LOWER BOUNDS ON THE NUMBERS OF SHADOW–BOUNDARIES AND ILLUMINATED REGIONS OF A CONVEX BODY

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ABSTRACT. There are determined sharp lower bounds for the number of shadow–boundaries, and illuminated regions of a convex body in E^n, exhibiting extremal properties of the simplex and the parallelotope, respectively of the j–fold both way infinite cylinder, over an $(n-j)$–simplex and an $(n-j)$–parallelotope.

1. INTRODUCTION

Kleinschmidt and Pachner [6] introduced the notion of shadow–boundary of a convex polytope $P \subset E^n$ with respect to a point $x \in E^n \setminus P$ as the intersection of P and the union of all its supporting rays with apex x. By analogy, H.Martini [8] considered the notion of shadow–boundary of a convex polytope $P \subset E^n$ with respect to a direction l in E^n as the intersection of P and the union of all lines having direction l and supporting K.

The shadow–boundary of a convex polytope $P \subset E^n$ with respect to a point z is called sharp provided z is not contained in any facet hyperplane; i.e., in the affine hull of a facet of P. Similarly, the shadow–boundary of P with respect to a direction l is called sharp if the direction l is not parallel to any facet hyperplane of P.

In [6], [7], [8] upper and lower bounds on the numbers of sharp shadow–boundaries of a convex polytope (with respect to exterior points, and directions) with a fixed number of facets have been obtained.

Below the notions of shadow-boundaries and sharp shadow–boundaries are extended for the case of a convex body. There are determined lower bounds for the numbers of shadow–boundaries, sharp shadow–boundaries, and illuminated regions of a convex body in E^n.

2. BASIC NOTIONS

The usual abbreviations aff, bd, int, relint, dim, card are used for affine hull, boundary, interior, relative interior (taken in the affine hull), dimension, and cardinality, respectively. The notations $[x, y]$, $]x, y[$, $\langle x, y\rangle$, $[x, y\rangle$ mean, respectively, closed line interval, open line interval, the line passing through different points x, y, and the (closed) ray with apex x passing through the point y. For any (oriented)

*This research work of both authors was (partially) supported by the Hungarian National Foundation for Scientific Research, grant no. 1817.

direction l in E^n and for any point $x \in E^n$, $l(x)$ means the ray with apex x having direction l, and $\tilde{l}(x)$ means the ray with apex x having the direction opposite to l.

Let K be a convex body (a proper closed convex set with nonempty interior) in the Euclidean space E^n. (When we speak about convex polyhedral sets, we always will involve the same properties.) Further C_K denotes the characteristic cone of K, i.e., $C_K = \cap\{K - x\colon x \in K\}$. A direction l in E^n is called exterior for K if the ray $\tilde{l}(0)$ with apex 0 having the direction opposite to l does not belong to C_K.

The notions of shadow–boundary (with respect to an exterior point or to an exterior direction) can be obviously extended to the case of a convex body $K \subset E^n$. Below we denote by $S_z(K)$ the shadow–boundary of K with respect to an exterior point z, and by $S'_l(K)$ the shadow–boundary of K with respect to an exterior direction l. Denote by $\sigma(K)$ the number of all pairwise different nonempty shadow–boundaries of K with respect to exterior points (put $\sigma(K) = \infty$ if their number is infinite). Similarly, denote by $\sigma'(K)$ the corresponding number of all pairwise different nonempty shadow–boundaries of K with respect to exterior directions.

The shadow–boundary $S_z(K)$ with respect to a point z is called sharp provided $\operatorname{card}(l(z) \cap K) \le 1$ for any ray $l(z)$ supporting K. By analogy, the shadow–boundary $S'_l(K)$ with respect to a direction l in E^n is called sharp if $\operatorname{card}(m \cap K) \le 1$ for any line m parallel to l and supporting K. Denote by $\delta(K)$ the number of all pairwise different nonempty sharp shadow–boundaries of K with respect to exterior points (put $\delta(K) = \infty$ if their number is infinite). Similarly, denote by $\delta'(K)$ the corresponding number of all pairwise different nonempty sharp shadow–boundaries of K with respect to exterior directions.

One can define the shadow–boundary $S_z(K)$ of a convex body $K \subset E^n$ with respect to a point $z \in \operatorname{bd} K$ word by word as for the case $z \in E^n \setminus K$, except for $n = 1$, when we define it as $\{z\}$. We denote by $\sigma_b(K)$ the number of all pairwise different shadow–boundaries of K, different from $\operatorname{bd} K$, with respect to points $z \in \operatorname{bd} K$ (put $\sigma_b(K) = \infty$ if their number is infinite). Note that these are nonempty.

Together with shadow-boundaries we consider illuminated regions of K. A point $x \in \operatorname{bd} K$ is called illuminated by a point $z \in E^n \setminus K$ provided $]z, x[\cap K = \emptyset$ and the ray $[z, x\rangle$ passes at x to $\operatorname{int} K$ (this is a slight variant of the notion of illumination according to L. Fejes Tóth [3]). For any point $z \in E^n \setminus K$, the set

$$I_z(K) := \{x \in \operatorname{bd} K\colon x \text{ is illuminated by } z\}$$

is called the illuminated region of K with respect to z. Denote by $\gamma(K)$ the number of all pairwise different illuminated regions of K, different from $\operatorname{bd} K$, with respect to exterior points (put $\gamma(K) = \infty$ if their number is infinite).

An exterior direction l in E^n illuminates a boundary point x of a convex body $K \subset E^n$ if the ray $\tilde{l}(x)$ intersects K at x only, and $l(x)$ passes at x to $\operatorname{int} K$. The set $I'_l(K)$ of all points $x \in \operatorname{bd} K$ illuminated by l is called the illuminated region of K with respect to the direction l. Denote by $\gamma'(K)$ the number of all pairwise different illuminated regions of K different from $\operatorname{bd} K$, with respect to exterior directions (put $\gamma'(K) = \infty$ if their number is infinite).

Lastly, let $\varepsilon(K)$ denote the number of all different ordered pairs $(S_z(K), I_z(K))$, where $z \in E^n \setminus K$ and $S_z(K) \neq \emptyset$ (and thus $I_z(K) \neq \operatorname{bd} K$). (Put $\varepsilon(K) = \infty$ if their number is infinite.)

Observation. Further, unless specified otherwise, any shadow–boundary and any illuminated region of a convex body $K \subset E^n$ is considered relative to exterior points and exterior directions only. In this connection the respective word "exterior" will be omitted.

Note that any two of the numbers $\sigma(K)$, $\sigma'(K)$, $\delta(K)$, $\delta'(K)$, $\sigma_b(K)$, $\gamma(K)$, $\gamma'(K)$, $\varepsilon(K)$ can be different. For example, if K is a regular hexagon in the plane, then $\sigma(K) = 45$, $\sigma'(K) = 6$, $\delta(K) = 15$, $\delta'(K) = 3$, $\sigma_b(K) = \gamma'(K) = 12$, $\gamma(K) = 18$, $\varepsilon(K) = 48$. Further, if $K \subset E^3$ is a pyramid over a square, $\sigma_b(K) = 18$, $\gamma'(K) = 22$.

