Handles & h-cobordism

① MANIFOLDS

Def: <u>TOPOLOGICAL MANIFOLD</u> M^n is a T2 + N2 topological space locally homeomorphic to $\{x^n \geqslant 0\} \subseteq \mathbb{R}^n$.

The boundary ∂M corresponds to $\{x^n = 0\}$; it is well-def.

Def: <u>PL MANIFOLD</u> $(M^n, \mathcal{A})$ is a topological manifold with a (maximal) PL atlas

$$\mathcal{A} = \{ \varphi : \underbrace{\Delta^n}_{\text{std simplex}} \longrightarrow M \text{ embedding} \}$$

whose interior covers M & with PL transition functions:

$$\begin{tikzcd}
\varphi(\Delta^n) \subseteq & M & \supseteq \psi(\Delta^n) \\
\Delta^n \arrow[u, "\varphi", "\wr"'] \arrow[rr, dashed, "\text{piecewise linear}"'] & & \Delta^n \arrow[u, "\wr", "\psi"']
\end{tikzcd}$$

(i.e., Δ^n can be written as a finite union of simplices & the map is linear on each of them)

Def: <u>SMOOTH MANIFOLD</u> $(M, \mathcal{A})$ is a top. mfd with a maximal smooth atlas $\{ \varphi : \underbrace{U}_{\text{open subset of } \{x^n \geqslant 0\}} \xrightarrow{\text{open } \subseteq \{x^n \geqslant 0\}} M \}$ and C^∞ transition functions.

We focus almost always on cpt manifolds.

Facts:

1) A smooth mfd has a unique canonical PL structure
(Cairns 1935, Whitehead 1940)
However, $\exists$ non-smoothable PL manifolds (Kervaire 1960)

2) A PL mfd (obviously) defines a unique top. manifold.
$\exists$ top. mfds with no PL structure ($n=4$ Freedman 1982, $n \geq 5$ Kirby-Siebenmann 1969).

3) Low-dimensional facts:

*) $n \leq 3$ TOP $\sim$ PL $\sim$ C^∞ $\left(\begin{array}{ll} n=2 & \text{Radó } 1925 \\ n=3 & \text{Moise } 1952 \end{array}\right)$

*) $n \leq 6$ PL $\sim$ C^∞, very diff. from TOP

(so we basically forget about PL for the rest of the class)

however, embeddings can be $\neq$
$\exists$ non-smoothable PL embeddings of S^2 into S^4

Triangulations

Def: A triangulation is a homeomorphism with a simplicial complex.

Facts:

1) PL manifolds (hence smooth mfds) admit a triangulation
[in fact they define an essentially unique one]

2) Not all top mfds are triangulable

[$n \leq 3$ yes ($C^\infty \sim PL \sim TOP$), $n=4$ no (Casson 1990),
$n \geq 5$ no (Manolescu 2013)]

3) $\exists$ triangulable, non-PL mfds (e.g. double suspensions of non-trivial $\mathbb{Z}HS^n$)

4) