

A colouring problem for the dodecahedral graph

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Abstract

We consider vertex colourings of the dodecahedral graph with five colours, such that on each face the vertices are coloured with all the five colours. We show that the total number of these colourings is 240. All such colourings can be obtained from any given such colouring, by permuting the colours, and possibly applying central symmetry with respect to the centre of the regular dodecahedron. For any such colouring, the colour classes form the vertex sets of five regular tetrahedra. These tetrahedra together form one of the two compounds of five tetrahedra, inscribed in the regular dodecahedron. We give two proofs: a combinatorial one, and a geometrical one. Our result is related to the result in W. W. Rouse Ball – H. S. M. Coxeter, stating that there are four such colourings, as follows. There are four such colourings, up to applying an arbitrary orientation-preserving congruence of the regular dodecahedron.

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1 Introduction

As well known, the most popular soccer balls (footballs) have the form of a spherical mosaic, consisting of twelve regular spherical pentagons and twenty regular spherical hexagons. If we retain all the vertices of all its faces, but we replace the regular spherical polygons with planar ones, then we obtain an *Archimedean polyhedron* (i.e., its faces are regular, and its symmetry (i.e., congruence) group acts transitively on its vertices). The group of symmetries (i.e., congruences) of this spherical mosaic coincides with the group of symmetries of this Archimedean polyhedron, and also with the group of symmetries of the regular dodecahedron. Therefore this group is sometimes called the *dodecahedral group*. Since the regular icosahedron is the dual (actually, polar reciprocal) of the regular dodecahedron, therefore *the symmetry group of the regular icosahedron* coincides with that of the regular dodecahedron. Therefore this group *is usually called the icosahedral group*, and is usually *denoted by I_h* . In our paper we will use this terminology and notation. Then *I denotes the subgroup of I_h consisting of the orientation preserving elements of I_h* . (I_h has 120, and I has 60 elements. Their descriptions cf. later in this introduction.) Both the above spherical mosaic, and the corresponding Archimedean polyhedron, are denoted by $(5, 6, 6)$ (cf. [1]).

This spherical mosaic was invented as a soccer ball by the former Danish football player Eigel Nielsen and was introduced in 1962. The usual colouring of the soccer balls is the following: the pentagons are coloured black, and the hexagons white, patented by M. Doss in 1964.

However, this mosaic was used for making balls much earlier. An archive home movie made at the end of the 1930s shows the appearance of such a ball (we call it the Wörthsee ball). However the name of the maker of this ball and the year of its making are unknown to us. The faces of this ball were coloured: all pentagons were green, while the hexagons were of five different colours (in our notation below in Fig. 1, white = 1, yellow = 2, red = 3, blue = 4, black = 5). Seemingly the intention of the maker was that each pentagon should be surrounded by hexagons of all five colours. This was achieved for several, but not all pentagons, cf. Fig. 1(A). For a more detailed history cf. T. Tarnai – A. Lengyel [14].

TO INCLUDE HERE FIGURE 1!

