem 2. (see [1], p. 209, Corollary) *Let $n \geq 1$ be an integer. Let X be a (real or complex) Banach space. Let for some number $c_n(X)$ each n-subspace Y^n of X^* admit a*

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projection $P : X^* \to X^*$, with range Y^n, of norm $\|P\| < c_n(X)$. Then each n-codimensional closed subspace X_n of X is the range of a projection $Q : X \to X$, of norm less than $1+c_n(X)$.

Actually, as observed by G. J. O. Jameson [5] in the Mathematical Reviews review of [1], Q was constructed in [1] via the complementary projection $R := I - Q : X \to X$, and actually $\|R\| < c_n(X)$ was proved in [1], p. 209, Corollary. This immediately implies $\|Q\| = \|I - R\| < 1 + c_n(X)$, as stated in Theorem 2. That is, there holds

Theorem 2'. (cf. [1], p. 209, Corollary, and [5]) *Let the hypotheses of Theorem 2 hold. Then each n-codimensional closed subspace X_n of X is the kernel of a projection $R : X \to X$, of norm less than $c_n(X)$.*

Evidently, Theorems 1 and 2' immediately imply the first cited result in our abstract. However, [1] uses for the proof of Theorem 2 (Theorem 2') a deeper statement on approximation of finite rank projections on the dual space X^* by adjoints of finite rank projections on X. In our paper we show that the first cited result in our abstract is an immediate consequence of the principle of local reflexivity (cf. the proof of our Theorem 3) and of Theorem 1 from [7], [4], and [1].

§ 2 Results and proofs

Instead of the form of the question given in our abstract and in our Theorems 1 and 2', we will investigate the equivalent form (cf. the end of the proof of Theorem 3) when we look for right inverses of quotient maps of n-dimensional range space, of possibly small norm. Actually we will consider real normed linear spaces rather than real Banach spaces (and all considered linear operators will be continuous).

Theorem 3. *Let $n \geq 1$ be an integer. Let X, X^n be real normed linear spaces, $\dim X^n = n$, and let $Q : X \to X^n$ be a quotient map. Let $\varepsilon > 0$ be arbitrary. Then there exists a map $S : X^n \to X$ such that QS is the identity map on X^n, and $||S|| \leq f(n) + \varepsilon$. Consequently, the (arbitrary) closed n-codimensional subspace $Q^{-1}(0)$ of X is the kernel of the projection $SQ : X \to X$, of norm at most $f(n) + \varepsilon$.*

Proof. First we investigate the case that X is complete, i.e., is a Banach space. Let us consider the dual spaces $X^*, (X^n)^*$ and the adjoint map $Q^* : (X^n)^* \to X^*$ of Q. Since Q is a quotient map, Q^* is an isometric embedding of $(X^n)^*$ onto an n-subspace of X^*, namely to $Q^*((X^n)^*)$. Then by [7] there exists a projection $P : X^* \to X^*$ with range $Q^*((X^n)^*)$, and of norm $\|P\| \leq f(n)$. This, considered as a map $X^* \to Q^*((X^n)^*)$, composed on the left with the inverse of Q^* (defined on $Q^*((X^n)^*)$), gives a map $R : X^* \to (X^n)^*$ such that RQ^* is the identity map on $(X^n)^*$ and $\|R\| \leq f(n)$. Turning to adjoints again, we have $R^* : X^n = (X^n)^{**} \to X^{**}$, with $Q^{**}R^* = (RQ^*)^*$ being the identity map on $(X^n)^{**} = X^n$ and $\|R^*\| = \|R\| \leq f(n)$.