3. Main Results

Theorem 1. *For a convex body $K \subset E^n$, the following conditions are equivalent:*

1) K is polyhedral,

2) $\sigma(K) < \infty$,

3) $\sigma'(K) < \infty$,

4) $\delta(K) < \infty$,

5) $\delta'(K) < \infty$,

6) $\sigma_b(K) < \infty$,

7) $\gamma(K) < \infty$,

8) $\gamma'(K) < \infty$,

9) $\varepsilon(K) < \infty$.

Theorem 2. *For a convex body $K \subset E^n$ which is not a parallel slab, we have:*

$$\sigma'(K) \le \sigma(K), \quad \delta'(K) \le \delta(K),$$

$$\max\{\gamma'(K), \sigma_b(K)\} \le \gamma(K), \quad \delta(K) \le \min\{\sigma(K), \gamma(K)\},$$

$$\delta'(K) \le \min\{\sigma'(K), \gamma'(K)\}, \quad \max\{\sigma(K), \gamma(K)\} \le \varepsilon(K).$$

Theorem 3. *Let $K \subset E^n$ be a convex body having a j–dimensional characteristic cone $(0 \le j \le n-2)$. Then the following hold:*

1) $\sigma(K) \ge (3^{n-j+1} - 2^{n-j+2} + 1)/2$;

1') $\sigma'(K) \ge (3^{n-j} - 1)/2$;

2) $\delta(K) \ge 2^{n-j} - 1$;

2') $\delta'(K) \ge 2^{n-j-1}$;

3) $\sigma_b(K) \ge 2^{n-j+1} - 2$;

4) $\gamma(K) \ge 2^{n-j+1} - 2$;

4') $\gamma'(K) \ge 2^{n-j+1} - 2$;

5) $\varepsilon(K) \ge 3^{n-j+1} - 2^{n-j+2} + 1$.

Equality in any of 1), 2), 3), 4), 4'), 5) holds if and only if K is a j–fold both way infinite cylinder over an $(n-j)$–simplex. Equality in any of 1'), 2') holds if and only if K is a j–fold both way infinite cylinder over an $(n-j)$–parallelotope.

Note that for $j = n$, for K a half–space all above quantities are 0; for $j = n-1$, for K a parallel slab

$$\sigma(K) = \sigma'(K) = \delta(K) = \delta'(K) = \varepsilon(K) = 0,$$

$$\sigma_b(K) = \gamma(K) = \gamma'(K) = 2,$$

and evidently each of these is the minimum value for the given j. (For $\gamma'(K)$, where K is polyhedral, use the direction of a segment with endpoints in the relative interior points of two facets.) Moreover, one easily sees that these are the only convex bodies for $j = n$, respectively $j = n-1$, for which these minima are attained.

The assertions about $\sigma'(K)$ and $\delta'(K)$ formulated in Theorem 3 are implied by the following generalization of Satz 1 from [8]. This can be proved by the method of [8], Satz 1 (and [7], Theorem 5), noting for $\delta'(K)$ that the dimension of the intersection of all facet hyperplanes of K and the infinite hyperplane (in the n–dimensional projective space) is less than the dimension of the characteristic cone of K, and for $\sigma'(K)$ using beside this Lemma 3, 2) and the sharp lower bounds for the number of projective cells of any given dimension in arrangements of m hyperplanes with empty intersection in the projective k–space, given by [9].

Theorem 4. *Let a convex polyhedral set $K \subset E^n$ have exactly m $(m > 1)$ pairwise non–parallel facets, and let it have j–dimensional characteristic cone (thus $j \ge n - m$). Then the following holds.*

1) If $0 \le j \le n-2$, then

$$\sigma'(K) \ge \big(3^{n-j-2}(4m - 4n + 4j + 9) - 1\big)/2,$$

$$\delta'(K) \ge 2^{n-j-2}(m - n + j + 2);$$

equality in any of these inequalities holds if and only if K is a j–fold both way infinite cylinder over an $(n-j-2)$–fold finite prism over a bounded planar convex polygon with exactly $m - n + j + 2$ pairwise non–parallel sides.

2) If $j = n-1$ (respectively, $j = n$), then $\sigma'(K) \ge 2m$ and $\delta'(K) \ge m$ (respectively, $\sigma'(K) \ge 2m - 1$ and $\delta'(K) \ge m - 1$); equality in any of these inequalities holds if and only if K is an $(n-2)$–fold both way infinite cylinder over an unbounded planar convex polygonal set with exactly m pairwise non–parallel sides, having one pair of parallel sides (respectively, with no pair of parallel sides). $\square$

There remains the problem, analogous to Theorem 4, to find the minima of $\sigma(K)$, $\sigma_b(K)$, $\gamma(K)$, $\gamma'(K)$, $\varepsilon(K)$, if beyond j also the number of facets of K is given (for $\delta(K)$ this is done in [7, Theorem 5]).

4. Auxiliary lemmas

First we repeat some considerations of [6] and [8], and give some considerations of [7] in a more detailed way, in our slightly more general context. For a convex body $K \subset E^n$, let $z \in E^n \setminus K$, and let a ray $l(z)$ intersect $\operatorname{bd} K$. Then either $\operatorname{aff} l(z)$ is a supporting line to K, or else $\operatorname{card}\big(l(z) \cap \operatorname{bd} K\big) \le 2$, and for $x \in l(z) \cap \operatorname{bd} K$ either $l(z)$ passes at x from $\operatorname{int}(E^n \setminus K)$ to $\operatorname{int} K$, or conversely (depending on the position of the ray $l(x)$ relative to the supporting cone of K at x). Thus $(\operatorname{bd} K) \setminus S_z(K)$ is the union of two open (in $\operatorname{bd} K$) sets, namely

$$I_z(K) = \{x \in \operatorname{bd} K \colon [z, x\rangle \text{ passes at } x \text{ from } \operatorname{int}(E^n \setminus K) \text{ to } \operatorname{int} K\},$$

$$D_z(K) = \{x \in \operatorname{bd} K \colon [z, x\rangle \text{ passes at } x \text{ from } \operatorname{int} K \text{ to } \operatorname{int}(E^n \setminus K)\}.$$

The set $I_z(K)$ is nonempty and connected. In fact, let $x_1, x_2 \in I_z(K)$ (actually, it suffices to suppose $x_1 \in I_z(K)$, $[z, x_2] \cap K = \{x_2\}$). Then for any $\lambda \in \,]0, 1[$

and for the direction l directly parallel to $\lambda(x_1 - z) + (1 - \lambda)(x_2 - z)$, the ray $l(z)$ intersects $\operatorname{bd} K$ first in a point x, that belongs to $I_z(K)$. Similarly for the case $z \in E^n \setminus K$, $[z, x_1] \cap \operatorname{bd} K = \{x_1\}$, $[z, x_2] \cap \operatorname{bd} K = \{x_2\}$, one has $[z, x] \cap \operatorname{bd} K = \{x\}$ with the above λ, l and x. Therefore we have

$$S_z(K) \supset \operatorname{bd} I_z(K) \supset \{x \in S_z(K) \colon \,]z, x[\cap \operatorname{bd} K = \emptyset\}$$

($\operatorname{bd} I_z(K)$ taken relative to $\operatorname{bd} K$). If K is bounded then also $D_z(K)$ is nonempty and connected. In this case $S_z(K)$ determines the unordered pair $\{I_z(K), D_z(K)\}$, namely as the set of the connected components of $(\operatorname{bd} K) \setminus S_z(K)$.

However, for K unbounded, $D_z(K)$ may be empty, and also may have several connected components. E. g. for $K \subset E^2$ and $z \in E^2$ given in (ξ, η) coordinate system by

$$K = \{(\xi, \eta) \colon \xi \geq 0, \ \eta \geq 0, \ \xi + \eta \geq 3\}, \quad z = (1, 1),$$

$D_z(K)$ has two components. Already in E^3, $D_z(K)$ can have any finite number of components. Departing from a one–way infinite cone over a convex $2k$–gon F, let us cut its apex off by the plane of F. Thus we obtain a convex polyhedral set K, and we choose for z the apex of the originally considered cone. Then the facets F_i of K lying on the boundary of the cone lie in $S_z(K)$. By a small rotation in suitable sense of each facet plane $\operatorname{aff} F_i$ about the corresponding edge of F, we can achieve that for the obtained convex polyhedral set K' with facets F, F_i' (in corresponding notations), every second F_i' will have its relative interior in $D_z(K')$, and all other F_i'-s in $I_z(K')$. Thus $D_z(K')$ has exactly k components.

Therefore we only conjecture that also for unbounded K the set $S_z(K)$ determines the pair $\{I_z(K), D_z(K)\}$.

Anyway, for K polyhedral and $S_z(K)$ sharp, $S_z(K)$ determines $\{I_z(K), D_z(K)\}$. Namely if $(\operatorname{bd} K) \setminus S_z(K)$ is connected, it equals $I_z(K)$, and then $D_z(K) = \emptyset$. If $(\operatorname{bd} K) \setminus S_z(K)$ has two connected components, then one is $I_z(K)$, the other is $D_z(K)$. If $(\operatorname{bd} K) \setminus S_z(K)$ has at least three connected components (which implies that K is not a parallel slab and thus $\operatorname{bd} K$ is connected), then one is $I_z(K)$, and let those contained in $D_z(K)$ (which are finitely many) be $D_1, \ldots, D_k$, where $k \geq 2$. The connectedness of $\operatorname{bd} K$ implies that for any D_i and any facets F_i, F_i' of K, such that $\operatorname{relint} F_i \subset D_i$ and $\operatorname{relint} F_i' \subset I_z(K)$, there exists a sequence $F_i = F_{i,1}, F_{i,2}, \ldots, F_{i,l} = F_i'$ of facets of K, such that

$$\dim(F_{i,j} \cap F_{i,j+1}) = n - 2 \quad \text{for all} \quad 1 \leq j \leq l - 1.$$

Let j_0 be the minimal j such that $\operatorname{relint} F_{i,j} \subset I_z(K)$. By induction one sees that for each $j < j_0$, thus in particular, for $j = j_0 - 1$ we have

$$\operatorname{relint}(F_{i,j-1} \cap F_{i,j}) \subset D_z(K), \quad \text{t}hus \quad \operatorname{relint}(F_{i,j-1} \cap F_{i,j}) \subset D_i.$$

Hence $\operatorname{relint} F_{i,j} \subset D_i$. Thus

$$\operatorname{relint} F_{i,j_0-1} \subset D_i, \quad \operatorname{relint} F_{i,j_0} \subset I_z(K), \quad \dim(F_{i,j_0-1} \cap F_{i,j_0}) = n - 2.$$

However, for $i_1 \neq i_2$ there are no facets F_{i_1}, F_{i_2} of K, such that

$$\operatorname{relint} F_{i_1} \subset D_{i_1}, \quad \operatorname{relint} F_{i_2} \subset D_{i_2}, \quad \dim(F_{i_1} \cap F_{i_2}) = n - 2.$$

Thus in this case the sharp shadow boundary $S_z(K)$ actually determines even $I_z(K)$ (as the unique connected component of $(\mathrm{bd}\, K) \setminus S_z(K)$, neighbourly to all other components in the above sense).

Let now l be an exterior direction for K. In this case

$$\begin{cases} I'_l(K) = \{x \in \mathrm{bd}\, K \colon l(x), \text{ respectively } \tilde{l}(x) \text{ pass at } x \text{ to } \mathrm{int}\, K, \\ \text{respectively } \mathrm{int}(E^n \setminus K)\}. \end{cases}$$

Let

$$\begin{cases} D'_l(K) = \{x \in \mathrm{bd}\, K \colon l(x), \text{ respectively } \tilde{l}(x) \text{ pass at } x \text{ to } \mathrm{int}(E^n \setminus K), \\ \text{respectively } \mathrm{int}\, K\}. \end{cases}$$

Then, depending on whether the direction opposite to l is an exterior direction for K or not, we have that both $I'_l(K)$, $D'_l(K)$ are nonempty and connected, or $I'_l(K)$ is nonempty and connected and $D'_l(K) = \emptyset$. Moreover,

$$S'_l(K) \supset \mathrm{bd}\, I'_l(K) \supset \left\{x \in S'_l(K) : \tilde{l}(x) \cap K = \{x\}\right\}$$

($\mathrm{bd}\, I'_l(K)$ taken relative to $\mathrm{bd}\, K$). (For these observe that for any two directed lines of direction l, one passing into, respectively out of $\mathrm{int}\, K$ (or K), the other passing into, respectively out of K, any further parallel, similarly directed line strictly between them passes into, respectively out of $\mathrm{int}\, K$ (or K).) Thus $S'_l(K)$ determines the unordered pair $\{I'_l(K), D'_l(K)\}$, as the set of the connected components of $(\mathrm{bd}\, K) \setminus S'_l(K)$, or as $\{(\mathrm{bd}\, K) \setminus S'_l(K),\ \emptyset\}$.

From the above statements on the boundary of the illuminated domain there follows readily

Lemma 1. *Let a shadow boundary $S_z(K)$ (respectively, $S'_l(K)$) be sharp. Then $S_z(K) = \mathrm{bd}\, I_z(K)$ (respectively, $S'_l(K) = \mathrm{bd}\, I'_l(K)$), relative to $\mathrm{bd}\, K$.* ■

Lemma 2. *If a shadow-boundary $S_z(K)$ (respectively, $S'_l(K)$) is nonempty, then in any neighbourhood of the point z (respectively, of the direction l), there is a point v (respectively, a direction m) such that the shadow–boundary $S_v(K)$ (respectively, $S'_m(K)$) is nonempty and sharp.*

Proof. The assertion about the sharpness follows immediately from [5] and [1], respectively: the point–set union A (respectively, the set B of directions) of the lines determined by the line segments on the boundary of a convex body in E^n has σ–finite $(n-1)$–dimensional (respectively, $(n-2)$–dimensional) Hausdorff measure. Thus A (respectively, B) contains no nonempty open set in E^n (respectively, in the space of directions, with the natural topology). So it suffices to show that any neighbourhood of z (respectively, of l) contains some nonempty open set G (respectively, H), such that for any $v \in G$ (respectively, $m \in H$), we have $S_v(K) \neq \emptyset$ (respectively, $S'_m(K) \neq \emptyset$).

Let us choose $y \in S_z(K)$. Then K has a supporting cone K_y at y, and z is an exterior or boundary point of K_y.

1) If z is an exterior point of K_y, then any point v in a neighbourhood of z is an exterior point of K_y, thus $y \in S_v(K)$, and hence $S_v(K) \neq \emptyset$.

2) If $z \in \mathrm{bd}\, K_y$, then in a neighbourhood of z choose a point $v \in (\mathrm{int}\, K_y) \setminus K$ (note that by $z \notin K$, any open neighbourhood of z intersects $(\mathrm{int}\, K_y) \setminus K$; their intersection can be chosen for G). Then $z \in D_v(K)$, thus $D_v(K) \neq \emptyset$. Also we have $I_v(K) \neq \emptyset$.

If $S_v(K) = \emptyset$, then $\operatorname{bd} K$ is the union of its two disjoint, nonempty relatively open subsets U, V, i.e., $\operatorname{bd} K$ is disconnected. Therefore K is a slab. However a slab admits no nonempty shadow–boundary $S_z(K)$. This contradiction shows that $S_v(K) \neq \emptyset$.

The case of directions is analogous, choosing in case 2) an exterior direction m such that the ray $\tilde{m}(y)$ points from y to the interior of K_y, and noting that $I'_m(K) \neq \emptyset$. ■

Let $K \subset E^n$ be a convex polyhedral set. Let $A_E(K)$, respectively $A_P(K)$ denote the arrangement of the facet hyperplanes of K in E^n, respectively in the projective n–space, and let $A_\infty(K)$ denote the arrangement, which is the trace of $A_P(K)$ on the infinite hyperplane. We always consider the cells of arrangements as relatively open.

It is known ([6], [8]) that for a convex polyhedral set $K \subset E^n$, which is not a parallel slab, one has:

1) $\delta(K)$ equals the number of the projective n–cells disjoint to $\operatorname{int} K$, in the arrangement $A_P(K)$;

2) $\delta'(K)$ equals the number of those projective $(n-1)$–cells in the arrangement $A_\infty(K)$, which are different from the intersection of the infinite hyperplane and all open outer half–spaces bounded by the facet hyperplanes of K (i.e., the set of those directions l, which themselves or their opposites illuminate each facet of K).

Denoting by $\tilde{K}$ the intersection of all closed outer half–spaces of the facet hyperplanes of K (if this is not empty, or, equivalently, if its interior is not empty), we have similarly

Lemma 3. *Let $K \subset E^n$ be a convex polyhedral set, which is not a parallel slab. We have*

1) if K is compact, $\sigma(K)$ equals the number of those projective cells in the arrangement $A_P(K)$, which do not lie in K or in the infinite hyperplane;

2) $\sigma'(K)$ equals the number of those projective cells in the arrangement $A_\infty(K)$, which are different from the intersection of the infinite hyperplane and all facet hyperplanes of K (if this is not empty) and also from the intersection of the infinite hyperplane and all open outer half–spaces bounded by the facet hyperplanes of K (if this is not empty);

3) $\sigma_b(K)$ equals the number of all those faces of K, of dimensions $0, 1, \ldots, n-1$, which are different from the unique minimal face of K, of dimension in $\{0, 1, \ldots, n-1\}$ (if it exists);

4) $\gamma(K)$ equals the number of those Euclidean n–cells in the arrangement $A_E(K)$, which are different from $\operatorname{int} K$ and also from $\operatorname{int} \tilde{K}$ (if this is not empty);

5) $\gamma'(K)$ equals the number of those unbounded Euclidean n–cells in the arrangement $A_E(K)$, which contain an unbounded part of some ray with apex some interior point of K, and are different from $\operatorname{int} K$, and also from $\operatorname{int} \tilde{K}$ (if this is not empty);

6) $\varepsilon(K)$ equals the number of those Euclidean cells in the arrangement $A_E(K)$, which are not contained in K and are different from $\operatorname{int} \tilde{K}$ (if this is not empty).

Proof. 1) As we have seen above, for K compact, $S_z(K)$ determines $I_z(K)$ and $D_z(K)$, up to order. Thus $S_z(K)$ determines — beside the set of facets of K lying in $S_z(K)$ — the set of facets of K on whose interior side z lies, and the set of facets of K on whose exterior side z lies, up to order. That is, $S_z(K)$ determines in which projective cell z lies, in the arrangement $A_P(K)$. Point z cannot lie in K, by definition, and also cannot lie in the infinite hyperplane. Conversely, without using compactness of K, if we know in which projective cell z lies (subject to the above restrictions), then $S_z(K)$ is uniquely determined, namely it is the union of the facets of K whose hyperplanes contain the cell containing z, and of the boundary of the union of the facets of K on whose exterior side the cell containing z lies (or of those facets of K on whose interior side the cell containing z lies).

2) The arguments in 1), applied to a light source a direction l rather than a point z, prove the statement of 2). We have to exclude cells consisting only of directions l which satisfy both $l(0) \subset (\operatorname{int} C_K) \cup (-C_K)$ and $\tilde{l}(0) \subset (\operatorname{int} C_K) \cup (-C_K)$. These directions correspond just to the points of the two excluded projective cells.

3) If the light source z lies on $\operatorname{bd} K$, then $S_z(K)$ is the union of the faces of K containing z, which is uniquely determined by the face F of K containing z in its relative interior. Further, for faces $F_1 \neq F_2$ of K and $z_1 \in \operatorname{relint} F_1$, $z_2 \in \operatorname{relint} F_2$, the respective unions are different, and $S_z(K) = \operatorname{bd} K$ if and only if F is the minimal face of K.

4) Let us consider an illuminated region $I_z(K) \neq \operatorname{bd} K$. If we move z towards some generic interior point of K through a small distance, then the illuminated region remains the same, and z does not get to any facet hyperplane of K. Further, if $u, v \in E^n \setminus K$ do not lie on any facet hyperplane of K, then $I_u(K) = I_v(K)$ if and only if u, v belong to the same open Euclidean n–cell in the arrangement $A_E(K)$. By definition $z \notin K$, and, since $I_z(K) = \operatorname{bd} K$ is excluded, z does not belong to the (eventual) cell $\operatorname{int} \tilde{K}$.

5) Similarly as in 4), for an illuminating direction l we choose an illuminating point z sufficiently far on a ray $\tilde{l}(x_0)$, where x_0 is a generic point of $\operatorname{int} K$. Then $I'_l(K) = I_z(K)$. All sufficiently far such points z lie in an unbounded open Euclidean n–cell of the arrangement $A_E(K)$. However only such Euclidean n–cells can arise this way, which contain an unbounded part of the ray $\tilde{l}(x_0)$, and which are different from $\operatorname{int} K$ and $\operatorname{int} \tilde{K}$.

6) Proceed analogously as for $\sigma(K)$. ■

Let $P \subset E^n$ be a convex polytope, and $F_1, \dots, F_q$ be all the facets of P. The facet hyperplanes $\operatorname{aff} F_1, \dots, \operatorname{aff} F_q$ form an arrangement of hyperplanes in E^n, which divides E^n into relatively open convex cells of dimensions $0, 1, ..., n$. Let the set of all these cells be denoted by $C(P)$. Denote by $L_i(P)$, $i = 1, \dots, n$ the set of those unbounded Euclidean i–cells C^i of this arrangement, for which $\operatorname{aff} C^i = \operatorname{aff} F^i$ for some i–face F^i of P and which contain an unbounded part of some ray $l(x)$, where $x \in \operatorname{relint} F^i$. Let $\eta_i(P) = \operatorname{card} L_i(P)$.

Lemma 4. *For any convex polytope $P \subset E^n$,*

$$\eta_i(P) \geq (2^{i+1} - 2) \cdot \binom{n+1}{i+1}, \quad i = 1, \dots, n.$$

Equality holds, for any $i \in \{1, \dots, n\}$, if and only if P is a simplex.

Proof. It suffices to deal with the case $i = n$. Namely, for $i \leq n-1$ the number of i–faces of P is at least $\binom{n+1}{i+1}$, with equality if and only if P is a simplex (cf. [4], p. 184, **2**, case $r = 2$), and for each i–face we can apply the statement of the lemma, with n replaced by i.

Now we turn to the case $i = n$. For P a simplex, the equality $\eta_n(P) = 2^{n+1} - 2$ is easily verified. Therefore we suppose P is not a simplex (thus $n \geq 2$), and prove $\eta_n(P) > 2^{n+1} - 2$. We use induction on n. In the case $n = 2$ some four side lines of

P bound a convex quadrangle Q ($\supset P$). We have $\eta_2(P) \geq \eta_2(Q)$. One easily sees that in each of the cases that Q has two, one or no pair of parallel sides, we have $\eta_2(Q) = 8 > 6 = 2^{2+1} - 2$.

Suppose now that $n \geq 3$ and the assertion of the lemma in the case $i = n$ is true for all $k \leq n - 1$ rather than n. Let P be a convex polytope in E^n, which is not a simplex. Let the facets of P be $F_1, \dots, F_q$. Consider the facet F_1 and the supporting hyperplane $\operatorname{aff} F_1$ of P. By inductive assumption, the $(n-2)$–planes $L_1, \dots, L_t$ generated by the facets of F_1 determine in $\operatorname{aff} F_1$ at least $2^n - 2$ unbounded relatively open $(n-1)$–cells which contain an unbounded part of some ray $l(x)$, where $x \in \operatorname{relint} F_1$. Here equality stands if and only if F_1 is an $(n-1)$–simplex. Denote by N the set of all unbounded relatively open Euclidean $(n-1)$–cells in the arrangement of the planes

$$N_i = (\operatorname{aff} F_1) \cap \operatorname{aff} F_i, \quad (i \in \{2, \dots, q\},\ F_i \text{ is not parallel to } F_1),$$

which contain an unbounded part of some ray $l(x)$, where $x \in \operatorname{relint} F_1$. Because of the inclusion $\{L_1, \dots, L_t\} \subset \{N_2, \dots, N_q\}$, we have $\operatorname{card} N \geq \eta_{n-1}(F_1) \geq 2^n - 2$.

Let $G \in N$. Then G is a common facet of two unbounded n–cells $D', D'' \in C(P)$, one on each side of $\operatorname{aff} F_1$, where D' (respectively, D'') lies in the inner (respectively, outer) half–space bounded by $\operatorname{aff} F_1$. We show that $D', D'' \in L_n(P)$. In fact, if an unbounded part of $l(x)$ belongs to G, where $x \in \operatorname{relint} F_1$, then we find a point $y \in \operatorname{int} P$ near x such that an unbounded part of $l(y)$ belongs to D'. For D'' define l_ε as the direction of the vector sum of the unit vector of direction l and ε times the outer normal unit vector of F_1, where $\varepsilon > 0$ is small. Then an unbounded part of $l_\varepsilon(x)$ lies in D'', hence the same holds for $l_\varepsilon(y)$, for some point $y \in \operatorname{int} P$ such that $l_\varepsilon(x) \subset l_\varepsilon(y)$. Further, for any two different cells $G_1, G_2 \in N$, the respective cells $D_1', D_1'', D_2', D_2'' \in L_n(P)$ are pairwise different. Hence

$$\eta_n(P) \geq 2 \cdot \operatorname{card} N \geq 2\eta_{n-1}(F_1) \geq 2(2^n - 2),$$

and if F_1 is not an $(n-1)$–simplex, then

$$\eta_n(P) \geq 2 \cdot \operatorname{card} N \geq 2\eta_{n-1}(F_1) \geq 2^{n+1} - 2.$$

A new n–cell in $L_n(P)$ will be found in the following way. Denote by H the hyperplane in E^n supporting P, parallel to and different from $\operatorname{aff} F_1$. Put $M = P \cap H$. Denote by $F_{i(1)}, \dots, F_{i(s)}$ all the facets of P containing M, and let $Q_{i(1)}, \dots, Q_{i(s)}$ be the open half–spaces bounded by the hyperplanes $\operatorname{aff} F_{i(1)}, \dots, \operatorname{aff} F_{i(s)}$ and disjoint to P. Obviously, the open set $Q = Q_{i(1)} \cap \ldots \cap Q_{i(s)}$ is unbounded. The set Q is (possibly) divided by the remaining hyperplanes $\operatorname{aff} F_i$, $i \neq i(1), \dots, i(s)$ into open n–cells. Let $u \in M$, and let l be a direction such that $l(u) \subset Q \cup \{u\}$ and l is not parallel to any facet hyperplane of P. Then an unbounded part of $l(u)$ belongs to some unbounded n–cell $C \in C(P)$, where $C \subset Q$. Thus $C \in L_n(P)$. Hence $\eta_n(P) \geq 2 \cdot \operatorname{card} N + 1$, and in case that F_1 (or any facet of P) is not an $(n-1)$–simplex, $\eta_n(P) \geq 2 \cdot \operatorname{card} N + 1 \geq 2^{n+1} - 1$.

From now on we may suppose that P is simplicial. If P were also simple, then P would be a simplex [4, p. 65, exercise 11]. Since P was supposed to be no simplex, this implies P cannot be a simple polytope.

Suppose that e.g. the vertex x of the facet F_1 of P is not a simple vertex of P, i.e., the number of facets of P, adjacent to x and different from F_1, is at least

n. The hyperplanes of these facets intersect $\operatorname{aff} F_1$ in at least n $(n-2)$–planes P_i lying in $\operatorname{aff} F_1$, each containing x, and each being a supporting plane of F_1 in $\operatorname{aff} F_1$. Since the $(n-1)$–simplex F_1 has $n-1$ facets containing x, one of the $(n-2)$–planes P_i, say P_1, is not a facet hyperplane of F_1. Let further F_x denote the facet of F_1 opposite to x.