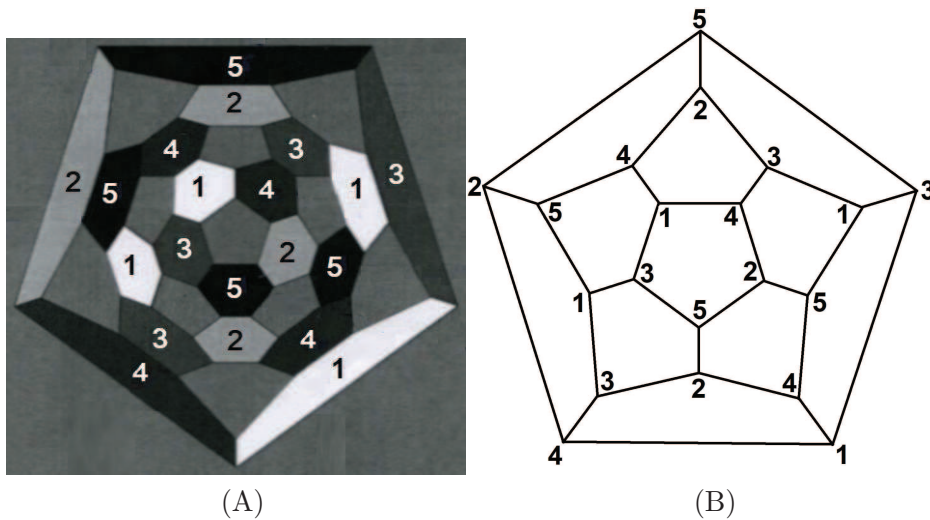


Figure 1. Colouring of the Wörthsee ball. (A) Schlegel diagram of $(5, 6, 6)$ with coloured faces. (B) Schlegel diagram of the corresponding regular dodecahedron with coloured vertices.

This raises the following mathematical problem. Does there exist a colouring of the hexagonal faces of $(5, 6, 6)$ with five colours, such that each pentagon has hexagonal neighbours of all five colours? Clearly in this problem the size of the pentagons is not important. We may shrink them simultaneously to their centres, while the hexagonal faces become regular triangles. Then our problem becomes the following. Colour the twenty faces of a regular icosahedron with five colours, so that each vertex is incident to faces of all five colours.

It will be more convenient to consider the dual problem. Colour the twenty vertices of the regular dodecahedron so that each face should have vertices of all five colours. The colours will be denoted by 1, 2, 3, 4, 5.

The actual colouring of the dual of the Wörthsee ball is shown in Fig. 1(B), where colours are replaced by numbers as given above.

This colouring problem was considered later in mathematics as well.

H. M. Cundy – A. P. Rollett, [4], 1985, wrote on pp. 82–83 the following (slightly rewritten for shortening). “The faces of the icosahedron can be coloured by five colours so that the five faces at every vertex are coloured differently, but opposite faces cannot then be coloured alike.”

W. W. Rouse Ball – H. S. M. Coxeter, [13], 1987, wrote on p. 242 the following. “L. B. Tuckerman remarks that the faces of an icosahedron can be coloured with five colours so that each face and its three neighbours

have four different colours. For five given colours this can be done in four ways, consisting of two enantiomorphous pairs. One pair can be derived from the other by making any odd permutation of the five colours (e.g. by interchanging two colours). The faces coloured alike belong to five regular tetrahedra forming the compound described on page 135 (cf. Coxeter, *Regular polytopes*, pp. 50, 106).”

Our theorem below asserts that there are 240 colourings of the vertices of the regular dodecahedron which satisfy our requirement. The “paradox” that this is different from 4 such colourings, as given in [13], will be resolved later in this paper. We give two proofs of our theorem: first a combinatorial one, and second a geometrical one.

We remark that our problem is equivalent to the problem about the number of the five-colourings of the vertices of the graph whose vertices are the vertices of the regular dodecahedron, and whose edges are the face-diagonals and the edges of the regular dodecahedron.

In our paper we will use the concepts and statements from [1], [2], [7], [8], [10], [11], [12]. In particular, a compound of five regular tetrahedra, mentioned above, is explained in them. However, for completeness, below we will give its description. Fig. 2 shows a photograph of a cardboard model of the compound of five regular tetrahedra, taken by A. Lengyel.

TO INCLUDE HERE FIGURE 2!



Figure 2. The compound of five tetrahedra. Quite recently (in 2018) the concept of regular compounds was defined

in a strict mathematical sense by P. McMullen in [9], as follows. A *vertex-regular compound* consists of copies of a regular polytope Q with vertex sets in the vertex set of a regular polytope P , such that some subgroup of the symmetry group of P is transitive both on the vertex set of P and on the copies of Q . A *face-regular compound* is defined in a dual way, replacing vertices by face-planes. Then a *regular compound* means either a vertex-regular, or a face-regular compound. In this sense, there are five regular compounds in \mathbb{R}^3 . These are: regular compounds of 2 tetrahedra inscribed in the cube (and escribed to an octahedron — this is called *Kepler's stella octangula*, although, according to [5], Part 1, Ch. II, §11, p. 82 (in the German edition §I.2.6, p. 83), it was described by L. Pacioli (or Pacciuolo), as well), regular compounds of 5 and of 10 tetrahedra inscribed in the dodecahedron (and escribed to an icosahedron), of 5 cubes inscribed in the dodecahedron, and of 5 octahedra escribed to an icosahedron (dual to the former one). L. Fejes Tóth, [5], Part 1, Ch. II, §9, p. 78 describes these five regular compounds (in the German edition under the name “Körperkomplexe”, in §I.2.4, p. 79), without giving the strict definition of [9]. At the end of [5] there are spectacular anaglyphs I-III, displaying these compounds three-dimensionally. [5] refers, in Part 1, Ch. II, §11, p. 82 (in the German edition §I.2.6, p. 83), to E. Hess [6], 1876, for thoroughly investigating these five polyhedral compounds. For the compound of five tetrahedra cf. [6], §6, p. 45.

Since we will use only the regular dodecahedron and tetrahedron, *usually we will drop the attribute “regular”*. Later on, *we will suppose that the centre of the dodecahedron is the origin 0, and that the dodecahedron is inscribed in the unit sphere S^2 about 0*.

We turn to some elementary group theoretical considerations, and with their help we will describe the compound of five tetrahedra.

The groups I_h and I have been introduced above. They are subsets of the group of congruences of \mathbb{R}^3 . Then all congruences in I_h (and thus also in I) preserve 0, the centre of the dodecahedron, i.e., are linear transformations. Further we will denote by *id* the *identical transformation on \mathbb{R}^3* , and hence *−id* is the *central symmetry with respect to 0*. Both are congruences of \mathbb{R}^3 , and are linear transformations.

Next we introduce some abstract groups, and some permutation groups. We write S_5 for the *symmetric group on five symbols*, i.e., the group of all permutations of the five symbols, and A_5 for the *alternating group on*

five symbols, i.e., the subgroup of S_5 consisting of all even permutations of the five symbols. If we want to give the five symbols on which these permutations act, then we will write $S_5(\dots)$ and $A_5(\dots)$, where the five symbols are given in the brackets. Moreover, $Z_2 \cong \{1, -1\}$ denotes the *two-element cyclic group* (written multiplicatively). The sign \cong means here isomorphism of groups. We will apply this sign only in cases when this isomorphism is in some sense “canonical”.

Now we are going to describe the compound of five tetrahedra. For this aim first we recall the description of the icosahedral group I_h . (Cf. [7] for its subsequent discussion, till the end of our introduction.) We describe it in terms of the dodecahedron. As an abstract group, $I_h \cong A_5 \times Z_2$. Therefore I_h has 120 elements (as mentioned earlier).

Next we are going to describe I_h concretely, as a group of congruences. We begin with the description of the group I . As an abstract group, it is isomorphic to A_5 . Now recall that the dodecahedron has ten inscribed tetrahedra, with vertices among the vertices of the dodecahedron. They can be found in Figures 3 and 4: the vertices with the same numbers (colours) are the vertices of five tetrahedra in Fig. 3, and of other five tetrahedra in Fig. 4. Then I_h acts transitively on the set of these ten tetrahedra, i.e., any of these tetrahedra can be taken over to any other one of these tetrahedra by some element of I_h .

However, this does not hold any more for the group I . Any tetrahedron depicted in Fig. 3 can be taken over to any other tetrahedron depicted in Fig. 3 by some element of I , and the same statement holds for Fig. 4. However, one tetrahedron from Fig. 3 (Fig. 4) cannot be taken over by any element of I to a tetrahedron from Fig. 4 (Fig. 3). That is, I acts transitively on both sets of five tetrahedra, and both these sets are *orbits for I*. Both of these sets of five tetrahedra are called a *compound of five tetrahedra*. These two sets are congruent, but only via an orientation reversing congruence. Thus these compounds have a *chirality*, i.e., left/right-handedness. The action of I on any of these compounds, as a group of permutations of the five tetrahedra T_1, \dots, T_5 in the compound, is the action of $A_5(T_1, \dots, T_5)$. Conversely, for any even permutation of the five tetrahedra there is a unique (orientation preserving) congruence, belonging to I , which acts on these five tetrahedra as the given even permutation. Therefore,

$$I \cong A_5(T_1, \dots, T_5). \tag{1}$$

Then we have, by letting to $\pm\text{id}$ correspond ± 1 , that

$$I_h = I \times \{\text{id}, -\text{id}\} \cong A_5(T_1, \dots, T_5) \times \{1, -1\}, \quad (2)$$

for T_1, \dots, T_5 being the tetrahedra in any of the two compounds.

Geometrically, $I = I \times \{\text{id}\}$ can be given as the set of all 4 rotations of order 5 about 6 axes through the centres of *antipodal* (i.e., opposite) faces, all 2 rotations of order 3 about 10 axes through antipodal vertices, all (i.e., 1 for each of the axes) rotations of order 2 about 15 axes through midpoints of antipodal edges, and of id . Then $I \times \{-\text{id}\}$ is a set of 24 rotatory reflections of order 10, and 20 rotatory reflections of order 6, and 15 rotatory reflections of order 2 (i.e., symmetries with respect to planes), with the same axes as above, and of $-\text{id}$.

Suppose that we have a permutation $p \in S_5(1, \dots, 5)$. This is by definition a bijective function from $\{1, \dots, 5\}$ to itself. We define the *action of such a permutation p on a colouring of the vertices of the dodecahedral graph with colours $1, \dots, 5$* as follows. The action of p on the above colourings changes the colours of all vertices of colour i to the colour $p(i)$, for each $i \in \{1, \dots, 5\}$. (Observe that, anyhow natural this definition is, we need it, since a bijective function on $\{1, \dots, 5\}$ acts by definition on the set $\{1, \dots, 5\}$, and not on the set of the above colourings.) Thus we have defined the *action of the group $S_5(1, \dots, 5)$ on the set of all our colourings*.

We also refer to [5] and [3] for ample material on regular figures, and to [15] for its nice figures of cardboard models of regular polyhedra and of compounds of regular polyhedra (these latter ones are not defined there in a strict way). The compound of five tetrahedra is in [15] (p. 44 in the 1989 reprint of the first paperback edition in 1974 of [15]).

2 The theorem and its two proofs

TO INCLUDE HERE FIGURE 3!

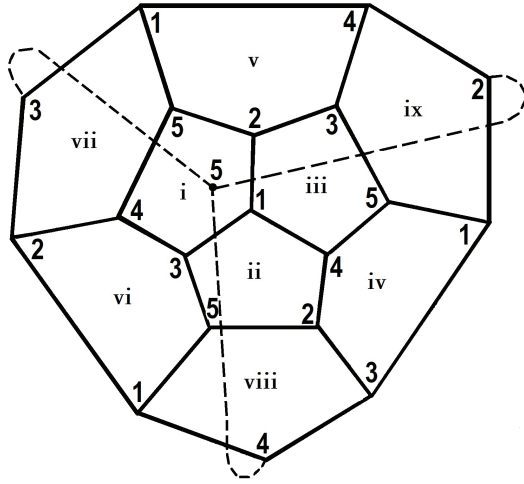


Figure 3. The first solution.
 TO INCLUDE HERE FIGURE 4!

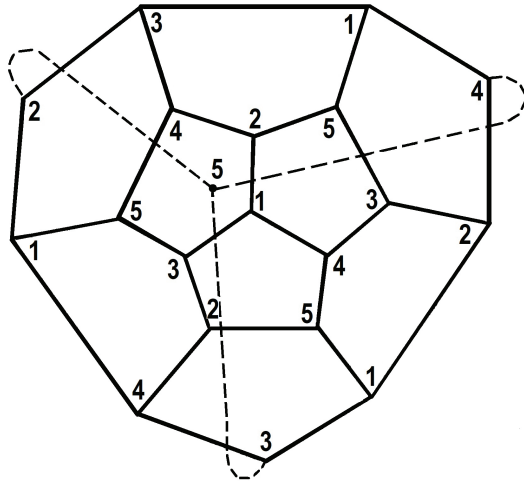


Figure 4. The second solution.

First we give two colourings of the dodecahedral graph, satisfying our requirements, in Figures 3 and 4. In these figures the middle vertex, both times of colour 1, is the north pole of the unit sphere S^2 about 0, which is the circumsphere of the dodecahedron. The three neighbours of the north pole, both times of colours 2, 3, 4 (in the same order), lie on a circle of latitude C_1 on the northern hemisphere.

The numbers of the 2nd, 3rd and 4th neighbours of the north pole are 6, 6 and 3, and they lie on circles of latitude C_2 , C_3 and C_4 , respectively. The unique 5th neighbour of the north pole is the south pole. C_1 and C_2 lie on the open northern hemisphere, and $C_3 = -C_2$ and $C_4 = -C_1$ lie on the open southern hemisphere. (These are some analogues of the two polar and two tropical circles.) The Euclidean distance between the points of C_3 and the north pole is the edge-length of an inscribed (regular) tetrahedron of the unit sphere S^2 .

We list several remarkable properties of these colourings.

P1) For Fig. 3 from any vertex, of colour i , we can reach the other vertices of colour i by passing on an edge incident to this vertex, then at the other endpoint of this edge turning to the left-hand side edge, and then at the other endpoint of this second edge turning to the right-hand side edge: a *left-right zigzag*. (For the north pole we pass thus, e.g., through vertices of colours 1, 2, 5, 1.) For any $i \in \{1, \dots, 5\}$ there are exactly four vertices of colour i , and they form together the vertices of a tetrahedron inscribed in the dodecahedron. For $i \in \{1, \dots, 5\}$ these five tetrahedra form a compound of five tetrahedra. The colour of the vertex antipodal to any vertex v is the colour different from the colours of v and its three neighbours. Moreover, the colours of the three neighbours of any vertex and its antipodal vertex are the same. For Fig. 4 all above statements of **P1)** hold with exchanging “left” and “right”. The respective compound of five tetrahedra is obtained from the one in Fig. 3 by taking its centrally symmetric image with respect to 0.

Before giving **P2)**, we recall that a *cyclic permutation of* $\{1, 2, 3, 4, 5\}$ is an equivalence class of permutations $(ijklm)$, where the five permutations $(ijklm)$, $(mijkl)$, \dots , $(jklmi)$ are considered equivalent. All the five above permutations have the same parity (even or odd), which is called the *parity of the cyclic permutation*.

P2) For Fig. 3, on the twelve faces of the dodecahedron, when looking back from the direction of their outer normal unit vectors (i.e., if we take the faces in the positive orientation), we have that the cyclic permutations of the colours of the vertices are exactly all twelve odd cyclic permutations. (Observe that this statement does not depend on the orientation of the faces; the same statement holds for the negatively oriented faces.) On opposite faces we have inverse cyclic orders (i.e., the same cyclic orders, but in opposite orientation). For Fig. 4 all above statements of **P2)** hold with changing “odd” to “even”.

P3) For each $i \neq j$ there are three edges with endpoints of colours i, j . For the three edges with endpoints of colours i, j the antipodal edges have vertices of colours k, l , where $k, l \in \{1, \dots, 5\} \setminus \{i, j\}$, and all three such pairs $\{k, l\}$ occur for these three edges. These four vertices of colours i, j, k, l are first neighbours of the four vertices of the fifth colour.

For a graph whose vertices are coloured by some colours, a *colour class* is the set of all vertices coloured by a particular colour. All the colour classes form a partition of the vertex set of the graph.

Theorem. *The number of the five-colourings of the vertices of the dodecahedral graph, such that on each face of the regular dodecahedron the vertices are coloured with all five colours, is 240. All such colourings can be obtained from any given such colouring, by permuting the colours (cf. the explanation at the end of the introduction), and possibly applying central symmetry with respect to the centre of the regular dodecahedron (i.e., the new colour of a vertex is the original colour of the antipodal vertex). For any such colouring, the colour classes form the vertex sets of five regular tetrahedra. These regular tetrahedra together form one of the two compounds of five regular tetrahedra, inscribed in the regular dodecahedron.*

First proof of the Theorem. We suppose that the circumsphere of the (regular) dodecahedron is the unit sphere S^2 with centre 0, and one of its vertices is at the north pole. We write “face” for a face of the dodecahedron.

We choose a permutation of the five colours, such that the vertex of the dodecahedron at the north pole has colour 1, and its three neighbours have colours 2, 3, 4, in the order as in Figures 3 and 4. We will extend the colouring to C_2, C_3, C_4 , and the south pole, consecutively.

The face F_1 with one vertex at the north pole, and neighbouring vertices of colours 2, 3, has two yet uncoloured vertices, which must have colours 4, 5. These can be coloured in two ways: as in Fig. 3, and as in Fig. 4. (In Fig. 3 the face F_i is denoted by writing in it its subscript i , but in Roman numerals.)

We begin with the case shown in Fig. 3. The remaining two faces F_2, F_3 , having the north pole as a vertex, and having neighbouring vertices of colours 3, 4, and 4, 2, both have two yet uncoloured vertices, which must have colours 2, 5, and 3, 5, respectively. The vertex of F_1 of colour 5 forces that F_3 has vertices of cyclic order (21453) in the positive sense. Then the vertex of F_3 of colour 5 forces that F_2 has vertices of cyclic order (41352) in the positive sense. Thus the vertices on C_2 are uniquely coloured.

We turn to the six vertices on C_3 . We call F_4, F_5, F_6 the faces neighbourly to F_2, F_3 , and F_3, F_1 , and F_1, F_2 , but different from F_1, F_2, F_3 . Then each of F_4, F_5, F_6 has two yet uncoloured vertices, which have to have colours 1, 3, and 1, 4, and 1, 2, respectively. Among these six vertices on C_3 , taken in cyclic order, there cannot be two neighbourly vertices of colour 1. Therefore each second of them in this order has colour 1. By the vertex of colour 4 of F_1 , the cyclic order of the vertices of F_5 is (52341) in the positive sense, and then the cyclic orders of the vertices of F_6 and F_4 are (53421), and (54231), in the positive sense, respectively. Thus the vertices on C_3 are uniquely coloured.

We turn to the three vertices on C_4 . We call F_7, F_8, F_9 the faces neighbourly to F_6, F_1, F_5 , and F_4, F_2, F_6 , and F_5, F_3, F_4 . These have unique yet uncoloured vertices, which therefore have colours 3, 4, 2, respectively. Thus the vertices on C_4 are uniquely coloured.

At last the south pole has neighbours of colours 3, 4, 2 and second neighbours of colours 1, 2, and 1, 3, and 1, 4, respectively. Therefore its colour is 5.

Thus for Fig. 3 the colouring is uniquely determined.

For Fig. 4 an analogous simple argument shows the same, which we leave to the reader.

Since at the beginning of the proof we fixed a permutation of the five colours, both Fig. 3 and Fig. 4 represent in fact $5! = 120$ solutions.

Clearly Figures 3 and 4 have a chirality (left- or right-handedness), left-right or right-left zigzag in Property **P1**), and odd or even cyclic permutations in Property **P2**). Thus the 120 colourings represented by Fig. 3 are different from the 120 colourings represented by Fig. 4.

From this there follows that applying an even permutation to the five colours takes the 120 colourings represented by Fig. 3 or Fig. 4 to some colouring represented by the same figure from Figures 3 and 4, and preserves the cyclic orientation of the faces (i.e., the parity of the cyclic permutation). However, applying an odd permutation to the five colours takes the 120 colourings represented by Fig. 3 or Fig. 4 to some colouring represented by the same figure from Figures 3 and 4, but changes the cyclic orientation of the faces.