Now we apply the so-called principle of local reflexivity, cf. [9], 28.1.3. For the operator $R^* : X^n \to X^{**}$, for any finite dimensional subspace Y of X^* and any $\varepsilon > 0$ this principle asserts that there exists an $S : X^n \to X$ such that for each $y \in X^n$ and each $x^* \in Y$ we have $\langle x^*, R^*y \rangle = \langle Sy, x^* \rangle$, and $||S|| \leq ||R^*|| \cdot (1 + \varepsilon)$. (The mentioned principle asserts the analogous statement for any operator from a finite dimensional Banach space to a bidual Banach space.) Now let us choose a base $y_1, \ldots, y_n$ in X^n, and consider the coordinate functionals $x_i^* \in X^*$ $(1 \leq i \leq n)$ of the map $Q : X \to X^n$ with respect to this base, i.e., for each $x \in X$ we have $Qx = \sum_{i=1}^{n} \langle x, x_i^* \rangle y_i$. (This implies that for each $y^* \in (X^n)^*$ we have $Q^* y^* = \sum_{i=1}^{n} \langle y_i, y^* \rangle x_i^*$, and similarly for each $x^{**} \in X^{**}$ we have $Q^{**} x^{**} = \sum_{i=1}^{n} \langle x_i^*, x^{**} \rangle y_i$.) These x_i^*'s span a subspace Y of X^*, of finite dimension, to which we will apply the above principle of local reflexivity. Then, in particular, for each $y \in X^n$ and each $i \in \{1, \ldots, n\}$ we have $\langle x_i^*, R^* y \rangle = \langle Sy, x_i^* \rangle$. Therefore, for each $y \in X^n$ we have $QSy = \sum_{i=1}^{n} \langle Sy, x_i^* \rangle y_i = \sum_{i=1}^{n} \langle x_i^*, R^* y \rangle y_i = Q^{**} R^* y = y$. Moreover, $\|S\| \leq \|R^*\| \cdot (1 + \varepsilon) \leq f(n) \cdot (1 + \varepsilon)$.

Second, if X is not complete, we consider its completion $\tilde{X}$, to which Q has a unique extension $\tilde{Q} : \tilde{X} \to X^n$ (since X^n is complete), that also is a quotient map. Then, from what has been said above, we obtain a map $S_1 : X^n \to \tilde{X}$ with $\tilde{Q} S_1$ identity on X^n and $||S_1|| \leq f(n) \cdot (1 + \varepsilon)$. We claim that for each $y \in X^n$ we have that $Q^{-1}(y)$ is dense in $\tilde{Q}^{-1}(y)$. In fact, in the opposite case $Q^{-1}(0)$ would not be dense in $\tilde{Q}^{-1}(0)$. Then, for some n-subspace Z of X, we would have $Q(Z) = X^n$, implying by closedness of $Q^{-1}(0)$ or $\tilde{Q}^{-1}(0)$ in X or $\tilde{X}$, respectively, that $X = Z \oplus Q^{-1}(0)$ and $\tilde{X} = Z \oplus \tilde{Q}^{-1}(0)$ ($\oplus$ meaning direct sum). Hence X would not be dense in $\tilde{X}$, a contradiction. Now, using our claim, an approximation argument gives S: $X^n \to X$ (by defining it on a base of X^n), such that QS is identity on X^n and $\|S\| \leq \|S_1\| \cdot (1 + \varepsilon) \leq f(n) \cdot (1 + \varepsilon)^2$.

Lastly, SQ is a projection by QS being the identity map on X^n, and $\ker(SQ) = \ker Q$ by $SQx = 0 \Rightarrow QSQx = 0$.

(Also conversely, if for a quotient map $Q : X \to X^n$, we have that $Q^{-1}(0)$ is the kernel of a projection $P_0 : X \to X$ of norm $\|P_0\| \leq f(n) + \varepsilon$, then Q has a right inverse of norm at most $f(n) + \varepsilon$. In fact, factoring P_0 via the quotient map $Q : X \to X/P_0^{-1}(0) = X/Q^{-1}(0) = X^n$ as $P_0 = S_0 Q$, we have $S_0 : X^n \to X$ and $\|S_0\| \leq f(n) + \varepsilon$, further $S_0 Q S_0 Q = P_0^2 = P_0 = S_0 Q$ implies, by epimorphism of Q and monomorphism of S_0 that QS_0 is the identity on X^n.) □

The proofs of the following corollaries are analogous to those of Lemmas 8.4 and 8.5 in [8].

Corollary 1. *Let $n \geq 1$ be an integer. Let X, X^n be real normed linear spaces,* $\dim X^n = n$, *and let $Q : X \to X^n$ be a quotient map. Let $\varepsilon > 0$ be arbitrary. Then there exists a subspace Y^n of X, of dimension n, such that for each $x \in Y^n$ we have $\|Qx\| \geq \|x\|/(f(n) + \varepsilon)$.*

Proof. By Theorem 3, we have $S : X^n \to X$, such that QS is the identity map on X^n, and $\|S\| \leq f(n) + \varepsilon$. Then $Y^n := S(X^n)$ is a n-dimensional subspace of X. For $x \in Y^n \setminus \{0\}$ we have $x = Sy$ with $y \in X^n \setminus \{0\}$, hence $\|Qx\|/\|x\| = \|QSy\|/\|Sy\| = \|y\|/\|Sy\| \geq 1/\|S\| \geq 1/(f(n) + \varepsilon)$. □

For the sake of completeness we prove the dual statement as well.