Let A denote the Euclidean arrangement of $(n-2)$–planes in $\operatorname{aff} F_1$, consisting of P_1 and the facet hyperplanes of F_1. Let further M denote the set of all unbounded relatively open Euclidean $(n-1)$–cells in the arrangement A, that contain an unbounded part of some ray $l(x)$, where $x \in \operatorname{relint} F_1$. Then $\operatorname{card} N \geq \operatorname{card} M$. Thus we have

$$\eta_n(P) \geq 2 \cdot \operatorname{card} N + 1 \geq 2 \cdot \operatorname{card} M + 1.$$

P_1 contains a ray $l(x)$ such that $l(x) \setminus \{x\}$ is disjoint to all facet hyperplanes of F_1. Let $l(y)$ be a subray of $l(x)$, where $y \neq x$. Then all sufficiently small perturbations $l_1(y_1)$ of $l(y)$ (i.e., y_1 is close to y, l_1 is close to l), lying in $\operatorname{aff} F_1$ and such that $l_1(y_1) \cap \operatorname{aff} F_x = \emptyset$, lie in the same relatively open $(n-1)$–cell of the arrangement $A_E(F_1)$. However such perturbations $l_1(y_1)$ can lie in different relatively open Euclidean $(n-1)$–cells of the arrangement A, namely they can lie on different sides of P_1. In fact, let u $(\notin P_1 - P_1)$ be the vector $x - c$, where c is the centroid of F_x. Then first we can translate $l(y)$ in the direction of $-u$ through a sufficiently small distance, so that for the obtained ray $l(y_1)$ the opposite ray $\tilde{l}(y_1)$ intersects $\operatorname{relint} F_1$. Second consider the direction l^ε of the vector sum of the unit vector of direction l and εu, and consider a subray $l^\varepsilon(y')$ of the ray $l^\varepsilon(y)$, where $\varepsilon > 0$ is small, and y' $(\neq y)$ is close to y, such that $\tilde{l^\varepsilon}(y')$ intersects $\operatorname{relint} F_1$. These two rays show that the cell $C_1 \in L_{n-1}(F_1)$ containing the ray $l(y)$ contains two cells in M. Hence $\operatorname{card} M \geq \eta_{n-1}(F_1) + 1 = 2^n - 1$. Therefore $\eta_n(P) \geq 2 \cdot \operatorname{card} M + 1 \geq 2^{n+1} - 1$. ■

5. Proofs of the results

Proof of Theorem 1. If K is polyhedral, then, obviously, any shadow–boundary and any illuminated region can be represented as a union of some relatively open faces of K. Since the number of faces of a polyhedral body is finite, each of the numbers $\sigma(K)$, $\sigma'(K)$, $\delta(K)$, $\delta'(K)$, $\sigma_b(K)$, $\gamma(K)$, $\gamma'(K)$, $\varepsilon(K)$ is finite.

Suppose that K is not polyhedral. Then for any natural number $m \geq 1$, it is possible to find inductively m regular points $x_1, \ldots, x_m \in \operatorname{bd} K$ such that the hyperplanes $H_1, \ldots, H_m$ supporting K at $x_1, \ldots, x_m$, respectively, are pairwise different and non–parallel. Let z be any point in $\operatorname{int} K$, and G_i be the hyperplane parallel to H_i and passing through z. The hyperplanes $G_1, \ldots, G_m$ divide $E^n \setminus K$ into open non-overlapping domains, say $D_1, \ldots, D_k$. By induction on m, it is easy to prove the inequality $k \geq m$. For any point

$$v \in E^n \setminus (K \cup G_1 \cup \ldots \cup G_m),$$

let $A(v)$ denote the intersection $\{x_1, \ldots, x_m\} \cap I'_{l(v)}$, where $l(v)$ is the direction of the ray $[v, z\rangle$. Then each $l(v)$ is an exterior direction, and for

$$v, w \in E^n \setminus (K \cup G_1 \cup \ldots \cup G_m),$$

one has $A(v) = A(w)$ if and only if v and w belong to the same domain D_r. By Lemma 2, we choose from each D_r, $r = 1, \ldots, k$ a point $v(r)$ such that each shadow–boundary $S'_{l(v(r))}$ is empty or sharp. For at least $k-2$ of the points $v(r)$, we have $\emptyset \neq A(v(r)) \neq \{x_1, \dots, x_m\}$, hence also $S'_{l(v(r))} \neq \emptyset$ (note that K is no slab, since it is not polyhedral). Thus we can choose $[(k-2)/2]$ points among the $v(r)$'s such that each corresponding set $A(v(r))$ is a nonempty proper subset of $\{x_1, \dots, x_m\}$, and no two $A(v(r))$'s are the complements of each other in $\{x_1, \dots, x_m\}$. Then the corresponding $[(k-2)/2]$ ($\geq [(m-2)/2]$) shadow–boundaries are sharp, nonempty, and pairwise different (since for an exterior direction l, $S'_l(K)$ uniquely determines $\{I'_l(K),\ D'_l(K)\}$, cf. the beginning of Section 4), and also the illuminated regions are pairwise different, and are also different from $\mathrm{bd}\, K$. Since m is arbitrary, $\sigma'(K) \geq \delta'(K) = \gamma'(K) = \infty$.

Choosing sufficiently far points $z(r)$ in some ε_0–neighbourhoods of the rays $\widetilde{l(v(r))}(z)$, they illuminate the same subsets of $\{x_1, \dots, x_m\}$ as the directions $l(v(r))$, hence similarly $\gamma(K) = \infty$. Since the $[(k-2)/2]$ above sharp shadow–boundaries $S'_{l(v(r))}(K)$ are different, the Hausdorff distances between any sufficiently large bounded portions of them are positive. However an easy compactness argument, using sharpness of $S'_{l(v(r))}(K)$, shows that in bounded portions of E^n the Hausdorff distances of $S'_{l(v(r))}(K)$ and $S_{z(r)}(K)$ are small (i.e., for any ball B about 0 and any $\varepsilon > 0$, the intersection of one of the above sets with B lies in the ε–neighbourhood of the other set, and vice versa, if $z(r)$ is sufficiently far). This implies that also the respective shadow–boundaries $S_{z(r)}(K)$ are different. Moreover we can choose $z(r)$ so that $S_{z(r)}(K)$ is sharp. Hence $\varepsilon(K) \geq \sigma(K) \geq \delta(K) = \infty$.

Evidently $x_i \in S_{x_i}(K) \not\ni x_j$ for $i \neq j$, hence the sets $S_{x_i}(K)$ are different, showing $\sigma_b(K) = \infty$. ■

Proof of Theorem 2. By Theorem 1, we may suppose K to be polyhedral.

1) Fix a point $y \in \mathrm{int}\, K$, and choose for any exterior direction l a point $z \in \tilde{l}(y)$ sufficiently far from y. Then $I'_l(K) = I_z(K)$, and, in case $S'_l(K)$ is sharp, $S'_l(K) = S_z(K)$. This implies $\gamma'(K) \leq \gamma(K)$, $\delta'(K) \leq \delta(K)$.

2) Now we prove that $\sigma'(K) \leq \sigma(K)$. Since nonempty sharp shadow–boundaries $S'_l(K)$ are equal to sharp shadow–boundaries $S_z(K)$ for suitable points z, we have to show that one can associate to each nonsharp shadow–boundary $S'_l(K)$ some nonsharp shadow–boundary $S_z(K)$, in such a way that to different $S'_l(K)$'s there are associated different $S_z(K)$'s.

