GENERALIZED FORMS OF AN OVERCONSTRAINED SLIDING MECHANISM CONSISTING OF TWO CONGRUENT TETRAHEDRA

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Abstract. The motions of a bar structure consisting of two congruent tetrahedra are investigated, whose edges in their basic position are the face diagonals of a rectangular parallelepiped. The constraint of the motion is the following: the originally intersecting edges have to remain coplanar. All finite motions of our bar structure are determined. This generalizes our earlier work, where we did the same for the case when the rectangular parallelepiped was a cube. At the end of the paper we point out three further possibilities to generalize the question about the cube, and give for them examples of finite motions.

1. Introduction

Let us draw all the diagonals of all faces of a cube. Thus we get the edges of two congruent regular tetrahedra. We call this position of these tetrahedra their *basic position.*

> One of the tetrahedra is kept fixed. The other one is moved under the following restriction. Each pair of edges of the two tetrahedra, originally having been diagonals of some face of the cube, have to remain coplanar.

It was L. Tompos, Jr., who invented, in 1982, the structure consisting of the above described two tetrahedra. He made then his undergraduate studies at the Hungarian Academy of Craft and Design. The physical model of the bar (i.e., edge) structure of these tetrahedra has been built by him, as follows. The bars of one

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tetrahedron touched the bars of the other tetrahedron from inside (cf. Fig. 1). **FIG. 1 ABOUT HERE.** His observation was the following: his structure

Figure 1. Tompos's tetrahedra: its bar-and-joint structure. The basic position of the physical model. (Photograph provided by András Lengyel.)

admits continuous motions. We note that [2], p. 7 contains a figure of these tetrahedra, but their mobility is not investigated there. We have found the same figure as a decoration of the dining room of Hotel Arcas de Agua in Spain, in village Arcas, Cuenca (Fig. 2). **FIG. 2 ABOUT HERE.**

Figure 2. The lamp decoration in the hotel in Arcas. (Photograph provided by Ampar López.)

A *motion* (sometimes called a *finite motion*) will not mean a continuous motion, beginning from the basic position, always satisfying the constraints, but *any position of our structure such that the constraints are satisfied*. (Possibly this position cannot be attained from the basic position, by a continuous motion, such that the constraints are always satisfied.) More detailed, this means the following. One tetrahedron is fixed. The other tetrahedron is got from its basic position by applying to it an isometry (i.e., congruence) of the space *with determinant* $+1$. Additionally, the coplanarity hypotheses are fulfilled. We will write an isometry of determinant $+1$ in the form $\mathbf{\Phi}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$. Here $\mathbf{A}$ is a 3×3 orthogonal matrix, which has determinant $+1$, and $\mathbf{b} \in \mathbb{R}^3$. Geometrically, $\mathbf{A}$ is a rotation about an axis containing the origin.

All motions of this pair of tetrahedra were determined by [10] and [12]. We briefly describe them, used for them the names from [12]. Observe that the bars of the model have non-zero width. Hence the physical model cannot realize all motions. Only such motions can be physically realized, for which each pair of edges, which has to be coplanar, has some common point. [1], Ch. 4 contains the description of the motions which are physically admissible, i.e., when this more restrictive requirement is satisfied. Later in this paper no distinction will be made between motions which are physically admissible, and physically inadmissible.

Suppose the following. In the basic position, the vertices of our tetrahedra are the points $(\pm 1, \pm 1, \pm 1)$. More detailed, the fixed tetrahedron has as vertices $P_1^0(1, -1, -1)$, $P_2^0(-1, 1, -1)$, $P_3^0(-1, -1, 1)$ and $P_4^0(1, 1, 1)$. The moving tetrahedron has as vertices Q_1^0, Q_2^0, Q_3^0 and Q_4^0. In the basic position, the following holds: Q_i^0 is the centrally symmetric image of P_i^0, w.r.t. the origin (i.e., the middlepoint of the cube with vertices $(\pm 1, \pm 1, \pm 1)$), see Fig. 3. **FIG. 3 ABOUT HERE.**

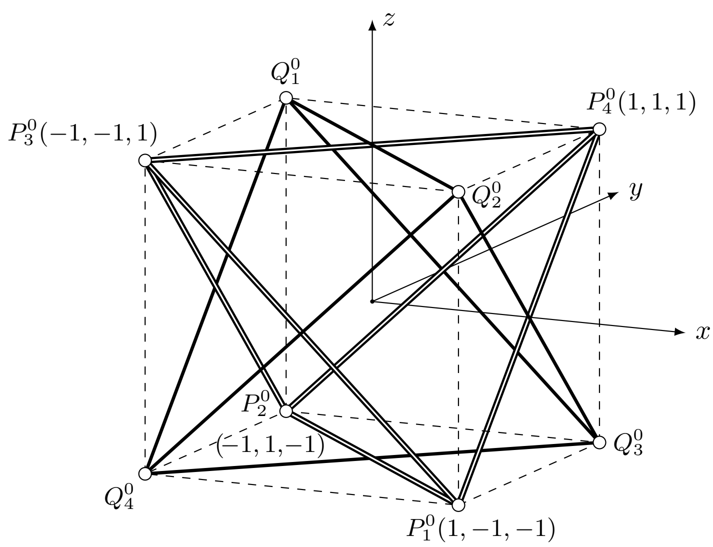

Figure 3. The notation of the vertices of our tetrahedra.

Tompos's tetrahedra admit motions $\mathbf{\Phi}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is a rotation, with axis $\mathbf{0}\mathbf{e}_i$, or $\mathbf{0}(\mathbf{e}_i \pm \mathbf{e}_j)$ or $\mathbf{0}(\mathbf{e}_i \pm \mathbf{e}_j \pm \mathbf{e}_k)$, resp. Here $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ are the usual basic unit vectors, with i, j, k being distinct. The names of these motions $\mathbf{\Phi}(\mathbf{x})$ are the *motions of the first, second and third kind*, resp. For the first case, when the angle of rotation is π, then $\mathbf{b}$ is not unique. However, only the case $\mathbf{b} = \mathbf{0}$ is counted to the motion of the first kind. The angle of rotation of $\mathbf{A}$ is arbitrary in the first case, it is arbitrary, except π, in the second case, and it is arbitrary, except $\pm\pi/2$, in the third case. (The angle of rotation is counted positive if, looking backwards from the axis vector, e.g., $\mathbf{0}\mathbf{e}_i$, it is positive.)

The following was proved. For each above rotation $\mathbf{A}$, with the exception of the rotation through the angle π in the first case, we had the following. There existed a unique $\mathbf{b} \in \mathbb{R}^3$, such that $\mathbf{\Phi}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ was a motion admitted by Tompos's tetrahedra. For an angle of rotation different from π, in the first case there held $\mathbf{b} = \mathbf{0}$.

Also for $i \neq j$, with C_1, C_2 real, not both 0, Tompos's tetrahedra admit motions $\mathbf{\Phi}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, with $\mathbf{A}$ a rotation of axis $\mathbf{0}(C_1\mathbf{e}_i + C_2\mathbf{e}_j)$. Let $C_1C_2 \neq 0$, i.e., let this be not a motion of the first kind. In this case, the rotation angle of $\mathbf{A}$ can be arbitrary, except π. Moreover, $\mathbf{b} \in \mathbb{R}^3$ is uniquely determined for each such rotation $\mathbf{A}$. We call *motions of the intermediate kind* the above motions, for $C_1 C_2 \neq 0$, together with the motions of the first kind (for which $C_1 C_2 = 0$), both these motions with angle of rotation different from π. The motions of the second kind are a special case of these motions.

Suppose that $\mathbf{A}$ is a rotation of axis $\mathbf{0e}_i$, through the angle π. Then let $\mathbf{b} \in \mathbb{R}^3$ be of the form $C\mathbf{e}_j$, or $C_1\mathbf{e}_j + C_2\mathbf{e}_k$, resp., with C, C_1, C_2 real and i, j, k distinct. Then $\mathbf{\Phi}(\mathbf{x}) = \mathbf{Ax} + \mathbf{b}$ is a motion admitted by Tompos's tetrahedra. We call them the *motion of the fourth, or fifth kind*, resp. The motions of the fourth kind are a special case of the motions of the fifth kind. Figs. 4a, 4b, 4c, 4d and 4e show the motions of the first, second, third, fourth and fifth kinds. **FIGS. 4a, 4b, 4c, 4d, 4e ABOUT HERE.**

The following was proved. The motions of each kind, for given $\mathbf{e}_i$, $\mathbf{e}_i \pm \mathbf{e}_j$, $\mathbf{e}_i \pm \mathbf{e}_j \pm \mathbf{e}_k$, $\{\mathbf{e}_i, \mathbf{e}_j\}$, $\mathbf{e}_j$ and $\{\mathbf{e}_j, \mathbf{e}_k\}$ (in the above order), resp., formed a smooth manifold in the six-dimensional manifold of all motions (i.e., isometries of determinant $+1$) of $\mathbb{R}^3$. Their dimensions were $1, 1, 1, 2, 1, 2$, resp. (in the above order). The motions of the third kind formed a manifold of two connected components, cf. [13], p. 141.

These manifolds exhibit certain bifurcation phenomena. These have been analyzed in [12]. Moreover, [10] and [12] proved that the above described motions are the only motions admitted by Tompos's tetrahedra. Further, the trajectories of the vertices during the physically admissible motions were described by [10]. Moreover, [4, 5, 6] (of which only [6] has been available to the authors) and [1] further investigated the motions admitted by Tompos's tetrahedra. Some possible mechanical engineering applications were pointed out by them.

The above investigations are generalized in our paper. Rather than with a cube, we begin with a general rectangular parallelepiped. Let us draw all the diagonals of all of its faces. Thus we get the edges of two congruent tetrahedra. We call this position of these two tetrahedra their *basic position*.

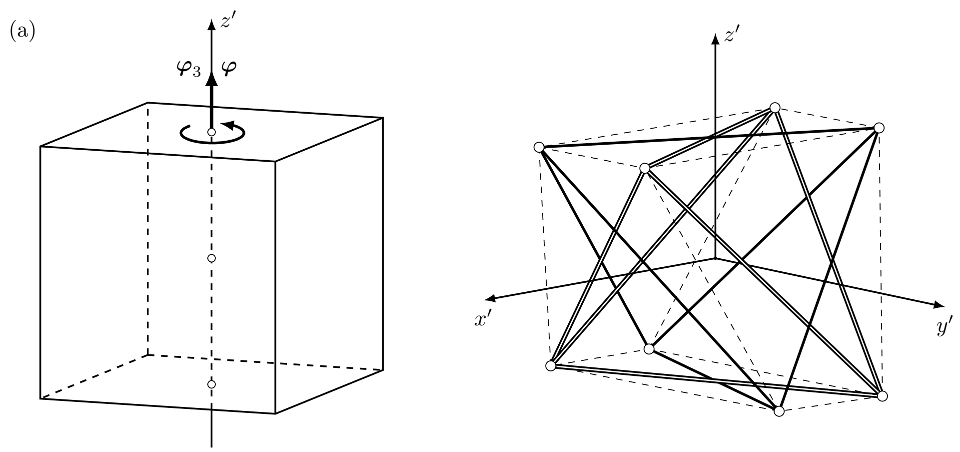

Figure 4a. The motion of the first kind.

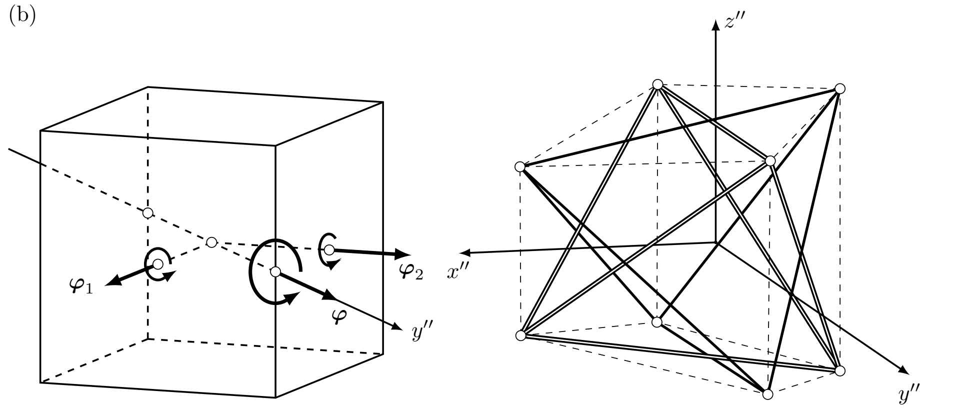

Figure 4b. The motion of the second kind.

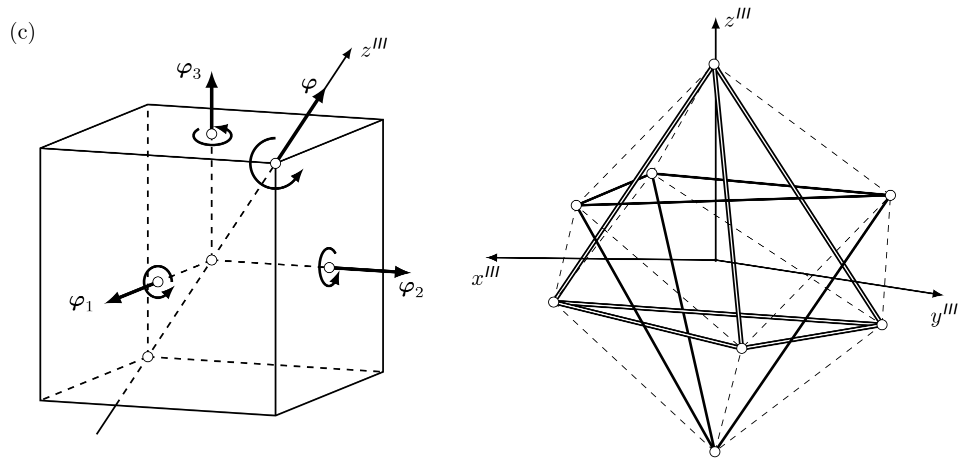

Figure 4c. The motion of the third kind.