Since by (2) the symmetry group of the dodecahedron is

$$I_h = I \times \{\text{id}, -\text{id}\} \cong A_5(T_1, \dots, T_5) \times \{1, -1\} \cong A_5(1, \dots, 5) \times \{1, -1\} \quad (3)$$

(here $A_5(T_1, \dots, T_5)$ is identified with $A_5(1, \dots, 5)$ by letting to T_i correspond i , and $\pm \text{id}$ is identified with ± 1), therefore we choose the “sim-

plest” orientation reversing symmetry $-id$ of the dodecahedron. Then all colourings of the dodecahedral graph, satisfying our requirements, can be obtained from that in Fig. 3 (alternatively, in Fig. 4), by an arbitrary permutation of the five colours, i.e., by applying an arbitrary element of $S_5(1, \dots, 5)$, and by possibly applying still the linear transformation $-id$, which we identify with -1 . We mean this in the following sense: id , *identified with 1, acts on a colouring identically* (i.e., preserves it), and $-id$, *identified with -1 , acts on a colouring so that in the new colouring the colour of any vertex $-v$ is the colour of the vertex v in the original colouring.* (Of course, this action is the natural one, but observe that by definition $\pm id$ act on \mathbb{R}^3 , and not on the set of our colourings. Cf. also the definition of the action of $S_5(1, \dots, 5)$ on our colourings, at the end of the Introduction.) Observe that here we have the direct product $S_5(1, \dots, 5) \times \{1, -1\}$, since application of an element of $S_5(1, \dots, 5)$ leaves the set of the colour classes invariant (only permutes them), but the application of $-id$ changes the set of the colour classes. Therefore the number of all colourings is $2 \cdot 120 = 240$.

This proves the first two statements of the theorem. The statement about the colour classes can be verified from Figures 3 and 4. \square

Observe that both the statement and the proof of our Theorem are combinatorial. However, one can prove our Theorem also geometrically, as follows.

Second proof of the Theorem. Recall that the circumsphere of the dodecahedron is the unit sphere S^2 about 0. Therefore any colour class of a colouring satisfying the hypotheses of our Theorem is a subset of S^2 . Moreover, among its points there occur no smallest and second smallest Euclidean distances among the vertices of the dodecahedron (these are the the Euclidean distances between the north pole and the points of C_1 and C_2). In other words, *the Euclidean distances between points of any colour class are at least the third smallest Euclidean distance among the vertices of the dodecahedron* (i.e., the Euclidean distance between the north pole and the points of C_3), i.e., *the edge length of a (regular) tetrahedron inscribed in S^2* (cf. the beginning of §2).

However, a subset of S^2 with this italicized property has at most four points, with equality only if these four points are the vertices of a (regular) tetrahedron, cf. [5], Part 2, §35, p. 227 (in the German edition Ch. II.2.6, p. 215). Since all five colour classes have altogether 20 points, each colour class has four points, which are the vertices of a tetrahedron, and also belong to the vertex set of the dodecahedron. The number of all such tetrahedra is

10, and they are depicted in Figures 3 and 4. Then all five colour classes are either as depicted in Fig. 3, or as depicted in Fig. 4, or some colour class is as depicted in Fig. 3 and some other colour class is as depicted in Fig. 4. Since the colour classes are disjoint, from Figures 3 and 4 we see that in the last case any two such colour classes are centrally symmetric images of each other with respect to 0. Hence the number of colour classes is two, a contradiction. Knowing the colour classes, one can finish this proof as in the combinatorial proof. \square

Identifying $\pm\text{id}$ with ± 1 , we write

$$G := S_5(1, \dots, 5) \times \{\text{id}, -\text{id}\} \cong S_5(1, \dots, 5) \times \{1, -1\} \quad (4)$$

Observe that any element of G takes a colouring satisfying our requirements to another such colouring. Also conversely, any such colouring can be taken over to any other such colouring by an element of G . Therefore we can say that, up to the action of elements of G , we have “geometrically” a unique colouring.