Corollary 2. *Let $n \geq 1$ be an integer. Let X, X^n be real normed linear spaces,* $\dim X^n = n$, *such that X^n is a subspace of X. Then there exists a quotient map Q from X to an n-dimensional real normed linear space such that for each $x \in X^n$ we have $\|Qx\| \geq \|x\|/f(n)$.*

Proof. The completion $\tilde{X}$ of X admits by [7] a projection P_0 with range X^n, and of norm at most $f(n)$. Restricting P_0 to X yields a projection $P : X \to X$, with range X^n, and of norm $\|P\| \leq f(n)$. By closedness of $P^{-1}(0)$ the quotient normed linear space $X/P^{-1}(0)$ exists. Let Q denote the quotient map of X to $X/P^{-1}(0)$. Then we have $P = P_1 Q$, for a unique map $P_1 : X/P^{-1}(0) \to X$, and we have $\|P_1\| = \|P\|$. Then for $x \in X^n \setminus \{0\}$ we obtain $\|Qx\|/\|x\| = \|Qx\|/\|Px\| = \|Qx\|/\|P_1Qx\| \geq 1/\|P_1\| = 1/\|P\| \geq 1/f(n)$. □

Remark 1. When $\dim X < n(n+1)/2$ (where $n > 1$), there are sharper statements, based on estimates of the norm of projections to n-subspaces from [6]. For these statements cf. Lemmas 8.3, 8.4, and 8.5 in [8]. In the complex case we have statements analogous to the statements of this paper with $g(n) = [1 + (n-1)\sqrt{n+1}]/n$ rather than $f(n)$, cf. [7] (we also have $g(n) < \sqrt{n}$ for $n > 1$, and $g(n) = \sqrt{n} - 1/(2\sqrt{n}) + O(1/n)$); when $\dim X < n^2$ (where $n > 1$), there are sharper statements based on estimates of the norms of projections to n-subspaces from [6].

Remark 2. Of course, in Theorem 3, like it has been obtained also in [1], we have obtained the following localized result. Let $c_n(X)$ be a number such that each n-subspace Y^n of X^* admits a projection $P : X^* \to X^*$ with range Y^n, of norm $\|P\| < c_n(X)$. Then each quotient map $Q : X \to X^n$ with n-dimensional range space X^n admits a right inverse $S : X^n \to X$ of norm $\|S\| < c_n(X)$. There arises the question if one can dualize this result. That is, suppose that for some number $c_n'(X)$, for X^* each quotient map $Q' : X^* \to Z^n$, with n-dimensional range space Z^n, admits a right inverse S' of norm $\|S'\| < c_n'(X)$ (or, equivalently, each closed n-codimensional subspace of X^* is the kernel of a projection $P' : X^* \to X^*$, of norm $\|P'\| < c_n'(X)$). Does then each n-subspace X^n of X admit a projection P'' of X with range X^n, of norm $\|P''\| < c_n'(X)$. However, this follows immediately. In fact, dualizing the arguments of the first paragraph of the proof of Theorem 3, for the embedding $J : X^n \to X$, we obtain $S' : (X^n)^* \to X^*$, with J^*S' identity on X^*, and with $\|S'\| < c_n'(X)$. Thus for $(S')^* : X^{**} \to (X^n)^{**} = X^n$ the operator $(S')^*J^{**}$ is identity on $(X^n)^{**} = X^n$, with $\|(S')^*\| < c_n'(X)$. Then we have $(X^n)^{**} = X^n \subset X \subset X^{**}$, and the restriction $(S')^*|X : X \to X^n$ is a left inverse of $J : X^n \to X$ (since J and J^{**} pointwise coincide by naturality of the embedding into the second dual), with $\|(S')^*|X\| < c_n'(X)$. In other words, $P'' = J((S')^*|X) : X \to X$ is a projection with range X^n, of norm less than $c_n'(X)$. If the set of $c_n'(X)$'s satisfying the hypotheses of the statement about $c_n'(X)$ is an open half-line $(c_{n,0}'(X), \infty)$, then even there exists a projection $P'' : X \to X^n$ of norm $\|P''\| \leq c_{n,0}'(X)$, cf. [4], Theorem 3, and [1], p. 210. Of course, we have the complex analogues of the statements of Remark 2 as well.

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