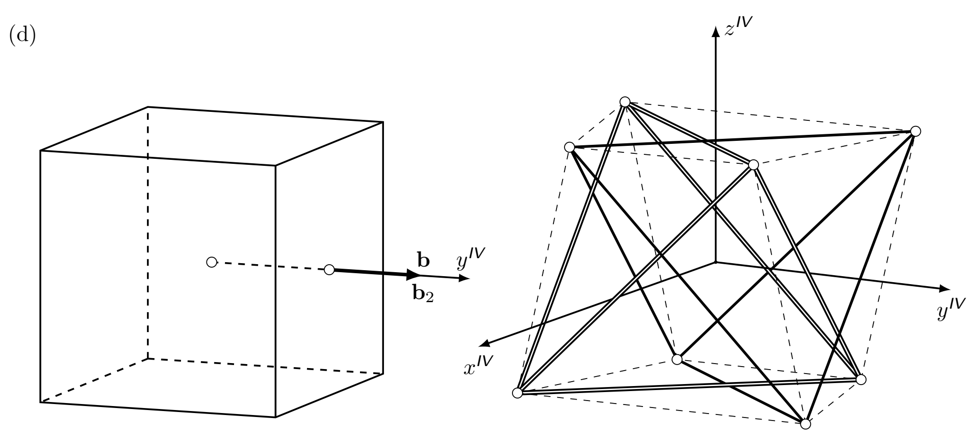

Figure 4d. The motion of the fourth kind.

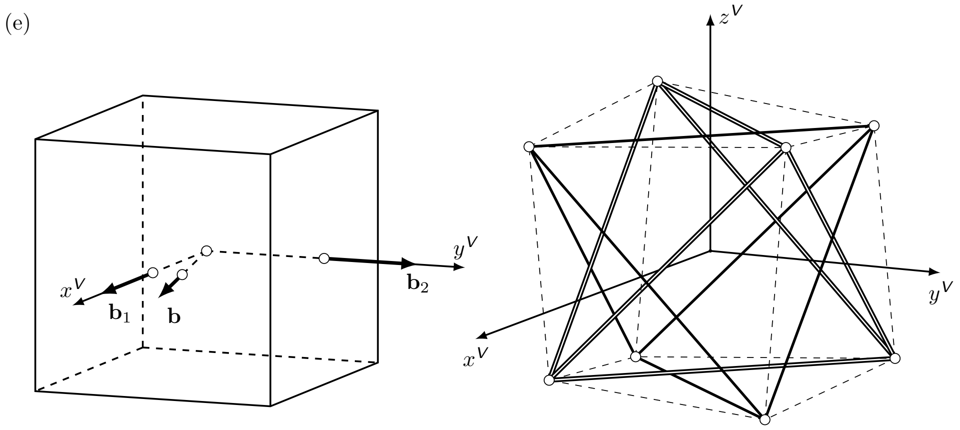

Figure 4e. The motion of the fifth kind.

> One of the tetrahedra is kept fixed. The other one is moved (i.e., we apply to it an isometry of the space, of determinant +1) under the following restriction. Each pair of edges of the two tetrahedra, originally having been diagonals of some face of the rectangular parallelepiped, has to remain coplanar. (1)

First all such motions will be described. Then we will give, in *Theorem 1,* a mathematical proof of the completeness of this list of motions. Our results are rather analogous to those in the case of the cube. More exactly, the motions for the case when the rectangular parallelepiped is a cube — which were described in [12], Theorem 1 — all have their analogues for the case of the rectangular parallelepiped. Even, the existence of most of them can be proved by analogous geometrical considerations, as in the case of the cube. There is one exception. The motion of the third kind, for the case of the cube, is given in the geometric way in [12], Theorem 1. However, in our paper we are able to describe it, for the case of the rectangular parallelepiped, only analytically (cf. (2)). Except for a special case, these are also the only admitted motions for the case of the rectangular parallelepiped.

In the above mentioned special case, we also have a motion of the sixth kind, which forms a 1-manifold (one-dimensional manifold). Here we only sketch its construction (details cf. in §2). Suppose that the rectangular parallelepiped has vertices $(\pm d_1, \pm d_2, \pm d_3)$, where, e.g., $d_3 = d_1 d_2/(d_1^2 + d_2^2)^{1/2}$ (here the coordinates can still be permuted). Consider its rectangular section with the xy-coordinate plane, and choose one of its diagonals. Then rotate one of the tetrahedra about this (oriented) diagonal through an angle $\pi/4$, and the other tetrahedron through an angle $-\pi/4$ (cf. Fig. 5). Then some suitable unique vertical translations of the already rotated tetrahedra, through some vectors $(0, 0, c)$ and $(0, 0, -c)$, resp., will produce an admitted motion. Then we can translate one of the tetrahedra in the direction of the rotation axis, through an arbitrary distance, producing an admitted motion. This is called the motion of the sixth kind. This completes the list of all possible motions for the case of all rectangular parallelepipeds.

In the physical model, the bars (edges) of one of the tetrahedra touch those of the other tetrahedron from inside, as in Fig. 1. This hints to a generalization of our Theorem 1. In fact, the physical constraint is not intersection, or coplanarity of the respective edges, but only the following. The edges of one tetrahedron lie "inside" the edges of the other tetrahedron. (A mathematical definition for this will be given in (9).) We take this condition as the definition of the *generalized* *admitted motions* of the two tetrahedra. Our *Theorem 2* asserts that the generalized admitted motions, for the case of two tetrahedra derived from a rectangular parallelepiped, coincide with the admitted motions for this case. This generalizes [12], Theorem 2, which stated the same for the case of the cube.

Our question can be further generalized to the case of two tetrahedra derived from a general parallelepiped. For this see §2, **2.3** and §3, **3.1**. Here we cannot determine all admitted motions of the two tetrahedra. However, still we can show, in our *Theorem 3*, that also for the case of an arbitrary parallelepiped, the generalized admitted motions of the two tetrahedra derived from it (defined analogously as above) are identical with the motions admitted by them.

We still analyze the bifurcation properties of the solution manifolds. Each of Theorems 1, 2 and 3 of this paper has been announced in [8].

Second, in §3, we give three further generalizations of the pair of regular tetrahedra. These generalizations are the following. (1) A pair of tetrahedra derived from a general parallelepiped. (2) A pair of regular n-gonal pyramidal frames (only the edge lengths are fixed, and the bases are allowed to change their shape, and also to become non-planar; cf. Fig. 6). (3) A pair of regular tetrahedra with congruent circular arc edges, whose endpoints are the endpoints of the face-diagonals of some cube, and which have all symmetries of this cube. For these generalizations we will not be able to determine all finite motions, but we will be able to give certain finite motions.

In case (1) these examples are partly analogues of some motions for the case of the rectangular parallelepiped. Partly experiments show probable existence of some new motions. In case (2), for $n \geq 4$, experiments produce only one one-parameter family of finite motions, where the two pyramids move as rigid bodies, and which is the analogue of the motion of the third kind for the case of the cube. (The case $n = 3$ is covered by case (1).) We will present some numerical evidence that, for $n \geq 4$, all continuous finite motions from the basic position, always satisfying the constraints, might form this family of motions. In case (3) we have loose analogues of the motions for the case of the cube, from [12], Theorem 1.

Some of these results, namely concerning cases (1) and (2), also have been announced in [8]. Related results on pairs of polyhedra or polyhedral frames moving with sliding constraints cf. in [7] and [9].

2. The motions of the two tetrahedra obtained from a rectangular parallelepiped

2.1. Analogous notations will be used as for the case when we have the cube. The vertices of the rectangular parallelepiped will be denoted by $(\pm d_1, \pm d_2, \pm d_3)$. Here $d_1, d_2, d_3 > 0$ are constant. Now, $P_1(d_1, -d_2, -d_3)$, $P_2(-d_1, d_2, -d_3)$, $P_3(-d_1, -d_2, d_3)$ and $P_4(d_1, d_2, d_3)$ are the fixed vertices. Similarly, Q_1, Q_2, Q_3 and Q_4 are the moving vertices. Here, in the basic position, the vertex Q_i is the centrally symmetric image of the vertex P_i w.r.t. the origin. Thus the fixed tetrahedron is $P_1P_2P_3P_4$, and the moving tetrahedron is $Q_1Q_2Q_3Q_4$. The moving tetrahedron is moved under restriction (1), which is the identical as for the case of the cube. First the motions $\mathbf{\Phi}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ of this moving tetrahedron will be described.

First we define the motions of the fourth and fifth kinds exactly as for the case of the cube. They exist evidently (i.e., the originally intersecting edges in fact remain coplanar).

Further, we define the motion of the first kind exactly as for the case of the cube. For any value of the rotation angle this exists. Moreover, $\mathbf{b} = \mathbf{0}$ is satisfied for this motion. These can be shown analogously as in [12], p. 428. Namely, let the axis of rotation be, e.g., $\mathbf{0e}_3$. Then the edges of the moving tetrahedron, originally lying on some horizontal face of the rectangular parallelepiped, contain $\pm d_3\mathbf{e}_3$. Hence they remain coplanar. Now consider the straight lines spanned by the edges of the moving tetrahedron, originally lying on some two opposite vertical faces of the rectangular parallelepiped. Their rotated copies, through all angles, constitute a ruling of a (doubly) ruled surface, a one-sheet hyperboloid of revolution. The other diagonals of these two faces belong to the other ruling of this hyperboloid. However, any two lines from different rulings of this hyperboloid are coplanar. Hence this motion exists for all φ.

We define the motion of the intermediate kind also exactly as for the case of the cube. (Now it is not necessary to treat separately any analogue of the motion of second kind.) For any value of the rotation angle, different from π, there exists the motion of the intermediate kind. (But sometimes it may exist also for rotation angle π, cf. below.) We apply the analogues of the arguments in [12], p. 429. Let the axis of rotation be $\mathbf{0u}$, where $\mathbf{u} = C_1\mathbf{e}_1 + C_2\mathbf{e}_2$ has length 1.

Following [12], rather than taking one of the tetrahedra as fixed, and the other tetrahedron as moving, we make the following. Both tetrahedra are rotated about the rotation axis, through angles $\pm\varphi/2$, in such a way that their symmetry w.r.t. the xy-plane is preserved (thus they are rotated in opposite senses). In this way the edges of our two tetrahedra, originally lying on one vertical face of the rectangular parallelepiped, will remain symmetric w.r.t. the horizontal coordinate plane. Hence they remain coplanar. This symmetry property, hence also the coplanarity property, continue to hold if both of these tetrahedra are still translated vertically, in a way symmetric w.r.t. the horizontal coordinate plane.

Consider the edges of the already rotated (but not yet translated) two tetrahedra, originally lying on one horizontal face of the rectangular parallelepiped. They may have projections to the xy-plane, spanning properly intersecting lines. In this case, some unique symmetric (w.r.t. the xy-plane) vertical translations of the already rotated tetrahedra will make the lines spanned by these two already rotated, and translated edges intersecting. Moreover, this happens simultaneously for the pairs of edges, originally lying on both horizontal faces.

Suppose $-\pi \le \varphi \le \pi$.

First we investigate the case $-\pi < \varphi < \pi$. We are going to show that in this case we have the following. The edges of the already rotated (but not yet translated) two tetrahedra, originally lying on one horizontal face of the rectangular parallelepiped, satisfy the following. They have projections to the xy-plane, which span properly intersecting lines.

The lines spanned by the diagonals of the upper horizontal face of the rectangular parallelepiped divide the plane spanned by this face to four angular domains. We translate these angular domains to the xy-plane, so that their vertices are translated to $\mathbf{0}$. We may suppose that $\mathbf{u}$ lies, e.g., in the closed angular domain $(Q_2-Q_1)\mathbf{0}(P_4-P_3)$ (in the unrotated position). We investigate the projections of the rotated diagonals Q_1Q_2 and P_3P_4 to the xy-plane. Let us replace these rotated diagonals by their translated copies $\mathbf{0}(Q_2 - Q_1)$ and $\mathbf{0}(P_4 - P_3)$. Then the projections of these translated copies of the rotated diagonals Q_1Q_2 and P_3P_4, to the xy-plane, are translates of the projections of the rotated diagonals Q_1Q_2 and P_3P_4, to the

xy-plane.

Now we investigate the respective rotations of the segments $\mathbf{0}(Q_2 - Q_1)$ and $\mathbf{0}(P_4 - P_3)$, about the axis $\mathbf{0u}$. Their points $\mathbf{0}$ remain fixed by these rotations. Moreover, these segments move on directrices of two semi-infinite circular cones, with apex $\mathbf{0}$, and rotation axis $\mathbb{R}\mathbf{u}$, and with sum of semiapertures less than π. (One of these cones, but not both, may degenerate to a half-line. This occurs exactly if $\mathbf{u}$ lies on a side of the angular domain $(Q_2 - Q_1)\mathbf{0}(P_4 - P_3)$, in the unrotated position.) This rotation axis $\mathbb{R}\mathbf{u}$ divides the xy-plane to two half-planes. Then the projections of the respective rotated copies of the segments $\mathbf{0}(Q_2 - Q_1)$ and $\mathbf{0}(P_4 - P_3)$ to the xy-plane satisfy the following. They lie, except for $\mathbf{0}$, in one open half-plane, and in the complementary closed half-plane, bounded by this rotation axis, resp. Hence, by the semiaperture condition, the lines spanned by them in fact properly intersect, as stated. Therefore this motion exists for all $\varphi \in (-\pi, \pi)$.

There remained to investigate the case $\varphi = \pm\pi$. Then the edges of the two tetrahedra, originally lying on a horizontal face of the rectangular parallelepiped, are rotated to positions lying on two parallel vertical planes. These planes have as distance the height $2d_3$ of our rectangular parallelepiped. So no vertical translations can make these rotated edges of the two tetrahedra intersecting. However, these vertical translates still can be coplanar, namely if they are parallel. To check their possible parallelity, it is sufficient to replace them by their already rotated and translated copies $\mathbf{0}(Q_2 - Q_1)$ and $\mathbf{0}(P_4 - P_3)$, and check their parallelity. Since both of them contain $\mathbf{0}$, therefore their parallelity means their coincidence. They lie in a vertical plane containing $\mathbb{R}\mathbf{u}$. They coincide exactly if $\mathbf{u}$ lies on the bisector of the angular domain $(Q_2 - Q_1)\mathbf{0}(P_4 - P_3)$, in the unrotated position. We have the analogous statements for any of the above four angular domains. Since the horizontal face of the rectangular parallelepiped is a rectangle, this means that $\mathbf{u}$ (of length 1) equals $\pm\mathbf{e}_1$ or $\pm\mathbf{e}_2$. Thus we have a motion of the first kind, which exists for all φ, so, in particular, for $\varphi = \pi$.

For a motion of the intermediate kind (in particular, for a motion of the first kind, with $\varphi \neq \pi$), for any given rotation $\mathbf{A}$, the translation $\mathbf{b}$ is uniquely determined. This is also shown in the same way as in [12], p. 429 (or cf. above, at the "properly intersecting lines"). However, possibly for such an $\mathbf{A}$ there exist several other $\mathbf{b}$'s yielding a motion of another kind, see the next paragraph.

We continue with describing the novel motion of the sixth kind. Let us suppose that our rectangular parallelepiped satisfies $d_k = d_i d_j/(d_i^2 + d_j^2)^{1/2}$, where i, j, k are different. Here d_k is half the length of the altitude belonging to the hypotenuse of the right triangle bounded by two sides and a diagonal of a face perpendicular to $\mathbf{e}_k$. Let, e.g., $d_3 = d_1 d_2/(d_1^2 + d_2^2)^{1/2}$. Let us take an axis of rotation passing through $\mathbf{0}$ and parallel to one of the diagonals of a horizontal face, say, to P_1P_2, with axis vector $\mathbf{0}(P_2 - P_1)$. Let us consider a rotation about this axis through the angle $\pm\pi/2$.

For convenience, rather than taking one tetrahedron as fixed, the other one as moving, we argue as at the motion of the intermediate kind. We rotate both tetrahedra about this axis, through an angle $\pm\pi/4$, in a way symmetric w.r.t. the xy-plane. Then we translate vertically both of these tetrahedra, in a way symmetric w.r.t. the xy-plane, through a suitable distance. Thus we achieve that all the pairs of the edges of the tetrahedra, which originally lay on the same face of the parallelepiped, will be, simultaneously, coplanar. Thus we obtain a position corresponding to a motion of the intermediate kind. This is already known to exist, even for all $\varphi \neq \pi$, so, in particular, for $\varphi = \pm\pi/2$. Cf. Fig. 5, where $P_1'P_2'P_3'P_4'$ and $Q_1'Q_2'Q_3'Q_4'$ denote the rotated, but not yet translated tetrahedra. **FIG. 5 ABOUT HERE.** Here P_i' and Q_i' are the rotated copies of P_i and Q_i, resp. It is supposed that the sense of rotation is such as drawn in Fig. 5 (the other case is analogous, only we have to change the role of the indices).

In the figure the orthogonal projection of the rectangular parallelepiped along the axis of rotation is shown. Since $d_3 = d_1 d_2/(d_1^2 + d_2^2)^{1/2}$, we obtain the following. This projection is a rectangle, with horizontal side twice as long as its vertical side. Observe that because of this the edges $P_1'P_3'$ and $P_2'P_3'$ together with their mirror images $Q_2'Q_4'$ and $Q_1'Q_4'$ w.r.t. the xy-plane are horizontal. Hence $P_1'P_3'$ and $Q_2'Q_4'$ are parallel, and similarly, $P_2'P_3'$ and $Q_1'Q_4'$ are parallel. Therefore, any of their translated copies are coplanar, as well, resp. Moreover, the edges $P_1'P_4'$ and $P_2'P_4'$, together with their mirror images $Q_2'Q_3'$ and $Q_1'Q_3'$ w.r.t. the xy-plane lie in a vertical plane (the projection of this vertical plane is a vertical line in the figure). Hence they will remain coplanar after any translation in this vertical plane. Last, the edge $P_1'P_2'$ (the projection of this edge in the figure is a point) and the edge $Q_3'Q_4'$ will intersect, therefore will be coplanar, if we translate vertically the two tetrahedra, in a way symmetric w.r.t. the xy-plane, through a suitable (unique) distance. In this case, by reason of symmetry, also the edges $Q_2'Q_1'$ and $P_4'P_3'$ will be intersecting. Therefore, by any subsequent translation of the moving tetrahedron in the direction of the edge $P_1'P_2'$ (= the direction of the edge $Q_2'Q_1'$ = the direction of the axis of rotation), both of these last two pairs of edges remain coplanar.

Recapitulating: any of the pairs of originally intersecting edges retains its coplanarity if we make the following operations. First we make two rotations — sym-

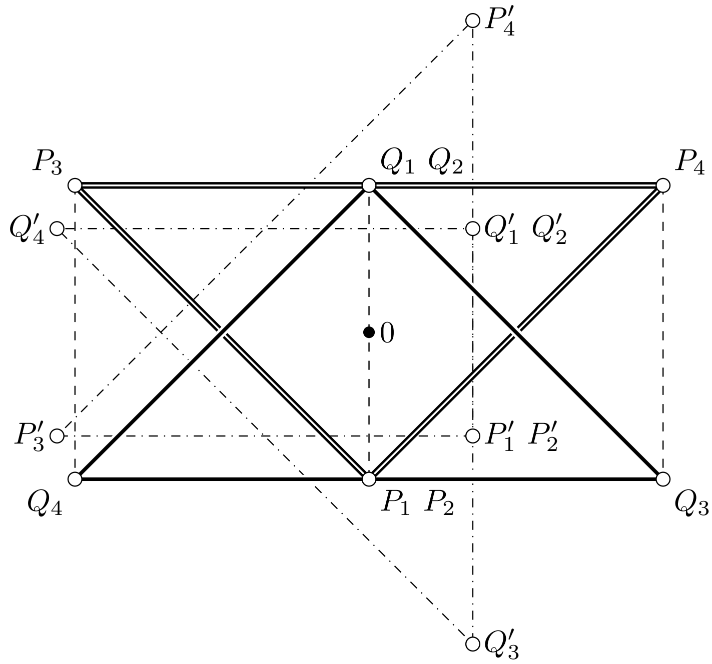

Figure 5. The motion of sixth kind.

metric w.r.t. the xy-plane — about the rotation axis described above, through angles $\pm\pi/4$. Second we make two (unique) vertical translations — symmetric w.r.t. the xy-plane — such that the edges $P_1'P_2'$ and $Q_3'Q_4'$ become intersecting. Third, the moving tetrahedron is translated, in the direction of the axis of rotation, through any distance. Hence, this is an affine 1-manifold of solutions for **b**. (An *affine manifold* is a translate of a linear subspace, or the empty set.) This will be called, for any permutation (ijk) (for which $d_k = d_i d_j/(d_i^2 + d_j^2)^{1/2}$ is satisfied) a *motion of the sixth kind*. (Observe that then $d_k < d_i, d_j$, hence if this condition holds, then it can hold only for one k.)

Last we describe the motion of the third kind. We can present it only in an analytic form. Let $\mathbf{u} := [u_1 \;\; u_2 \;\; u_3]^T$ and $u_1^2 + u_2^2 + u_3^2 = 1$. We write $D_i := d_i^{-2}$. Let $\mathbf{\Phi}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ be a motion. That is, $\mathbf{A}$ is a rotation, with axis of rotation some $\mathbf{0u}$, through some angle φ, and $\mathbf{b} \in \mathbb{R}^3$. Observe that $(\mathbf{u}, \varphi)$ and $(-\mathbf{u}, -\varphi)$ give the same rotation. Let $0 < \varphi < 2\pi$, and $\varphi \neq \pi/2, 3\pi/2$. Moreover, we write $s := \cot(\varphi/2)$ $(\neq \pm 1)$. In this case, for $i = 1, 2, 3$, let

$$\begin{aligned} u_i^2 &:= \frac{(s^4+3s^2+1) - (3s^4+7s^2+1)D_i/(D_1+D_2+D_3)}{2(s^2+1)}, \\ \text{if} \quad &\frac{s^4+3s^2+1}{3s^4+7s^2+1} > \frac{\max D_i}{D_1+D_2+D_3} \end{aligned} \tag{2}$$

(in this case, for each i there holds $u_i^2 > 0$). Thus the rotation part $\mathbf{A}$ of the motion $\mathbf{\Phi}(\mathbf{x})$ is determined. Then, among all admitted motions, the translation part $\mathbf{b}$ of this motion is uniquely determined. For given signs of the u_i's, this gives a 1-manifold of solutions. (But note that actually $\pm(\mathbf{u}, \varphi)$ give the same rotation.)

We exclude the case $D_1 = D_2 = D_3$, since it has been settled in [10] and [12]. First, our attention will be restricted to the case when $u_1, u_2, u_3 > 0$. In this case, if $\max D_i/(D_1+D_2+D_3) \geq 5/11$, then this solution manifold is connected. On the other hand, if $\max D_i/(D_1+D_2+D_3) < 5/11$, then this solution manifold has three connected components. Namely, one component for $s < -1$ (with s bounded), one component for $-1 < s < 1$, and one component for $s > 1$ (with s bounded).

Suppose $D_i \geq D_j, D_k$ (thus earlier $\max D_i$ will be replaced by D_i). Here and later till Theorem 1, (ijk) is a permutation of $\{1, 2, 3\}$. If $D_i/(D_1+D_2+D_3) \geq 5/11$, then this solution manifold ends (at the infimum or supremum of s) at two points which satisfy $u_i = 0$ and $|s| \in (0, 1]$ (with $|s| = 1$ exactly for $D_i/(D_1 + D_2 + D_3) = 5/11$). If $D_i/(D_1 + D_2 + D_3) < 5/11$, then the component for $s < -1$ begins (at the infimum of s) and the component for $s > 1$ ends (at the supremum of s) at two points which satisfy $u_i = 0$ and $|s| > 1$. Further, we have no endpoints at $s = \pm 1$, namely there the components of the manifold go to infinity. For any of the above two cases, these two endpoints are not considered as lying on this solution manifold. They correspond to two motions of the intermediate kind.

If $D_i > D_j, D_k$, then the above mentioned two endpoints (for any of the above two cases) do not correspond to motions of the first kind. If $D_i = D_j > D_k$, then the above mentioned two endpoints (for any of the above two cases) correspond to motions of the first kind.

This motion of the third kind never passes through the basic position, unless $D_1 = D_2 = D_3$, when it does (for $s^2 = \infty$).

These formulas in fact describe a motion of our tetrahedra. This fact, and the mentioned properties of this motion will follow from the proof of Theorem 1.

For $\min u_i^2 > 0$ this 1-manifold of motions of the third kind is smooth. However, unless $D_1 = D_2 = D_3$, this 1-manifold, parametrized by φ, or by s, is in general only topologically a manifold with boundary (i.e., its above two endpoints). Namely, at its endpoints it is in general not differentiable. For $D_i \geq D_j, D_k$, and for s yielding $u_i = 0$, we have $du_i/ds = (du_i^2/ds)/(2u_i) = \pm\infty$, provided that $du_i^2/ds \neq 0$. Here

du_i^2/ds exists, and is finite. E.g., for $D_i = D_j = 5/11$ and $D_3 = 1/11$, an easy calculation shows that for s yielding $u_i = 0$ we have $du_i^2/ds \neq 0$. Therefore in general, for this s, we have $du_i^2/ds \neq 0$. Hence this 1-manifold with boundary is, in general, not differentiable at its endpoints.

Even for $D_i = D_j > D_k$, in general, for s yielding $u_i = u_j = 0$, we have $du_i^2/ds = du_j^2/ds \neq 0$. This is shown by the same example. Similarly, for $D_i \geq D_j, D_k$, we can specialize to $D_i/(D_1+D_2+D_3) = 5/11$, or to $D_i/(D_1+D_2+D_3) = D_j/(D_1+D_2+D_3) = 5/11$, and the same example works. Then we see that also in these two special cases, in general, for s yielding $u_i = 0$, we have $du_i^2/ds \neq 0$. So, even in each of these three special cases, we have in general non-differentiability of this 1-manifold with boundary, at its endpoints.

Now we allow any signs of the u_i's. The number of 1-manifolds of motions of the third kind is $2^3/2 = 4$. Namely the number of sign combinations of the u_i's, when they are non-zero, is 2^3, but $\pm(\mathbf{u}, \varphi)$ give the same rotation. Then for $D_i > D_j, D_k$ these four 1-manifolds by twos have both endpoints in common. However, for different twos these both endpoints are disjoint pairs. These endpoints correspond to motions of the intermediate kind (but not of the first kind). For $D_i = D_j > D_k$ all four of these 1-manifolds have both endpoints in common. These endpoints correspond to motions of the first kind.

At both endpoints of this motion (at the infimum or supremum of s) we have bifurcations. First suppose $D_i > D_j, D_k$. Then at an endpoint the solution set locally consists of two 1-manifolds with boundary (this endpoint), of motions of the third kind. Moreover, of one 2-manifold, of motions of the intermediate kind. Second suppose $D_i = D_j > D_k$. Then at an endpoint the solution set locally consists of all four 1-manifolds with boundary (this endpoint), of the third kind. Moreover, of one 1-manifold of the motions of the first kind, and of two 2-manifolds of the motions of the intermediate kind.

Observe that the motions of the fifth and sixth kinds exist only for $\varphi = \pi$, i.e., $s = 0$, and for $\varphi = \pi/2, 3\pi/2$, i.e., $s = \pm 1$, resp. However, $s = 0$ does not yield an endpoint of the 1-manifold of motions of the third kind. Hence the motion of the fifth kind cannot occur locally at the endpoints of the 1-manifold of motions of the third kind. Further, $s = \pm 1$ yield endpoints of the 1-manifold of motions of the third kind if and only if $\max D_i/(D_1 + D_2 + D_3) = 5/11$. The motion of the sixth kind exists only for $D_i = D_j + D_k$ (where (ijk) is a permutation of $\{1, 2, 3\}$; cf. **(7)** of the proof of Theorem 1). If an endpoint of the 1-manifold of motions of the third kind, and a motion of the sixth kind coincided, then we had $1/2 = D_i/(D_i + D_j + D_k) = 5/11$, a contradiction. Hence the motion of the sixth kind cannot occur locally at the endpoints of the 1-manifold of the motions of the third kind.

Also at each *counterbasic position*, i.e., a motion of the first kind with rotation angle $\varphi = \pi$ (i.e., $s = 0$), we have bifurcations. We may suppose $\mathbf{u} = (0, 0, 1)$. At a counterbasic position there locally occur one 1-manifold of the motions of the first kind, two 2-manifolds of the motions of the intermediate kind, two 1-manifolds of the motions of the fourth kind, and one 2-manifold of the motions of the fifth kind. However, manifolds of the motions of the third kind cannot occur at a counterbasic position. Namely, for $s = 0$, (2) simplifies to $u_i^2 = (1 - D_i/(D_1 + D_2 + D_3))/2$, for $i = 1, 2, 3$. Then $0 = u_1^2 = (1 - D_1/(D_1 + D_2 + D_3))/2 > 0$, a contradiction. Also manifolds of the motions of the sixth kind cannot occur at a counterbasic position, because for them we have $\varphi = \pm\pi/2$.

We recall that we mean by a motion an isometry (congruence) of the space, of determinant $+1$.

Theorem 1. *Consider the two tetrahedra $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$, derived above from the rectangular parallelepiped of vertices $(\pm d_1, \pm d_2, \pm d_3)$. The only finite motions admitted by these tetrahedra — i.e., all positions of the moving tetrahedron, satisfying* (1) *— are the following: The motions of the first, intermediate, third, fifth kinds and, provided $d_k = d_i d_j/(d_i^2 + d_j^2)^{1/2}$ for some permutation (ijk) of $\{1, 2, 3\}$, of the sixth kind, described above.*

Proof. **(1)** Let $\mathbf{\Phi}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ be a finite motion admitted by our tetrahedra. I.e., $\mathbf{A} = [a_{ij}]$ is an orthogonal 3×3 matrix of determinant $+1$, and $\mathbf{b} = [b_1 \quad b_2 \quad b_3]^T$ is a vector in $\mathbb{R}^3$, and condition (1) is satisfied. Here $\mathbf{A}$ is a rotation about some axis $\mathbf{0u}$, where $\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T$ and $u_1^2 + u_2^2 + u_3^2 = 1$, through an angle φ (with sense of rotation as described in §1).

For the vertices $Q_i(x_i, y_i, z_i)$ of the moving tetrahedron we have, using the coordinates of the basic position of the Q_i's from the beginning of **2.1**,

$$\begin{aligned}
Q_1(x_1, y_1, z_1) &= \mathbf{A}[-d_1 \quad d_2 \quad d_3]^T + \mathbf{b},\\
Q_2(x_2, y_2, z_2) &= \mathbf{A}[d_1 \quad -d_2 \quad d_3]^T + \mathbf{b},\\
Q_3(x_3, y_3, z_3) &= \mathbf{A}[d_1 \quad d_2 \quad -d_3]^T + \mathbf{b},\\
Q_4(x_4, y_4, z_4) &= \mathbf{A}[-d_1 \quad -d_2 \quad -d_3]^T + \mathbf{b}.
\end{aligned}$$

Let us denote

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}.$$

Using the notations P_i^0, Q_i^0 from §1 (e.g., Fig. 3), and P_i from the beginning of **2.1**, we have $P_i = \mathbf{D}P_i^0$. Further we have $Q_i = \mathbf{\Phi}(\overline{Q}_i)$, where $\overline{Q}_i$, or $\overline{Q}_i^0$, is the basic position of Q_i, or Q_i^0, resp. These satisfy $\overline{Q}_i = \mathbf{D}\overline{Q}_i^0$. The coplanarity, e.g., of the fixed vertices $P_1 = \mathbf{D}P_1^0$ and $P_2 = \mathbf{D}P_2^0$, and of the moving vertices $Q_3 = \mathbf{\Phi}(\overline{Q}_3) = \mathbf{\Phi}\big(\mathbf{D}(\overline{Q}_3^0)\big)$ and $Q_4 = \mathbf{\Phi}(\overline{Q}_4) = \mathbf{\Phi}\big(\mathbf{D}(\overline{Q}_4^0)\big)$, is equivalent to the following. The points P_1^0, P_2^0, and $\mathbf{D}^{-1}\big(\mathbf{\Phi}(\mathbf{D}(\overline{Q}_3^0))\big) = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}(\overline{Q}_3^0) + \mathbf{D}^{-1}\mathbf{b}$ and $\mathbf{D}^{-1}\big(\mathbf{\Phi}(\mathbf{D}(\overline{Q}_4^0))\big) = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}(\overline{Q}_4^0) + \mathbf{D}^{-1}\mathbf{b}$ are coplanar. We have analogous equivalent conditions for the coplanarity of the other quadruples of vertices to be considered.

So $\mathbf{\Phi}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$ represents a motion of our tetrahedra if and only if the following holds. The transformation $\mathbf{D}^{-1}\mathbf{A}\mathbf{D}(\mathbf{x}) + \mathbf{D}^{-1}\mathbf{b}$ of the vertices $Q_1^0, \ldots, Q_4^0$ preserves coplanarity of the four vertices of any face of the cube with vertices $(\pm 1, \pm 1, \pm 1)$. Here two of these vertices, namely the vertices P_i^0, are fixed, and only two of them, namely the vertices Q_i^0, are transformed by this transformation. We write $[a_{ij}^0] := \mathbf{D}^{-1}\mathbf{A}\mathbf{D} = [d_i^{-1}a_{ij}d_j]$, and $[b_1^0 \quad b_2^0 \quad b_3^0]^T := \mathbf{D}^{-1}\mathbf{b} = [d_1^{-1}b_1 \quad d_2^{-1}b_2 \quad d_3^{-1}b_3]^T$. Then, like in [12], p. 435, by these coplanarities we have

$$-(a_{22}^0 + a_{33}^0)b_1^0 + a_{12}^0 b_2^0 + a_{13}^0 b_3^0 = a_{21}^0 a_{13}^0 + a_{31}^0 a_{12}^0 + (a_{23}^0 + a_{32}^0)(1 - a_{11}^0), \tag{I/1}$$

$$-(a_{23}^0 + a_{32}^0)b_1^0 + a_{13}^0 b_2^0 + a_{12}^0 b_3^0 = a_{12}^0 a_{21}^0 + a_{31}^0 a_{13}^0 + (a_{22}^0 + a_{33}^0)(1 - a_{11}^0). \tag{II/1}$$

There hold the analogous equations obtained from these ones by the permutation of the indices $1 \mapsto 2 \mapsto 3 \mapsto 1$. These equations will be denoted by (I/2) and (II/2), resp. Similarly, using the permutation $1 \mapsto 3 \mapsto 2 \mapsto 1$ we get equations (I/3) and (II/3). Thus we obtain a system of six linear equations for b_1, b_2 and b_3. This system of equations expresses the coplanarity of the respective six quadruples from the fixed points P_i and the moved points Q_i, cf. [12], p. 435.

(2) In these six equations (I/i), (II/i), where $i = 1,2,3$, we replace a^0_{ij} by $d_i^{-1} a_{ij} d_j$, and b^0_i by $d_i^{-1} b_i$. Then we express a_{ij} by u_1, u_2, u_3 and φ, by the well-known formula $\mathbf{Ax} = \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u} + \cos\varphi \cdot (\mathbf{x} - \langle \mathbf{x}, \mathbf{u} \rangle \mathbf{u}) + \sin\varphi \cdot (\mathbf{x} \times \mathbf{u})$ (cf., e.g., [12], p. 436).

Then let us consider the 6×4 matrix $\mathbf{B}$ formed by the coefficients of b_1, b_2, b_3, and the right-hand sides of these six equations, rewritten as indicated just above. (Its entries depend on $\mathbf{u}$, φ, and the constants d_i.)

We call its rows I/i and II/i, according to the equation they correspond to. We will not write out $\mathbf{B}$ explicitly, but rather we will make some simplifications in it. In matrix $\mathbf{B}$ we multiply rows I/i, II/i by d_i, then divide the fourth column by $2d_1d_2d_3$, and then divide row II/i by $d_1d_2d_3d_i^{-1}$. Like in **2.1**, before Theorem 1, denote $D_i := d_i^{-2}$. Note that for $\varphi = 0$, thus $\mathbf{A} = \mathbf{I}$, we have the unique motion with $\mathbf{b} = \mathbf{0}$. Henceforward we will assume $0 < \varphi < 2\pi$. Letting $s := \cot(\varphi/2)$, as in **2.1**, before Theorem 1, we still multiply each entry of the matrix obtained last time from $\mathbf{B}$ by $(s^2+1)/2$. Thus we obtain the matrix

$$\begin{bmatrix} u_1^2 - s^2 & u_1u_2 + su_3 & u_3u_1 - su_2 & u_2u_3(D_2+D_3) \\ u_1u_2 - su_3 & u_2^2 - s^2 & u_2u_3 + su_1 & u_3u_1(D_3+D_1) \\ u_3u_1 + su_2 & u_2u_3 - su_1 & u_3^2 - s^2 & u_1u_2(D_1+D_2) \\ \begin{matrix} -u_2u_3(D_2+D_3)+ \\ +su_1(D_3-D_2) \end{matrix} & (u_3u_1 - su_2)D_2 & (u_1u_2 + su_3)D_3 & 0 \\ (u_2u_3 + su_1)D_1 & \begin{matrix} -u_3u_1(D_3+D_1)+ \\ +su_2(D_1-D_3) \end{matrix} & (u_1u_2 - su_3)D_3 & 0 \\ (u_2u_3 - su_1)D_1 & (u_3u_1 + su_2)D_2 & \begin{matrix} -u_1u_2(D_1+D_2)+ \\ +su_3(D_2-D_1) \end{matrix} & 0 \end{bmatrix}. \tag{3}$$

(Note that for $D_1 = D_2 = D_3 = 1$ this reduces to (B) in [12], p. 437, up to a factor 1/2 in the fourth column.) The rows of matrix (3) corresponding to equations (I/i), (II/i) will be called rows I/i, II/i of (3). Later, unless stated otherwise, we will consider only rows I/i, II/i of matrix (3), and not of the original matrix $\mathbf{B}$. Now suppose that $\mathbf{u}$ and s are fixed. Then the solvability of the system of equations corresponding to this new matrix (3), for $\mathbf{b}$, is equivalent to the solvability of the system of equations corresponding to the original matrix $\mathbf{B}$ (thus of our original system of equations (I/i), (II/i), for $i = 1,2,3$), for $\mathbf{b}$. More exactly, $\mathbf{u}$, s, $\mathbf{b}$ is a solution of the system of equations corresponding to $\mathbf{B}$ if and only if $\mathbf{u}$, s, $\mathbf{b}/(2d_1d_2d_3)$ is a solution of the system of equations corresponding to (3). Hence, for $\mathbf{u}$ and s fixed, the dimensions of the solution manifolds, for $\mathbf{b}$, of the two systems of equations (if they are not empty) are the same. Moreover, they can be obtained from each other by multiplication with a non-zero constant. Later we will not be interested in formulas for $\mathbf{b}$, therefore, *unless stated otherwise, we will use the system of equations corresponding to* (3).

(3) The upper left 3×3 submatrix of (3) is independent of D_i, hence it is singular in the same case when it is singular for $D_1 = D_2 = D_3 = 1$. Observe that the left-hand side of equation (I/1) equals $d_1^{-1}[-(a_{22}+a_{33})b_1+a_{12}b_2+a_{13}b_3]$, and similarly for (I/2), (I/3). Hence the determinant of the considered 3×3 submatrix of (3) is a non-zero number times the determinant $\big|a_{ij}-\delta_{ij}(a_{11}+a_{22}+a_{33})\big|$. By [11], p. 270 or [12], p. 438, this determinant is 0 if and only if $\varphi = \pm\pi/2$ or $\varphi = \pi$. (We do not distinguish between angles differing by multiples of 2π.) Hence our equations (for matrix (3)) can have a non-unique solution for $\mathbf{b}$ only for $\varphi = \pm\pi/2$ and $\varphi = \pi$, i.e., for $s = \pm 1$ and $s = 0$.

The last three rows of matrix (3) are linearly dependent. Namely, multiplying row II/i by D_i, and then summing them, we obtain the zero row.

Multiplying row I/i of (3) by u_i, and then summing them, we obtain

$$\big[u_1(1-s^2) \quad u_2(1-s^2) \quad u_3(1-s^2) \quad u_1u_2u_3 2(D_1+D_2+D_3)\big].$$

The corresponding equation implies that for $s^2 = 1$, i.e., for $\varphi = \pm\pi/2$, for any solution of our equations (for matrix (3)) we have $u_1u_2u_3 = 0$.

(4) The determinant of the submatrix of (3) formed by its rows I/1, I/2, I/3, II/1 is a homogeneous eighth degree polynomial of u_1, u_2, u_3 and s, with coefficients polynomials of the D_i's. A straightforward but somewhat lengthy calculation gives that it equals

$$\begin{aligned}&su_1u_2u_3(D_1+D_2+D_3)\{(u_1^2+u_2^2+u_3^2)\times\\&\times[u_1^2(D_3-D_2)+u_2^2(-D_2-D_3)+u_3^2(D_2+D_3)]+\\&+s^2[u_1^2(3D_3-3D_2)+u_2^2(-3D_2+D_3)+u_3^2(-D_2+3D_3)]+s^4(D_3-D_2)\},\end{aligned} \tag{4}$$

which equals 0. Now suppose $su_1u_2u_3 \neq 0$. Then the factor of (4) in braces is 0. Moreover, two more analogous expressions are equal to 0, which are obtained from this expression by cyclic permutations of the indices. (These arise analogously from the determinants of the submatrices formed by rows I/1, I/2, I/3, II/2, and I/1, I/2, I/3, II/3 of (3), resp.) These three equations are homogeneous linear in D_2, D_3, in D_3, D_1, and in D_1, D_2, resp., and can be written as

$$\begin{aligned}&\frac{(u_1^2+u_2^2+u_3^2)(-u_1^2+u_2^2+u_3^2)+s^2(u_1^2+3u_2^2+3u_3^2)+s^4}{D_1}=\\&=\frac{(u_1^2+u_2^2+u_3^2)(u_1^2-u_2^2+u_3^2)+s^2(3u_1^2+u_2^2+3u_3^2)+s^4}{D_2}=\\&=\frac{(u_1^2+u_2^2+u_3^2)(u_1^2+u_2^2-u_3^2)+s^2(3u_1^2+3u_2^2+u_3^2)+s^4}{D_3}\end{aligned} \tag{5}$$

(thus they are actually only two equations).

(5) First we discuss the case when the first factor of (4), i.e., s, equals 0. Consider the 4×4 matrix formed by rows I/1, I/2, II/1, and the sum of (D_1+D_2) times row II/2 and D_3 times row II/3 of our matrix (3). Its determinant is

$$-u_1u_2u_3^2D_1D_3(D_1+D_2+D_3)\big[u_2^2(D_2+D_3)-u_1^2(D_3+D_1)\big](u_1^2+u_2^2+u_3^2),$$

which equals 0. By cyclic permutation of rows I/i and II/i we get two more similar equations. Namely the expressions, obtained from the last expression by cyclic permutations of the indices, are equal to 0.

These three equations together imply $u_1u_2u_3 = 0$, or $u_1^2 : u_2^2 : u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2)$.

First let us suppose, e.g., $u_1 = 0$. Then the equation corresponding to row I/1 becomes $0 = u_2u_3(D_2 + D_3)$, thus $u_2u_3 = 0$. Let, e.g., $u_1 = u_2 = 0$ (thus $\mathbf{u} = \pm\mathbf{e}_3$ by $u_1^2 + u_2^2 + u_3^2 = 1$). By $s = 0$ we have $\varphi = \pi$. Then the pairs of the edges of the two tetrahedra, originally lying on some vertical face of the rectangular parallelepiped, are parallel. Thus they remain coplanar after any translation of the moving tetrahedron. The pairs of the edges of the two tetrahedra, originally lying on some horizontal face of the rectangular parallelepiped, are properly intersecting and horizontal. Thus they remain coplanar after a translation of the moving tetrahedron, through a vector $[b_1 \quad b_2 \quad b_3]^T$, exactly when $b_3 = 0$. Therefore the set of solution vectors $[b_1 \quad b_2 \quad b_3]^T$ of our equations (for matrix (3)) is given by all vectors with $b_3 = 0$. *This gives the motion of the fifth kind.*

Second let us suppose $u_1^2 : u_2^2 : u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2)$. Then by $u_1^2 + u_2^2 + u_3^2 = 1$ we have $u_1u_2u_3 \neq 0$. Observe that in this case the double equality (5) is satisfied (for $s = 0$). We will investigate this case further together with the investigation of (5), in **(10)** (where both cases $s = 0$ and $s \neq 0$ will be allowed). Now we only show that in this case there is a unique solution $\mathbf{b}$ of our equations (for matrix (3)). (The existence of the solution for $\mathbf{b}$ will be showed in **(10)**.) For this consider the 4×4 matrix from the beginning of **(5)**. Then take its 3×3 submatrix consisting of the first three entries of its first, third and fourth rows. Its determinant is a homogeneous polynomial of third degree of the D_i's, with coefficients homogeneous sixth degree polynomials of the u_i's. To test if it is zero or not, it suffices to substitute $D_2 + D_3 := u_1^2$, and $D_3 + D_1 := u_2^2$, and $D_1 + D_2 := u_3^2$. Thus this determinant becomes

$$u_1^2u_2u_3(u_1^2 + u_2^2 + u_3^2)^2(u_1^2 + u_2^2 - u_3^2)(-u_1^2 + u_2^2 + u_3^2)/8.$$

Here $u_1^2u_2u_3 \neq 0$, as shown above, and each other factor is positive. E.g., $(u_1^2+u_2^2-u_3^2)/(u_1^2 + u_2^2 + u_3^2) = D_3/(D_1 + D_2 + D_3) > 0$. Hence the considered determinant is non-zero, *showing unicity of the solution for* $\mathbf{b}$ *(for matrix* (3)*).*

(6) Second we discuss the case when the factor $u_1u_2u_3$ of (4) equals 0. Let, e.g., $u_1 = 0$. In this case we have the motions of the first, intermediate, fifth and sixth kinds. We have to show for $u_2u_3 \neq 0$, that if the motion of the intermediate kind does not exist — that is $\varphi = \pi$, i.e., $s = 0$ — then we do not have any solution. (Recall the following. The motion of the intermediate kind exists only for $\varphi \neq \pi$. The motion of the fifth kind — for $u_1 = 0$ — exists only for $u_2u_3 = 0$. The motion of the sixth kind exists only for $\varphi = \pm\pi/2$.) However, in **(5)** it has been shown that $s = 0$ and $u_1 = 0$ imply $u_2u_3 = 0$.

There remains the question of the unicity of the translation part $\mathbf{b}$ of the motion. However, in **(3)** it has been shown that our equations (for matrix (3)) can have a non-unique solution for $\mathbf{b}$ only for $s = 0$ and $s = \pm 1$. The case $s = 0$ has been dealt with in **(5)**, and the question of unicity has been completely settled there. We will deal with $s = \pm 1$ in **(7)**.

(7) We turn to discuss unicity of $\mathbf{b}$ for $s = \pm 1$, i.e., $\varphi = \pm\pi/2$. Replacing $\mathbf{u}$ by $-\mathbf{u}$ if necessary, we may assume $s = 1$. In **(3)** it has been shown that $s^2 = 1$ implies $u_1u_2u_3 = 0$. Let, e.g., $u_3 = 0$. Then our matrix (3) becomes a function only of u_1, u_2 and D_1, D_2, D_3. Loosing homogeneity, we will use $u_1^2+u_2^2 = 1$. Thus we see that rows I/1 and I/2 are proportional, they are $-u_2$, and u_1 times $[u_2 \quad -u_1 \quad 1 \quad 0]$, resp. Since u_1, u_2 are not both zero, we may replace rows I/1 and I/2 by one row $[u_2 \quad -u_1 \quad 1 \quad 0]$. Row I/3 is retained. As mentioned in **(3)**, rows II/1, II/2 and II/3 are linearly dependent, with non-zero coefficients. Hence we may omit from among them row II/3. Thus we obtain a 4×4 matrix $\mathbf{M}$. The question of the dimension of the solution manifold, for $\mathbf{b}$, of the equations corresponding to $\mathbf{M}$, at the considered rotation part of the motion, is equivalent to the same question regarding matrix (3). By $u_3 = 0$ and $\varphi = \pm\pi/2 \neq \pi$ one solution always exists, namely a motion of the intermediate kind. Thus the dimension of this solution manifold is $3 - r$, where r is the rank of the matrix

$$\mathbf{N} = \begin{bmatrix} u_2 & -u_1 & 1 \\ u_2 & -u_1 & -1 \\ u_1(D_3 - D_2) & -u_2 D_2 & u_1 u_2 D_3 \\ u_1 D_1 & u_2(D_1 - D_3) & u_1 u_2 D_3 \end{bmatrix},$$

obtained by omitting the last column from matrix $\mathbf{M}$.

Subtracting the second row of $\mathbf{N}$ from the first one, the first row becomes $[0 \quad 0 \quad 2]$. Hence $r = 1 + r'$, where r' is the rank of the 3×2 submatrix $\mathbf{N}'$ of $\mathbf{N}$ at the lower left corner. If u_1 or u_2 is 0, then we have $r' = 2$, thus $r = 3$. Then we have a unique solution for $\mathbf{b}$ (for matrix $\mathbf{M}$). Now let $u_1 u_2 \neq 0$. The determinants of the 2×2 submatrices of $\mathbf{N}'$, obtained by omitting its first, second or third row, resp., are the following: $u_1 u_2 D_3(D_1 + D_2 - D_3)$, and $u_2^2(D_1 - D_3) + u_1^2 D_1$, and $-u_2^2 D_2 + u_1^2(D_3 - D_2)$. If any of these expressions is not 0, then we have $r' = 2$, thus $r = 3$. Then again there is a unique solution of our equations for $\mathbf{b}$ (for matrix $\mathbf{M}$). If all these above expressions are equal to 0, then we have (equivalently) $D_3 = D_1 + D_2$ and $u_1^2 D_1 = u_2^2 D_2$. Hence, by $u_3 = 0$, we have

$$u_1^2 = D_2/(D_1 + D_2) = d_1^2/(d_1^2 + d_2^2) \text{ and } u_2^2 = D_1/(D_1 + D_2) = d_2^2/(d_1^2 + d_2^2).$$

In this case $r' = 1$, thus $r = 2$, and then the dimension of the affine manifold of solutions, for $\mathbf{b}$ (for matrix $\mathbf{M}$, or matrix (3)), is 1 (for this rotation part of the motion).

It remains to show that geometrically this is the motion of the sixth kind. Since $D_i = d_i^{-2}$, therefore $D_3 = D_1 + D_2$ means $d_3 = d_1 d_2/(d_1^2 + d_2^2)^{1/2}$. We have from above $[u_1 \quad u_2 \quad u_3]^T = [\pm d_1/(d_1^2 + d_2^2)^{1/2} \quad \pm d_2/(d_1^2 + d_2^2)^{1/2} \quad 0]^T$ (the $\pm$ signs being independent). Hence the axis of rotation of the rotation part $\mathbf{A}$ of the motion is parallel to a diagonal of a horizontal face of our rectangular parallelepiped in its basic position. Further, the angle of rotation is $\pm\pi/2$. This is just the rotation part of the motion of the sixth kind (for $k = 3$, cf. the description of the motion of the sixth kind, in **2.1**, before Theorem 1). At describing the motion of the sixth kind, we have exhibited an affine 1-manifold A_1 of solutions for $\mathbf{b}$, with the above $\mathbf{A}$ (for matrix $\mathbf{B}$). This is therefore a subset of the entire solution manifold A_2, for $\mathbf{b}$, with this $\mathbf{A}$ (for matrix $\mathbf{B}$). Now we have shown that the entire affine manifold of solutions $\mathbf{b}$, with this $\mathbf{A}$ (for matrix (3)), is exactly 1-dimensional. Recall that the solution manifolds, for $\mathbf{b}$, for matrices $\mathbf{B}$ and (3), are obtained from each other by multiplication with a non-zero constant, cf. **(2)**. Hence also A_2 is an affine 1-manifold, containing the affine 1-manifold A_1. Hence $A_2 = A_1$, and this is the manifold of the motions of the sixth kind.

(8) Recall that a non-unique solution for $\mathbf{b}$ (for matrix (3)) is possible only for $s = 0, \pm 1$ (cf. **(3)**). These non-unique solutions have been discussed in **(5)** and **(7)**, resp.

For the existence of solutions we have derived in **(4)** the equation that (4) equals 0, and some of its consequences. The case when the first factor of (4), i.e., s, equals 0, has been settled in **(5)**, except the case when $u_1u_2u_3 \neq 0$ and (5) is satisfied. The case when the factor $u_1u_2u_3$ of (4) equals 0, has been settled in **(6)**, except the case of unicity at $s = \pm 1$. This in turn has been settled in **(7)**. If $su_1u_2u_3 \neq 0$, then we have derived in **(4)** equations (5).

Therefore all that remains is the following. We have to solve equations (5), where s can be 0 or any non-zero number, and $u_1u_2u_3 \neq 0$. Moreover, we have to verify if they are solutions of our equations (for matrix (3)). Recall that by **(3)** $s^2 = 1$ implies $u_1u_2u_3 = 0$, hence we will suppose $s \neq \pm 1$.

(9) *Now we show that for* $u_1u_2u_3 \neq 0$ *and* $s \neq \pm 1$ *any solution of equations* (5) *is a solution of our equations for matrix* (3). This is necessary since equations (5) are only consequences of the equations for matrix (3). They have not been gained from these equations by equivalent transformations. Moreover, we do not have a geometrical description of the motion of the third kind, making its existence evident.

First we show that, for $u_1u_2u_3 \neq 0$ and $s \neq 0, \pm 1$, any solution of equations (5) is a solution of our equations for matrix (3). In fact, equations (5) have been derived for $su_1u_2u_3 \neq 0$ from the equation that expression (4) equals 0, and from two other analogous equations. These express linear dependence of rows I/1, I/2, I/3 and II/i ($i = 1, 2, 3$) of matrix (3), resp. For $s \neq 0, \pm 1$ the determinant of the matrix formed by the first three entries of rows I/1, I/2 and I/3 is not 0, cf. **(3)**. Hence (5) expresses linear dependence of rows II/1, II/2 and II/3 on the linearly independent rows I/1, I/2 and I/3. (Observe that already their first three entries form linearly independent row vectors.) Hence (5) implies that the rank of (3) is at most 3, thus that the four column vectors of (3) are linearly dependent. However, *at this linear dependence the fourth column vector of* (3) *must have a non-zero coefficient*. Namely the first three column vectors of (3) are linearly independent. (Their first three entries already form linearly independent column vectors.) *This just means that our equations have a unique solution for* **b** *(for matrix* (3)*).*

Now let $u_1u_2u_3 \neq 0$ and $s = 0$. (Recall that $s = \pm 1$ has been excluded, cf. **(8)**.) We show that also now any solution of equations (5) is a solution of our equations for matrix (3). For $s = 0$ (5) gives $(-u_1^2 + u_2^2 + u_3^2)/D_1 = (u_1^2 - u_2^2 + u_3^2)/D_2 = (u_1^2 + u_2^2 - u_3^2)/D_3$. Let their common value be λ, say. Then $u_i^2 = (\lambda D_j + \lambda D_k)/2$, for i, j, k different (hence $\lambda \neq 0$ by $u_1^2 + u_2^2 + u_3^2 = 1$), thus $u_1^2 : u_2^2 : u_3^2 = (D_2 + D_3) : (D_3 + D_1) : (D_1 + D_2)$. Then rows I/1, I/2 and I/3 of matrix (3) are all proportional to $[u_1 \quad u_2 \quad u_3 \quad 2u_1u_2u_3/\lambda]$. Further, rows II/1, II/2 and II/3 are linearly dependent by **(3)**. Hence (5) implies that the rank of (3) is at most 3. Thus after some row manipulations some 3×3 submatrix, contained in the first three columns, has a non-zero determinant by the last paragraph of **(5)** (for $s = 0$). Therefore we have, like at the case $s \neq 0, \pm 1$, that also in this case *our equations have a unique solution for* **b** *(for matrix* (3)*).* Cf. the italicized texts in the previous paragraph.

(10) By the second and third paragraphs of **(8)**, it remained to solve equations (5) for $u_1u_2u_3 \neq 0$, where s can be any real number different from ± 1.

Using $u_1^2 + u_2^2 + u_3^2 = 1$, equations (5) become

$$\frac{(1 - 2u_i^2) + s^2(3 - 2u_i^2) + s^4}{D_i} = \lambda, \text{ where } \lambda \text{ is independent of } i \ (i = 1, 2, 3). \quad (6)$$

Solving this for u_i^2, we obtain

$$u_i^2 = \frac{1}{2}\left(s^2 + 2 - \frac{\lambda D_i + 1}{s^2 + 1}\right). \tag{7}$$

Summing these for $i = 1, 2, 3$, we obtain

$$1 = \frac{1}{2}\left(3s^2 + 6 - \frac{\lambda(D_1 + D_2 + D_3) + 3}{s^2 + 1}\right),$$

from which we express λ and put it into (7). Thus we obtain

$$u_i^2 = \frac{(s^4 + 3s^2 + 1) - (3s^4 + 7s^2 + 1)D_i/(D_1 + D_2 + D_3)}{2(s^2 + 1)}, \tag{8}$$

provided of course that all these expressions are non-negative. Actually, by $u_1u_2u_3 \neq 0$, all these expressions have to be positive. It is easily seen that these expressions actually satisfy (6) and have sum 1, thus *in this paragraph we have made equivalent transformations* (i.e., (6) $\Leftrightarrow$ (8)).

Using (8), the condition $\min u_i^2 > 0$ is equivalent to $f(s^2) := (s^4 + 3s^2 + 1)/(3s^4 + 7s^2 + 1) > \max D_i/(D_1 + D_2 + D_3)$. Here $s^2 \mapsto f(s^2)$ strictly decreases in $[0, \infty)$, with image $(1/3, 1]$. Hence, except the case $D_1 = D_2 = D_3$, when this inequality is satisfied for all $s \in \mathbb{R}$, we have that this inequality is satisfied for $s^2 < f^{-1}[\max D_i/(D_1 + D_2 + D_3)] < \infty$. (Thus in this case this solution set is far from the basic position, which is characterized by $s^2 = \infty$.) Here f^{-1}, defined on $(1/3, 1]$, and strictly decreasing there, with image $[0, \infty)$, is the inverse of f, defined on $[0, \infty)$. We have $f(1) = 5/11$.

Exclude further the case $D_1 = D_2 = D_3$, which has been completely settled by [10] and [12] (where our Theorem 1 was proved for this case). We may restrict our attention to the case $u_1, u_2, u_3 \geq 0$. We write $s_0 := [f^{-1}(\max D_i/(D_1 + D_2 + D_3))]^{1/2}$. By $f(0) = 1 > \max D_i/(D_1 + D_2 + D_3)$, we have $s_0 > 0$. By positivity of (8), we have $s \in (-s_0, s_0)$.

First suppose $s_0 < 1$ (i.e., $\max D_i/(D_1+D_2+D_3) > 5/11$). We claim that s varies in $I := (-s_0, s_0)$. Recall from **(9)** that for $u_1u_2u_3 \neq 0$ and $s \neq \pm 1$ any solution of equations (5) is a solution for our equations (for matrix (3)). However, as we have seen in (8), also using the equivalence of (8) and (6), *each* $s \in [-s_0, s_0] \not\ni \pm 1$ *can occur for a solution for our equations (for matrix* (3)*).* Moreover, for $s \in (-s_0, s_0)$ we have $u_i^2 > 0$. Hence, also using **(3)** and **(9)**, *for* $s \in [-s_0, s_0]$, *the solution of our equations for* $\mathbf{b}$ *(for matrix* (3)*) exists and is unique, and hence is continuous in* s *and* $\mathbf{u}$, *hence in* s (since now, by (8), $\mathbf{u}$ is continuous in s).

Now we prove continuity in the last italicized statement. By existence and unicity of the solution for $\mathbf{b}$, for matrix (3), we have that the rank of the submatrix of (3), consisting of its first three columns, is 3. Otherwise said, some 3×3 submatrix of matrix (3), contained in its first three columns, has a non-zero determinant — this submatrix may depend on s. Then the system of the corresponding three linear equations for b_1, b_2 and b_3 can be uniquely solved by Cramer's rule, with the denominator being non-zero. Then the solution vector $\mathbf{b}$ for this submatrix depends continuously on the coefficients of these three linear equations, therefore on $\mathbf{u}$ and s, hence on s. By existence and unicity of the solution vector for matrix (3), the solutions for this submatrix and for matrix (3) coincide.

This continuity property implies the statements about the endpoints of this solution manifold.

Summing up: for $s_0 < 1$ we have a connected manifold of solutions. Moreover, adding to it its two endpoints, at $s = \pm s_0$, we obtain a topological 1-manifold with boundary (these two points), contained in the solution set.

Second suppose $s_0 > 1$ (i.e., $\max D_i/(D_1 + D_2 + D_3) < 5/11$). We claim that s varies in $J := (-s_0, s_0) \setminus \{-1, 1\}$. From above, we have $s \in (-s_0, s_0)$. Like in the case $s_0 < 1$, each $s \in J \cup \{-s_0, s_0\}$ actually can occur for a solution of our equations for $\mathbf{b}$ (for matrix (3)). Like in the first case, for $s \in J \cup \{-s_0, s_0\}$, this solution is also unique and hence is continuous in s and $\mathbf{u}$, hence in s. Then the statements about the endpoints of this solution manifold follow from this.

For $u_1u_2u_3 \neq 0$, we have that $s = \pm 1$ cannot occur for a solution, cf. **(3)**. We are going to show that $u_1u_2u_3 \neq 0$ *continues to hold also for* $s = \pm 1$*, hence* $s = \pm 1$ *cannot occur for a solution.* We have $f(s_0^2) = \max D_i/(D_1 + D_2 + D_3)$. Thus, applying (8) for $s = \pm 1$, we have

$$\min u_i^2 = \frac{5/11 - \max D_i/(D_1 + D_2 + D_3)}{4/11} = \frac{f(1) - f(s_0^2)}{4/11} > 0.$$

Then however, $u_1^2u_2^2u_3^2 \geq (\min u_i^2)^3 > 0$, hence $u_1u_2u_3 \neq 0$, and $s = \pm 1$ cannot occur for a solution, as asserted. Then this implies that for $0, \pm 1 \neq s \to \pm 1$ the uniquely existing solutions $\mathbf{b}$ of our equations (for matrix (3)) tend to infinity (recall that non-uniqueness for $\mathbf{b}$ can occur only for $s = 0, \pm 1$, cf. **(3)**). In fact, in the contrary case, a standard compactness argument would give that $s = \pm 1$ could occur for a solution.

Summing up: for $s_0 > 1$ we have a solution manifold of three connected components: one for $s \in (-s_0, -1)$, one for $s \in (-1, 1)$, and one for $s \in (1, s_0)$. At $s = \pm 1$ the manifold components go to infinity. Like for $s_0 < 1$, also now, adding to the solution manifold its two endpoints, at $s = \pm s_0$, we obtain a topological 1-manifold with boundary (these two points), contained in the solution set.

Third suppose $s_0 = 1$ (i.e., $\max D_i/(D_1 + D_2 + D_3) = 5/11$). Suppose $D_i \geq D_j, D_k$. Then for $s = \pm 1$ the only possibility of non-uniqueness of the solution for $\mathbf{b}$ for our equations (for matrix (3)) is when $D_i = D_j + D_k$ (cf. **(7)**). Then however $1/2 = D_i/(D_i + D_j + D_k) = 5/11$, a contradiction. Hence, also using **(5)**, for $s = \pm 1, 0$, thus also for all $s \in [-1, 1]$, the solution for $\mathbf{b}$ for our equations (for matrix (3)) is unique. However, for $s = \pm 1$ it also exists. Namely, for $s = \pm 1$ we have $u_i = 0$, and $\varphi = \pm\pi/2$, which is a motion of the intermediate kind, which exists even for all $\varphi \neq \pi$, so, in particular, for $\varphi = \pm\pi/2$. Like in the first case, s varies in $(-s_0, s_0) = (-1, 1)$. Hence, like in the first case, for $s \in [-1, 1]$ the solution of our equations for $\mathbf{b}$ (for matrix (3)) exists and is unique. Hence, like in the first case, for $s \in [-1, 1]$ it is continuous in $\mathbf{u}$ and s, hence in s. Then the statements about the endpoints of this solution manifold follow from this.

Summing up: for $s_0 = 1$ we have a connected manifold of solutions. Moreover, adding to it its two endpoints, at $s = \pm s_0 = \pm 1$, we obtain a topological 1-manifold with boundary (these two points), contained in the solution set. □

2.2. In [12], pp. 438–440 a slight generalization of the question of Tompos's tetrahedra has also been considered. Now we present the corresponding question for the tetrahedra $P_1P_2P_3P_4$ and $Q_1Q_2Q_3Q_4$, derived from a rectangular parallelepiped. In the physical model of these tetrahedra, we have the following. The bars (edges) of one tetrahedral frame (of the fixed tetrahedron $P_1P_2P_3P_4$, say) touch the corresponding bars (edges) of the other tetrahedral frame (of the moving tetrahedron $Q_1Q_2Q_3Q_4$) from inside (cf. §1). Thus the actual physical constraint is only that each edge P_iP_j lies "inside Q_jQ_k" (here $(ijkl)$ is any permutation of $\{1,2,3,4\}$). This can be defined mathematically as follows (cf. [12], p. 439).

> For each permutation $(ijkl)$ of $\{1,2,3,4\}$ we have the following. The signed volume of the tetrahedron $P_iP_jQ_kQ_l$ is either 0, or has the opposite sign as the signed volume of the tetrahedron $P_iP_jR_kR_l$. Here R_kR_l is that translate of the segment Q_kQ_l in the basic position (i.e., of the segment $\overline{Q}_k\overline{Q}_l = (-P_k)(-P_l)$) which satisfies the following. The midpoint of R_kR_l coincides with the centre of the rectangular parallelepiped in its basic position. (9)

We take (9) as the definition of a *generalized motion of the moving tetrahedron* $Q_1Q_2Q_3Q_4$ (while the tetrahedron $P_1P_2P_3P_4$ is fixed). We prove

Theorem 2. *Consider the pair of tetrahedra, derived from a rectangular parallelepiped, considered in Theorem 1. For them the generalized admitted finite motions — i.e., all positions of the moving tetrahedron (obtained from its basic position by applying to it an isometry of the space, of determinant* $+1$*), satisfying* (9) *— are identical with the finite motions admitted by them (described in Theorem 1).*

Proof is analogous to that of [12], Theorem 2, pp. 439–440. Details cf. there, we only indicate the differences.

Let $\mathbf{\Phi x} = \mathbf{Ax} + \mathbf{b}$ be a generalized admitted finite motion. For $\mathbf{A} = \mathbf{I}$ we have $\mathbf{b} = \mathbf{0}$. From now on we suppose $0 < \varphi < 2\pi$. Observe that now the constraints are expressed by six inequalities (corresponding to equalities (1g)–(1l) in [12], p. 423). Namely three expressions (the left-hand sides of (1g), (1h), (1i)) are non-negative, three expressions (the left-hand sides of (1j), (1k), (1l)) are non-positive. In [12] the moving vertices Q_i were obtained by the motion $\mathbf{Ax} + \mathbf{b}$, from the respective points $(\pm 1, \pm 1, \pm 1)$. Differing from this, now the images by $\mathbf{D}^{-1}$ of the moving vertices Q_i are obtained by the transformation $\mathbf{D}^{-1}\mathbf{AD}(\mathbf{x}) + \mathbf{D}^{-1}\mathbf{b}$, from the respective points $(\pm 1, \pm 1, \pm 1)$, while $P_i^0 = (\pm 1, \pm 1, \pm 1)$ are fixed. Cf. the proof of our Theorem 1, **(1)**.

Subtract from a non-negative above expression a non-positive above expression, corresponding to pairs of edges which were originally diagonals of opposite faces of the rectangular parallelepiped. Then divide this difference by 2. Then like in [12], p. 439, we obtain the following. Rather than our equalities (II/i) in **(1)** of the proof of Theorem 1, we will have inequalities. Namely the left-hand sides of (II/i) are not less than their right-hand sides, which are equal to 0. However, by **(3)** of the proof of Theorem 1, a positive linear combination of rows II/i of matrix (3) is 0. Therefore the same holds for matrix $\mathbf{B}$. Hence, like in [12], in each of the inequalities, corresponding to equalities (II/1), (II/2) and (II/3), we have equalities.

Now recall that the left hand side of each of the inequalities corresponding to an equality (II/i) was obtained as follows. It was half the difference of a non-negative and a non-positive number. This half difference being 0 implies that both of these non-negative and non-positive numbers are 0. In other words, in all the six original constraint inequalities we have equalities. I.e., each pair of edges P_iP_j, Q_kQ_l (where $(ijkl)$ is any permutation of $\{1,2,3,4\}$) is coplanar. Thus $\mathbf{\Phi}$ is a motion admitted by our pair of tetrahedra. □

2.3. Let us start, rather than with a rectangular parallelepiped, with a general parallelepiped. All the diagonals of all of its faces constitute the edges of two congruent tetrahedra. This position of the two tetrahedra is called their *basic position.* We define the admitted motions as in (1), but deleting the word "rectangular". We have, with the notations from **(1)** in the proof of Theorem 1, that $P_i = \mathbf{D}P_i^0$ and $\overline{Q}_i = \mathbf{D}\overline{Q}_i^0$. However, now $\mathbf{D} = [d_{ij}]$ is a general non-singular matrix. In what follows, we will show (in **3.1**) that in certain cases the analogues of the motions for the case of the cube or the rectangular parallelepiped exist. Further we prove the generalization of Theorem 2 to the case of general parallelepipeds. Also we will investigate the unicity of the solutions of our equations for $\mathbf{b}$ (in **3.1**). In the physical model, the bars (edges) of the fixed tetrahedron touch those of the moving tetrahedron from inside (as in Fig. 1).