If H is an arbitrary subgroup of G , we can ask for the *number of our colourings, up to the action of an element in H* . This number is

$$|G|/|H| = 240/|H|. \quad (5)$$

For the statement of W. W. Rouse Ball – H. S. M. Coxeter [13], using (1), and identifying id with 1, we consider

$$H := I \times \{\text{id}\} \cong A_5(T_1, \dots, T_5) \times \{1\} \cong A_5(1, \dots, 5) \times \{1\} \quad (6)$$

(here for the second isomorphism sign cf. (3)), and then, using (5), we have the following

Corollary. ([13]) *The number of the colourings of the dodecahedral graph, satisfying the requirements of our Theorem, up to the application of any element of $I \times \{\text{id}\} \cong A_5(T_1, \dots, T_5) \times \{1\} \cong A_5(1, \dots, 5) \times \{1\}$, is 4. They can be distinguished as follows.*

1) *The colour classes are the sets of vertices of the five regular tetrahedra, forming either of the two compounds of five regular tetrahedra inscribed in the regular dodecahedron (these two possibilities are depicted in Figures 3 and 4).*

2) *Further they can be distinguished by odd/even cyclic permutations of the colours of the vertices on one (or all) positively oriented face(s) of the regular dodecahedron, or left-right/right-left zigzags in Figures 3 and 4.*

These two choices being independent, we have altogether four possibilities. \square

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References

- [1] *Archimedean solid*, Wikipedia
- [2] *Compound of five tetrahedra*, Wikipedia
- [3] Coxeter, H. S. M., *Regular polytopes*, 3rd edition, Dover, New York, 1973, MR **51**#6554
- [4] Cundy, H. M., Rollett, A. P., *Mathematical models*, 3rd edition, Tarquin Publications, Stradbroke, 1985, Zbl **0259**.0002 (Review of 2nd edition.)
- [5] Fejes Tóth, L., *Regular figures*, Pergamon, Macmillan, New York, 1964, MR **29**#2705; *Reguläre Figuren* (German), Akad. Kiadó, Verl. Ungar. Akad. Wiss., Budapest, 1965, MR **30**#3408
- [6] Hess, E., *Über die zugleich gleicheckigen und gleichflächigen Polyeder*, Schriften der Gesellschaft zur Beförderung der gesammten Naturwissenschaften zu Marburg, **11**, Kay Verlag, Cassel, 1876 (digitized version in Google Scholar, under the title “Hess 1876 Ueber die zugleich gleicheckigen”)
- [7] *Icosahedral symmetry*, Wikipedia
- [8] *List of regular polytopes and compounds*, Wikipedia
- [9] McMullen, P., *New regular compounds of 4-polytopes*. In: Ambrus, G., Bárány, I., Böröczky, K. J., Fejes Tóth, G., Pach, J. (eds.), *New Trends in Intuitive Geometry*, Bolyai Soc. Math. Studies **27**, 307-320. J. Bolyai Math. Soc., Budapest, and Springer, Berlin-Heidelberg, part of Springer Nature. Corrected publication, 2018.
- [10] *Regular dodecahedron*, Wikipedia

- [11] *Regular icosahedron*, Wikipedia
- [12] *Regular polytope*, Wikipedia
- [13] Rouse Ball, W. W., Coxeter, H. S. M., *Mathematical recreations and essays*, 13th edition, Dover, New York, 1987, MR **88m**:00013
- [14] Tarnai, T., Lengyel, A., *The truncated icosahedron as an inflatable ball*, Periodica Polytechnica Architecture **49**(2), 2018, 99-108, <https://doi.org/10.3311/PPar.12375>
- [15] Wenninger, M., *Polyhedron models*, Cambridge Univ. Press, London–New York, 1971, MR **57**#7350a