Let $F_1, \dots, F_m$ ($m \geq 1$) be the facets contained in a nonsharp shadow–boundary $S'_l(K)$. Let us choose among the intersections, different from the empty set, of subfamilies of $\{\mathrm{aff}\, F_1, \dots, \mathrm{aff}\, F_m\}$ one of minimal dimension. Let this intersection be G. We may suppose $G = \cap\{\mathrm{aff}\, F_i \colon 1 \leq i \leq p\}$, and $G \cap \mathrm{aff}\, F_j = \emptyset$ for $p+1 \leq j \leq m$, where $1 \leq p \leq m$. Evidently, $l(0) \subset G - G$, and

$$G - G \subset \mathrm{aff}\, F_j - \mathrm{aff}\, F_j \quad \textit{for} \quad p+1 \leq j \leq m.$$

Let us choose $z_0 \in G$. Since l is an exterior direction for K, the difference $\tilde{l}(z_0) \setminus K$ is unbounded. Choose $z \in \tilde{l}(z_0) \setminus K$ so far from z_0 that, for any facet F of K, $\mathrm{relint}\, F \subset I'_l(K)$ implies $\mathrm{relint}\, F \subset I_z(K)$, and $\mathrm{relint}\, F \subset D'_l(K)$ implies $\mathrm{relint}\, F \subset D_z(K)$. Then the facets of K contained in $S_z(K)$ are exactly $F_1, \dots, F_p$; in particular, $S_z(K) \neq \emptyset$.

We have to show that $S_z(K)$ uniquely determines $S'_l(K)$. Let us consider the set M (which contains l) of those exterior directions m, for which $m(0) \subset \mathrm{aff}\, F_i - \mathrm{aff}\, F_i$, $1 \leq i \leq p$.

These directions m are the exterior directions m such that $m(0) \subset G - G$. Thus for $m \in M$, we have $m(0) \subset \operatorname{aff} F_j - \operatorname{aff} F_j$ for $p+1 \le j \le m$ as well. Therefore for each $m \in M$, one has

$$\{F \in F(K) \colon F \subset S'_l(K)\} \subset \{F \in F(K) \colon F \subset S'_m(K)\},$$

where $F(K)$ is the set of all facets of K. Hence

$$\bigcap_{m \in M} \{F \in F(K) \colon F \subset S'_m(K)\} = \{F \in F(K) \colon F \subset S'_l(K)\},$$

showing that $S_z(K)$ uniquely determines the set of facets contained in $S'_l(K)$.

Observe that both of $S'_l(K)$ and $S_z(K)$ are the unions of certain (closed) $(n-2)$–faces and $(n-1)$–faces of K. An $(n-2)$–face of K, which does not lie in any $(n-1)$–face of K contained in $S'_l(K)$, belongs to $S'_l(K)$ if and only if it belongs to $S_z(K)$. Hence

$$S'_l(K) = S_z(K) \cup \left(\bigcup \{F \in F(K) \colon F \subset S'_l(K)\}\right),$$

thus $S_z(K)$ uniquely determines $S'_l(K)$.

3) Now we turn to the last three inequalities of Theorem 2. Since each nonempty sharp shadow–boundary $S_z(K)$ corresponds to at least one illuminated region $I_z(K) \neq \operatorname{bd} K$, and each illuminated region $I_z(K) \neq \operatorname{bd} K$ corresponds to at most one nonempty sharp shadow–boundary (see Lemma 1), one has $\delta(K) \le \gamma(K)$. The inequality $\delta(K) \le \sigma(K)$ is trivial. Hence $\delta(K) \le \min\{\sigma(K), \gamma(K)\}$. Similarly, $\delta'(K) \le \min\{\sigma'(K), \gamma'(K)\}$. The last inequality is trivial.

4) By Lemma 3, $\sigma_b(K)$ equals the number of all faces of K of dimensions $0, 1, \dots, n-1$, with the exception of the minimal face among them (if it exists), and $\gamma(K)$ equals the number of those Euclidean n–cells in the Euclidean arrangement of all facet hyperplanes of K, which are different from $\operatorname{int} K$ and $\operatorname{int} \tilde{K}$. Observe that if a face F of K, of dimension in $\{0, 1, \dots, n-1\}$, is the intersection of the facets $F_1, \dots, F_m$, then F is the intersection of K and the closure of the open Euclidean n–cell $C(F)$ which is the intersection of the open outer half–spaces bounded by $\operatorname{aff} F_1, \dots, \operatorname{aff} F_m$, and of the open inner half–spaces bounded by all the other facet hyperplanes of K. Therefore for different faces F also the cells $C(F)$ are different. Further, for F not a minimal face, $C(F) \neq \operatorname{int} \tilde{K}$. Hence $\sigma_b(K) \le \gamma(K)$. ■

Proof of Theorem 3. By Theorem 1, we can suppose that K is a polyhedral set. Because of $j \le n-2$, K is no parallel slab. For polyhedral sets the statements about $\sigma'(K)$, $\delta(K)$, $\delta'(K)$ are immediate consequences of our Theorem 4 and [7, Theorem 5]. However, below we give simple direct proofs also for them.

By [4, p.24, **5** and p.26, **1**] (and since K is the direct sum of some E^k, where $0 \le k \le n-1$, and a line–free closed convex set) we have $K = K_0 + C_K$, where K_0 is a compact convex polytope (possibly less than n dimensional). Let π denote the projection along $\operatorname{aff} C_K$ ($\ni 0$),

$$\pi : E^n \to E^n / \operatorname{aff} C_K \cong E^{n-j}.$$

Then $\pi(K) = \pi(K_0)$ is a compact convex $(n-j)$–polytope. Let F' be an i–face of $\pi(K)$ $\quad (0 \le i \le n-j-1)$, and let us choose $i+1$ affine independent points

$x'_1, \dots, x'_{i+1}$ of F'. Choose points $x_1, \dots, x_{i+1} \in K$, projecting by π to $x'_1, \dots, x'_{i+1}$. These span an i–plane P_i in E^n, for which $(P_i - P_i) \cap \operatorname{aff} C_K = \{0\}$. Further choose j linearly independent vectors $v_1, \dots, v_j \in C_K$. Then $x_1, \dots, x_{i+1}, x_1 + v_1, \dots, x_1 + v_j$ are $i + j + 1$ affine independent points of K projecting by π to F', further $\dim\big(\pi^{-1}(\operatorname{aff} F')\big) = i + j$. Hence

$$F := K \cap \pi^{-1}(\operatorname{aff} F') \ (= K \cap \pi^{-1}(F'))$$

is an $(i + j)$–face of K, and F' is uniquely determined by F via $\pi(F) = F'$. We have $\operatorname{aff} F - \operatorname{aff} F \supset \operatorname{aff} C_K$, and, conversely, each face G of K, of dimension at most $n - 1$, and satisfying $\operatorname{aff} G - \operatorname{aff} G \supset \operatorname{aff} C_K$, is of the form $G = K \cap \pi^{-1}(F')$, for a face F' $(= \pi(G))$ of $\pi(K)$, of dimension in $\{0, 1, \dots, n - j - 1\}$.

For the investigated eight quantities we have evidently

$$\begin{cases} \sigma\Big(\pi^{-1}\big(\pi(K)\big)\Big) = \sigma\big(\pi(K)\big), \ \sigma'\Big(\pi^{-1}\big(\pi(K)\big)\Big) = \sigma'\big(\pi(K)\big), \dots, \\ \varepsilon\Big(\pi^{-1}\big(\pi(K)\big)\Big) = \varepsilon(\pi(K)). \end{cases}$$

If for each facet F of K we have $\operatorname{aff} F - \operatorname{aff} F \supset \operatorname{aff} C_K$, then $K = \pi^{-1}\big(\pi(K)\big)$, i.e., K is a j–fold both way infinite cylinder over $\pi(K)$. Let now K have a facet F_0 such that $\operatorname{aff} F_0 - \operatorname{aff} F_0 \not\supset \operatorname{aff} C_K$. Then, taking in account that now by the above considerations the set of facet hyperplanes (respectively, face planes) of K strictly contains the set of facet hyperplanes (respectively, face planes) of $\pi^{-1}\big(\pi(K)\big)$, by statements 1) and 2) before Lemma 3 and by Lemma 3 we have seven strict inequalities

$$\sigma'(K) > \sigma'\Big(\pi^{-1}\big(\pi(K)\big)\Big), \dots, \varepsilon(K) > \varepsilon\Big(\pi^{-1}\big(\pi(K)\big)\Big).$$

(For $\sigma'(K)$ and $\delta'(K)$ use also that F_0 is not parallel to any facet F of K such that $\operatorname{aff} F - \operatorname{aff} F \supset \operatorname{aff} C_K$.) For $\sigma(K)$ observe that if $z'_i \in E^n / \operatorname{aff} C_K$ are points with different $S_{z'_i}\big(\pi(K)\big)$'s, then choosing any points $z_i \in \pi^{-1}(z'_i)$ the sets $S_{z_i}(K)$ will be nonempty and different. Namely the sets of faces of K of the form $K \cap \pi^{-1}(F')$, belonging to $S_{z_i}(K)$ (where F' is a face of $\pi(K)$, of dimension in $\{0, 1, \dots, n-j-1\}$), are nonempty and different. A further different nonempty shadow–boundary is $S_u(K)$, where $u \in \Big(\operatorname{int} \pi^{-1}\big(\pi(K)\big)\Big) \setminus K$. Namely the relative interior of each face of K of the form $K \cap \pi^{-1}(F')$ (where F' is a face of $\pi(K)$, of dimension in $\{0, 1, \dots, n - j - 1\}$) belongs to $D_z(K)$. Hence also

$$\sigma(K) > \sigma\Big(\pi^{-1}\big(\pi(K)\big)\Big).$$

Now by statements 1) and 2) before Lemma 3, by Lemma 3 and by Lemma 4 we obtain

$$\sigma\big(\pi(K)\big) \geq \frac{1}{2} \cdot \sum_{i=1}^{n-j} \eta_i\big(\pi(K)\big) \geq (3^{n-j+1} - 2^{n-j+2} + 1)/2,$$

$$\delta\big(\pi(K)\big) \geq \frac{1}{2} \cdot \eta_{n-j}\big(\pi(K)\big) \geq 2^{n-j} - 1, \quad \gamma'\big(\pi(K)\big) = \eta_{n-j}\big(\pi(K)\big) \geq 2^{n-j+1} - 2,$$

$$\varepsilon\big(\pi(K)\big) \ge \sum_{i=1}^{n-j} \eta_i\big(\pi(K)\big) \ge 3^{n-j+1} - 2^{n-j+2} + 1.$$

By [4, p.184, **2**, case $r = 2$] the number of i–faces of an $(n-j)$–polytope is at least $\binom{n-j+1}{i+1}$ $(0 \le i \le n-j-1)$, with equality, for any i, only for a simplex. Hence, also using the third inequality in Theorem 2,

$$\gamma\big(\pi(K)\big) \ge \sigma_b\big(\pi(K)\big) \ge 2^{n-j+1} - 2.$$

Since in these five chains of inequalities in any of the right hand side inequalities equality holds only if $\pi(K)$ is a simplex, any of

$$\sigma\big(\pi(K)\big),\ \delta\big(\pi(K)\big),\ \gamma'\big(\pi(K)\big),\ \varepsilon\big(\pi(K)\big),\ \gamma\big(\pi(K)\big),\ \sigma_b\big(\pi(K)\big)$$

is equal to the respective right hand side expression only if $\pi(K)$ is a simplex.

For $\sigma'\big(\pi(K)\big)$ and $\delta'\big(\pi(K)\big)$ we proceed analogously to [8]. The polytope $\pi(K)$ has some $n-j$ facets having a one–point intersection, and the corresponding facet hyperplanes subdivide the infinite hyperplane of $E^n/\operatorname{aff} C_K$ into $(3^{n-j}-1)/2$ projective cells, with 2^{n-j-1} projective $(n-j-1)$–cells among them. Hence

$$\sigma'\big(\pi(K)\big) \ge (3^{n-j}-1)/2, \quad \delta'\big(\pi(K)\big) \ge 2^{n-j-1},$$

with equality only if each facet of the polytope $\pi(K)$ is parallel to one of the mentioned $n-j$ facets of $\pi(K)$, and thus $\pi(K)$ is a parallelotope.

Putting together the above estimates proves all inequalities 1)–5) of the Theorem. Equality in any of these holds only if a) K is a j–fold both–way infinite cylinder over $\pi(K)$, and b) $\pi(K)$ is a simplex, except for the inequalities 1'), 2'), when $\pi(K)$ is a parallelotope. The fact that in the asserted cases of equality in fact equality holds, is easily verified. ■

The proof of Theorem 4 follows from the remarks preceding it, in Section 3, as noted already there.

References

[1] G. Ewald, D. G. Larman and C. A. Rogers, The directions of the line segments and the r-dimensional balls on the boundary of a convex body in Euclidean space, *Mathematika* **17**(1970), 1–20.

[2] L. Fejes Tóth, On primitive polyhedra, *Acta Math. Acad. Sci. Hungar.* **13** (1962), 379–382.

[3] L. Fejes Tóth, Illumination of convex discs, *Acta Math. Acad. Sci. Hungar.* **29**(1977), 355–360.

[4] B. Grünbaum, *Convex polytopes,* Wiley Interscience, New York, 1967.

[5] B. A. Ivanov, Straight line segments on the boundary of a convex body, *Ukrain. Geometr. Sb.* **13**(1973), 69–71 (Russian).

[6] P. Kleinschmidt and U. Pachner, Shadow-boundaries and cuts of convex polytopes, *Mathematika* **27**(1980), 58–63.

[7] E. Makai,Jr. and H. Martini, A lower bound on the number of sharp shadow-boundaries of convex polytopes, *Periodica Math. Hungar.* **20**(1989), 249–260.

[8] H. Martini, Über scharfe Schattengrenzen und Schnitte konvexer Polytope, *Beitr. Algebra Geom.* **19**(1985), 105–112.

[9] R. W. Shannon, A lower bound on the number of cells in arrangements of hyperplanes, *J. Combin. Theory, Ser. A* **20**(1976), 327–335.

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