Also now we have for $[a_{ij}^0] := \mathbf{D}^{-1}\mathbf{AD}$ and $[b_1^0 \quad b_2^0 \quad b_3^0]^T := \mathbf{D}^{-1}\mathbf{b}$ equations (I/i), (II/i), $i = 1, 2, 3$. Evidently the left-hand sides of (II/1), (II/2) and (II/3) have sum 0. Their right-hand sides have sum $2\mathrm{Tr}\,(\mathbf{D}^{-1}\mathbf{AD}) - 2m_2(\mathbf{D}^{-1}\mathbf{AD})$. Here, for any 3×3 matrix $\mathbf{B}$, we write $m_2(\mathbf{B})$ for the sum of the symmetric 2×2 subdeterminants of $\mathbf{B}$. We have $\mathrm{Tr}\,(\mathbf{D}^{-1}\mathbf{AD}) = \mathrm{Tr}\,(\mathbf{A})$. We also have $m_2(\mathbf{D}^{-1}\mathbf{AD}) = m_2(\mathbf{A})$. Namely these last two numbers are the coefficients of $-\lambda$ in the characteristic polynomials of $\mathbf{D}^{-1}\mathbf{AD}$ and $\mathbf{A}$, resp. However, these polynomials coincide. Hence the sum of the right-hand sides of equations (II/1), (II/2) and (II/3) is the same as for the case $\mathbf{D} = \mathbf{I}$, i.e. 0. (Cf. [12], p. 436, (II/1)$'$). Thus the sum of equations (II/1), (II/2) and (II/3) is the equation $0 = 0$. Hence, among our six linear equations for b_1, b_2 and b_3, there are at most five linearly independent ones.

Defining also *for the case of general parallelepipeds the generalized admitted finite motions* as in Theorem 2, but in (9) deleting the word "rectangular", we have

Theorem 3. *Consider the pair of tetrahedra, derived above from a general parallelepiped. For them the generalized admitted finite motions are identical with the finite motions admitted by them.*

Proof is the same as for Theorem 2, using that the sum of the linear equations (II/1), (II/2) and (II/3), for b_1, b_2, b_3, is the equation $0 = 0$. □

Further we will deal with the pair of tetrahedra, derived above from a general parallelepiped, in **3.1**.

3. The motions of a pair of tetrahedra derived from a general parallelepiped, of a pair of regular pyramidal frames, and of a pair of regular tetrahedra with circular arc edges

3.1. We continue the investigation of the two tetrahedra derived from a general parallelepiped. Like in the beginning of **2.1**, in **(1)** of the proof of Theorem 1, and in **2.3**, our parallelepiped is taken as the image of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, by the non-singular matrix $\mathbf{D}$.

Similarly like in **(3)** of the proof of Theorem 1, we have the following. A non-unique solution of equations (I/i), (II/i), $i = 1, 2, 3$, for $\mathbf{b}$, can occur only if $\mathrm{Tr}\,(\mathbf{D}^{-1}\mathbf{AD})$ is an eigenvalue of $\mathbf{D}^{-1}\mathbf{AD}$. Equivalently, $\mathrm{Tr}\,(\mathbf{A})$ is an eigenvalue of $\mathbf{A}$, i.e., $\varphi = \pm\pi/2$ or $\varphi = \pi$ (cf. [11], p. 270, or [12], p. 438).

Observe that for $[a_{ij}^0] := \mathbf{D}^{-1}\mathbf{AD}$ and $[b_i^0] := \mathbf{D}^{-1}\mathbf{b}$ the left hand sides of equations (I/1), (I/2) and (I/3) form the vector $[a_{ij}^0 - \delta_{ij}(a_{11}^0 + a_{22}^0 + a_{33}^0)][b_1^0 \quad b_2^0 \quad b_3^0]^T$.

Therefore, like in the first paragraph of **(3)**, and in the second paragraph of **(9)** of the proof of Theorem 1, for $\varphi \neq \pm\pi/2,\ \pi$, we have the following. The vanishing of the determinants of the system of equations (I/1), (I/2), (I/3), (II/1), and of the system of equations (I/1), (I/2), (I/3), (II/2), is also a sufficient condition for the solvability of equations (I/i), (II/i), for $i = 1, 2, 3$, for $\mathbf{b}$. (Recall from **2.3** the linear dependence among equations (II/1), (II/2) and (II/3). Namely, their sum is the equation $0 = 0$.)

Because of the linear dependence among our equations, it is to be expected that there is a 1-manifold of solutions. This exists — and is a motion of the third kind — if the parallelepiped has a threefold rotational symmetry about a spatial diagonal. Namely, one tetrahedron remains fixed. Beginning from the basic position, the other one is first rotated about this spatial diagonal. Then it is translated in the direction of this spatial diagonal, till the coplanarity conditions become, simultaneously, satisfied (this position is unique). The angle of rotation can be arbitrary, except $\pm\pi/2$.

Now suppose that the mid-plane between two parallel faces contains $\mathbf{0}$, and is a plane of symmetry of the parallelepiped. Let these faces be horizontal. Then, as follows from the considerations in **2.1**, we have the motions of the intermediate kind, for any $\varphi \neq \pi$, and we have the motions of the fifth kind. The axis of rotation $\mathbf{0u}$ lies in this plane of symmetry, i.e., the xy-plane (and beside this it can have an arbitrary direction), and is perpendicular to this plane of symmetry, resp. In this symmetric case, suppose that moreover the projection of the parallelepiped along a diagonal of one of the mentioned parallel faces is a rectangle of side ratio $2 : 1$. (The other diagonal of this face having a larger projection than the altitude belonging to this face.) Then we have a motion of the sixth kind.

However, in this symmetric case, with the symmetrical faces being horizontal, as follows from the considerations in **2.1**, the motion of the intermediate kind may exist also for $\varphi = \pi$. This happens exactly in the cases when the rotation axis $\mathbf{0u}$ (lying in the xy-plane) is parallel to an angle bisector of the diagonals of a horizontal face. Moreover, then there exists also a new motion, which we call a *motion of the seventh kind*. Namely, as in **2.1**, we rotate both tetrahedra about this rotation axis, in a way symmetric w.r.t. the xy-plane, through angles $\pm\pi/2$. Then we make arbitrary vertical translations of the two tetrahedra, in a way symmetric w.r.t. the xy-plane. Thus we get positions satisfying our constraints. For the case of a rectangular parallelepiped, this yields a motion of the fourth kind.

For the general case, suppose that the parallelepiped is nearly a cube — more exactly, $\mathbf{D}$ is near to $\mathbf{I}$. Let us choose a point of a solution manifold of the motions of the third kind for the cube (with $\mathbf{D} = \mathbf{I}$). Let it correspond to $\mathbf{u}^0 = (\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, 1/\sqrt{3})$ (the $\pm$ signs being independent), and to a fixed $\varphi \in (0, 2\pi)$, with $\varphi \neq \pi/2, \pi, 3\pi/2$ (with some unique $\mathbf{b}$). Then for any fixed $\varepsilon > 0$, for a sufficiently small perturbation $\mathbf{D}$ of $\mathbf{I}$, we have the following. There is a solution for the parallelepiped associated to $\mathbf{D}$, with $u_1^2 + u_2^2 + u_3^2 = 1$ and $\|(u_1, u_2, u_3) - \mathbf{u}^0\| < \varepsilon$, and with this fixed value of φ (with some unique $\mathbf{b}$). (Possibly these points (u_1, u_2, u_3), for φ varying in a fixed closed subinterval of $(0, 2\pi)$, avoiding $\pi/2, \pi, 3\pi/2$, and for a sufficiently small perturbation $\mathbf{D}$ of $\mathbf{I}$, form a smooth 1-manifold with boundary.)

In fact, our problem is now equivalent to solving the system of the two determinantal equations, mentioned in the third paragraph of **3.1**, for u_1, u_2, u_3 and s. For the case of the cube (with $\mathbf{D} = \mathbf{I}$), these equations say that non-zero multiples of $u_2 - u_3$, and of $u_3 - u_1$, are 0, cf. [12], p. 437, 3. Thus these multiples change their signs at the curves on S^2, given by $u_2 = u_3$ and $u_3 = u_1$, resp. Hence, after a small perturbation of the equations (i.e., for $\mathbf{D}$ near to $\mathbf{I}$), the zero-sets of the perturbed multiples will be near the above two curves, resp. Therefore, for each fixed $\varphi \in (0, 2\pi)$, where $\varphi \neq \pi/2, \pi, 3\pi/2$, and for a sufficiently small perturbation, the following holds. We have a solution of this perturbed system of our two equations, with $u_1^2 + u_2^2 + u_3^2 = 1$ and $\|(u_1, u_2, u_3) - \mathbf{u}^0\| < \varepsilon$, and with this fixed value of φ (with some unique $\mathbf{b}$). (Cf. [3], p. 40, Proposition D.)

Of course, for a general non-singular $\mathbf{D}$, the basic position is a solution as well. By the linear dependence of equations (I/i) and (II/i) (cf. **2.3**), it is to be expected that the basic position lies on a solution manifold of dimension at least 1. Experiences with models, far from the rectangular parallelepipeds (more exactly, with all edge lengths rather different), seem to confirm this, even with dimension exactly 1. (Observe that, for a rectangular perallelepiped, each opposite pair of edges of the tetrahedra have equal lengths.) However, we cannot identify this (assumed) solution manifold, which we can call the (assumed) *motion of the eighth kind.*

We have determined the infinitesimal degree of freedom, at the basic position, for several incongruent parallelepipeds having a threefold rotational symmetry about a spatial diagonal. (Details of this calculation will be given in **3.2**.) Except for Tompos's tetrahedra, this infinitesimal degree of freedom always turned out to be 1.

Hence we have the following. The infinitesimal degree of freedom, at a point of a solution manifold, which can be reached from the basic position by a continuous motion, always satisfying the constraints, is probably, in general, not greater than 1. Moreover, in general, the basic position does not lie on a smooth 2-manifold of solutions.

The simplest unsolved case is probably that of a parallelepiped P having a threefold rotational symmetry about a spatial diagonal. Using analogous notations as in Fig. 3, let the vertices be P_i (fixed) and Q_i (moving). We define the admissible motion, obtained from the basic position of the moving tetrahedron $Q_1Q_2Q_3Q_4$, by a rotation about the axis of rotation P_4Q_4, say, through an angle $\pm\pi/3$, and a subsequent (unique) translation in the direction of this axis. This motion is denoted by $\Phi_4^{\pm}$. (Only Q_j is moved by $\Phi_i^{\pm}$, and P_j remains fixed.) Then $\Phi_4^{\pm}$ degenerates P to a regular double pyramid, with base the triangle $P_1P_2P_3 = \Phi_4^+(Q_1)\Phi_4^+(Q_2)\Phi_4^+(Q_3) = \Phi_4^-(Q_1)\Phi_4^-(Q_2)\Phi_4^-(Q_3)$. Moreover, the plane spanned by this triangle is a plane of symmetry of this double pyramid. Also, Φ_4^+ and Φ_4^- lie on a connected component of a 1-manifold of solutions, of motions of the third kind (cf. above).

Now let, e.g., $i = 1$. Let S_1 be the symmetry w.r.t. the plane spanned by the face $P_2P_3P_4$. Then the symmetric double pyramid $(P_1P_2P_3P_4) \cup S_1(P_1P_2P_3P_4)$ is a degenerate image of P. It can be obtained by choosing any of the two orientation-preserving isometries Φ_i^+ and Φ_i^-, which are also admitted motions, and which are defined as follows. We have $\Phi_1^+(Q_1) := P_2$ and $\Phi_1^+(Q_2) := P_3$ and $\Phi_1^+(Q_3) := S_1(P_1)$ and $\Phi_1^+(Q_4) := P_4$. Similarly, we have $\Phi_1^-(Q_1) := P_3$ and $\Phi_1^-(Q_2) := S_1(P_1)$ and $\Phi_1^-(Q_3) := P_2$ and $\Phi_1^-(Q_4) := P_4$. Analogously we define Φ_i^+ and Φ_i^-, for $i = 2, 3$. Then Φ_i^+ and Φ_i^-, for any fixed $i \in \{1, 2, 3\}$, are probably points of assumed three analogues of the 1-manifold of the motions of the third kind, resp. Moreover, a model experiment indicates three connected smooth solution

1-manifolds, containing these three pairs of points, resp.

Concluding: it is to be awaited that the solution manifolds have in general dimension 1, and also the infinitesimal degree of freedom at their points is in general 1. Let $\mathbf{D}$ be a small perturbation of $\mathbf{I}$. Then probably, in some small neighbourhoods of the four 1-manifolds of the motions of the third kind for the cube (with $\mathbf{D} = \mathbf{I}$), there are four 1-manifolds of solutions. Possibly these exist even for each non-singular $\mathbf{D}$ (as they do for the rectangular parallelepipeds). Moreover, probably there is one 1-manifold of solutions, passing through the basic position. (For the case of a rectangular parallelepiped, this may degenerate to have length 0.)

3.2. Let us consider two congruent right pyramids, whose bases are regular n-gons (where $n \geq 3$). Suppose that their axes of rotation coincide, and the basic edges of one pyramid intersect the lateral edges of the other one, and also conversely, with the vectors from the centres of the bases to the respective apices being opposite. (Without this oppositeness property, the moving pyramid could coincide with the fixed pyramid.) Additionally, we suppose that the direction of some basic edge of one pyramid and the direction of some basic edge of the other pyramid enclose an angle π/n (Fig. 6). **FIG. 6 ABOUT HERE.** We call this position the *basic position* of this bar structure, consisting of these two pyramids. (Observe that the case $n = 3$ is a special case of **3.1** as well.)

> Let us consider these pyramids only as bar structures. Let us move each vertex of the bar structure consisting of these two pyramids under the following restriction. One triangular face of one of the pyramids is left fixed, each bar (edge) preserves its length, and each pair of edges, one from each pyramid, which were originally intersecting, is left coplanar. (10)

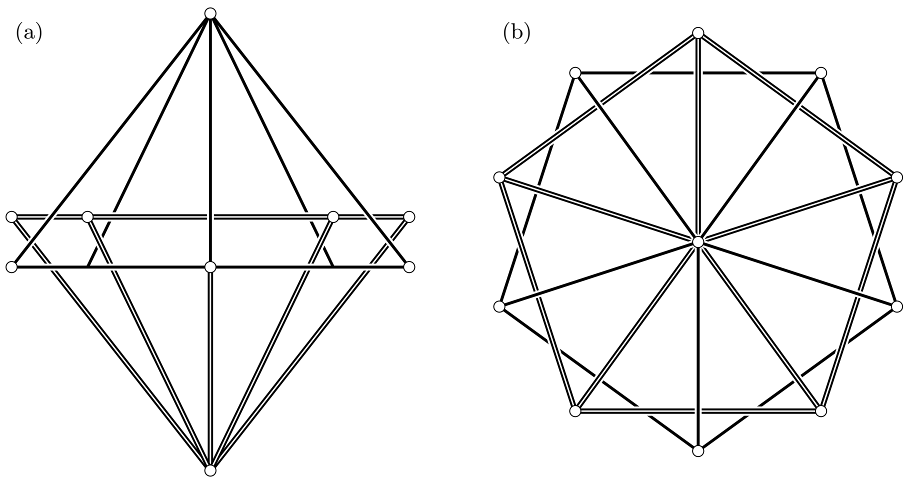

Figure 6. The pair of regular pyramids in basic position: (a) front view, (b) top view.

The physical model is built in such a way that the bars (edges) of one pyramidal frame touch those of the other one from inside (as in Fig. 1). Observe that it is not supposed a priori that the two pyramids undergo rigid motions (isometries of the space of determinant +1). Rather, the bases are allowed to change their shapes, and also to become non-planar. However, by experimenting with the respective physical models, *for positions attainable from the basic position by continuous motions, always satisfying the constraints*, the following seems probable. These conditions seem to enforce the rigid motion of the two pyramids (with one of the two pyramids remaining fixed), even with their axes of rotations coinciding, and with the vectors from the centres of the bases to the respective apices being opposite (as in Fig. 6).

Actually even the weakening of (10), analogously as in (9), seems to enforce this. Also cf. [9].

We make local investigations. Since the motions of Tompos's tetrahedra have already been described, we further exclude the case when $n = 3$ and the two tetrahedra are regular.

> We have a 1-manifold of finite motions, which are conjectured to be the only positions, attainable from the basic position by continuous motions, always satisfying the constraints.

(This is supported by experimenting with the models.) Namely, one pyramid remains fixed, and the other one undergoes a rigid motion, as follows. Its axis of rotation remains fixed, and it undergoes a certain rotation about this axis, followed by a suitable translation. This translation happens in the direction of the common axis of rotation, through a distance depending on the angle of rotation. We translate the moving pyramid till the coplanarity conditions become, simultaneously, satisfied (this position is unique). This is an analogue of the motion of the third kind for Tompos's tetrahedra. The angle of rotation can be arbitrary, except $\pm\pi/2$. For $n \geq 4$ (unlike as for $n = 3$, cf. **3.1**), we are unaware of any other motions, admitted by our bar structure.

We have considered the basic position of this motion, for $3 \leq n \leq 7$, and for several different values of the quotient of the lengths of the lateral and the basic edges. We have determined for these cases the infinitesimal degrees of freedom of our bar structure, consisting of these two pyramidal frames, as follows.

The number of the free parameters, i.e., of all the three coordinates of all but the fixed three vertices, is $6n-3$. The constraints are that the lengths of all but the fixed three edges are fixed, and $2n$ pairs of edges are coplanar, i.e., the tetrahedra spanned by their vertices have fixed signed volumes, namely 0. The total number of constraints is also $6n-3$. Thus we have a function $\mathbb{R}^{6n-3} \to \mathbb{R}^{6n-3}$. This maps a $(6n-3)$-tuple of the coordinates of the non-fixed vertices to the vector with coordinates the $6n-3$ constraints, as functions of the previous $6n-3$ coordinates. (The constraints are the lengths of the non-fixed edges, and the signed volumes of the above tetrahedra.) Then the number of infinitesimal degrees of freedom of our bar structure is the nullity of the Jacobian J of this map, i.e., $6n - 3 - \operatorname{rank} J$.

Having performed these calculations, like e.g. in [7], §2, we have found the following. This infinitesimal degree of freedom, at the basic position, is in all the cases considered by us equal to 1, except in the case of Tompos's tetrahedra. This can be considered as numerically supporting the above conjecture.

It would be interesting to clarify even that case, when the two pyramids move as rigid bodies. (Observe that for $n \geq 4$ this yields an overdetermined system of equations, namely we have $2n$ equations about coplanarities, for six unknowns.)

3.3. Another generalization of the pair of tetrahedra, derived in §1 from the cube with vertices $(\pm1, \pm1, \pm1)$, is the following. We replace each edge of both tetrahedra by congruent circular arcs of some fixed radius. These have the same endpoints as the respective edges. Moreover, each of them lies in the plane spanned by the respective edge and the centre of the cube. Further suppose that any congruence of the above cube to itself is also a congruence of this system of circular arcs.

Thus we obtain a figure roughly resembling Fig. 1 or Fig. 3. Thus both tetrahedra become tetrahedron-like frames. The arcs of circles, replacing diagonals of the same face of the cube, intersect. Their point of intersection lies on the straight line, connecting the midpoint of the cube with the midpoint of the considered face of the cube.

> Fixing one of these frames, we move the other one in the following way. Each pair of circles, containing the pairs of circular arcs, originally corresponding to diagonals of some face of the cube, continue to have a common point, in the complex projective sense. (11)

Namely, in this sense the condition is to be awaited simpler.

Now suppose that the frames both lie on the surface of the circumsphere of the cube. Then an arbitrary rotation about the centre of the cube, with translation part $\mathbf{b} = \mathbf{0}$, is an admitted motion. So now we have an at least 3-parameter set of motions.

Again we turn to the general case. We will show that the motions of the intermediate, third and fifth kinds generically exist. (These contain the motions of first, second and fourth kinds as special cases.)

We begin with the analogue of the motion of the fifth kind. At this motion the moving tetrahedron undergoes from the position of first kind — obtained by $\mathbf{A}$ being a rotation through the angle π, about the z-axis, say — a translation, through a vector $\mathbf{b}$. This happens in the following way. An arbitrarily fixed point of the moving circle, containing the circular arc corresponding to the edge $Q_1^0 Q_2^0$, in its rotated position, will coincide after translation with an arbitrarily fixed point of the fixed circle, containing the circular arc corresponding to the edge $P_3^0 P_4^0$ (in analogy with Fig. 4e).

This is a two-parameter motion. At this motion the circles, containing the arcs corresponding to the edges $P_1^0 P_2^0$ and $Q_3^0 Q_4^0$ (in its rotated position), also intersect. This follows by a simple argument using central symmetry. However, the set of $\mathbf{b}$'s for this $\mathbf{A}$ does not form an affine 2-manifold — on the contrary, it is bounded. (For Tompos's tetrahedra we had here an affine 2-manifold.) All other pairs of respective circles, which should have common points, in the complex projective sense, lie in respectively parallel or coincident planes. This guarantees that these pairs of circles in fact have common points, in the complex projective sense. (Observe that the complex projective extension of any circular line, lying in a horizontal plane, contains the points $(1, \pm i, 0, 0)$ of the complex projective space — and the analogous statement holds for any other plane, as well.)

However, for motions of the intermediate and third kinds, there is a difference as compared to **2.1**.

We turn to the analogue of the motion of the third kind. It will be convenient to rotate our original cube about $\mathbf{0}$, so that P_4^0 becomes $(0, 0, \sqrt{3})$. The fixed circle C_{14}, containing the circular arc replacing the edge $P_1^0 P_4^0$, should lie in the vertical plane $y = 0$. However, the moving circle C'_{23} (obtained by a rotation, through an angle φ, about the z-axis, from its basic position) containing the circular arc replacing the edge $Q_2^0 Q_3^0$, will lie in a not vertical plane.

We denote the projection map to the xy-coordinate plane by π. Recalling the definition of the motion of the third kind, we want to find a λ, such that $C_{14} \cap (C'_{23} + \lambda \mathbf{e}_3) \neq \emptyset$. (Then, by reason of symmetry, with the evident notations, also $C_{23} \cap (C'_{14} + \lambda \mathbf{e}_3) \neq \emptyset$.)

We have that C_{14} lies in the xz-coordinate plane, hence $\pi(C_{14})$ is contained in the x-coordinate axis. The equation system of C_{14} is $y = 0$, and an equation of the form $(x - a)^2 + (z - b)^2 = R^2$. Hence, in the complex case, $\pi(C_{14})$ also contains the x-coordinate axis, hence equals the x-coordinate axis. Even generically $x\mathbf{e}_1$ is the projection, by π, of two points $(x, 0, z_1(x)), (x, 0, z_2(x)) \in C_{14}$ (C_{14} meant in the complex sense): namely the two endpoints of a chord of C_{14}, whose difference lies in the z-coordinate axis.

On the other hand, C'_{23} (meant in the complex sense) does not lie in a vertical plane, hence the projection from the affine hull of C'_{23} to the xy-coordinate plane is a bijection. Hence $\pi(C'_{23})$ is a non-degenerate ellipse, of an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Its intersection with the x-axis has an equation $Ax^2 + Dx + F = 0$, which has generically two zeroes x_1, x_2. Therefore, generically, the number of $(x, 0, z)$'s, for which $(x, 0, z) \in C_{14} \cap (C'_{23} + \lambda\mathbf{e}_3)$, for some λ (depending on x and z), is four. Namely, $(x_i, 0, 0)$ is the projection, by π, of one point $(x_i, 0, z'(x_i)) \in C'_{23}$, and then we choose $\lambda_{ij} = z_j(x_i) - z'(x_i)$ (for $i, j = 1, 2$). Moreover, these λ_{ij}'s are generically different. This can be seen from the example with radius of the circles $\sqrt{2}$, and $\varphi := \pi/2$, where these λ_{ij}'s are all different.

We turn to the analogue of the motion of the intermediate kind. Let the axis of rotation $\mathbf{0u}$ lie in the xy-plane. Let us apply symmetric rotations, w.r.t. the xy-plane, through angles $\pm\varphi/2$, about the rotation axis, to the two tetrahedron-like frames. Then the pairs of the curved edges (circles), corresponding to the two diagonals of any of the originally vertical faces of the cube, remain symmetric images of each other w.r.t. the xy-plane. The (possibly complex projective) intersection points of the xy-plane and one of the circles lie also on the other circle.

Moreover, after these symmetric rotations, we have the following. Each pair of the curved edges, corresponding to the two diagonals of an originally horizontal face of the cube, generically have, as projections to the xy-plane, two elliptical lines. These projections have generically four intersection points, in the complex projective sense. Therefore generically, for any of the four vertical lines, containing some of these four intersection points, the following holds. Suitable vertical translations, symmetric w.r.t. the xy-plane — in general through different distances for different lines (cf. below) — produce common points of these two curved edges, lying on this particular vertical line. Further, these distances, for the two originally horizontal faces, pairwise coincide. Namely, by reason of symmetry, the distances, associated to the same vertical line, (pairwise) coincide, so that these common points, on any of these, generically four vertical lines are produced, for the two originally horizontal faces, simultaneously.

It remained to show that the four vertical translations are generically different. Again, it suffices to give one example for this. We let the radius of the circles to be $\sqrt{2}$, and $\mathbf{u} := (1/\sqrt{2}, 1/\sqrt{2}, 0)$, and $\varphi := \pi$. Then, as at the analogue of the motion of the third kind, the moving circle, containing the arc replacing the diagonal $Q_1^0 Q_3^0$, has as a projection to the xy-plane, in the complex sense, a fixed straight line. Therefore we can repeat the considerations at the analogue of the motion of the third kind, obtaining that the four λ_{ij}'s, being different for this example, are generically different.

This question perhaps could be handled in analogy with [12], Theorem 1. We have six pairs of circles in the space, one fixed and one moving, which pairwise intersect (in the complex projective sense). This for each pair means an equation of degree six for the coefficients of the equations of our circular lines, hence for our parameters a_{ij} and b_i. Thus we have a system of six equations of degree twelve for our parameters $\mathbf{u}$, s and $\mathbf{b}$. Unfortunately this is not linear in the b_i's. Namely, for the analogues of the intermediate and the third kinds of motions, there are generically four solutions for **b** (in the complex projective sense, cf. above). Therefore, rather than calculating determinants, as in [12], one needs to calculate resultants of polynomials (cf. [14]). Possibly some symbolic algebraic calculations, like with Mathematica or Maple, and efficient algorithms from computational algebraic geometry could help.

3.4. In [9] the following general model was considered, which contains the examples in **3.1** and **3.2** as special cases. Let us have two convex polyhedra, which are combinatorially dual. Let one of them have f faces, e edges and v vertices. Then the other one has v faces, e edges and f vertices.

> Consider these polyhedra as bar structures only. Move each vertex of the bar structure consisting of these two polyhedra under the following condition. Each bar (edge) retains its length, and each pair of combinatorially corresponding edges is coplanar. (12)

The faces may change their shapes, and may become non-planar, and also convexity may not hold any more. The number of free parameters (all three coordinates of all vertices) is $3v + 3f$. The number of constraints (edge lengths and coplanarity conditions) is $3e$. By Euler's theorem, these numbers have a difference 6, i.e., the number of parameters of all rigid motions of the space. This used to indicate that there are not even infinitesimal motions. However, in example **3.2**, and sometimes (possibly always) in example **3.1**, there are finite motions, so here intuition fails.

An example is a pair of congruent tetrahedra with the same orientation, with the combinatorially corresponding pairs of edges being those induced by a fixed orientation-preserving congruence. Then a rotation of the fixed tetrahedron about any of its altitudes, or a translation of the fixed tetrahedron by any vector, yields a moving tetrahedron satisfying the constraints.

Problem 1. Is there some general theorem behind these examples, that under suitable hypotheses, the model described in **3.4** always has a finite motion?

Problem 2. Determine the finite motions of the examples in **3.1**, **3.2** and **3.3**.

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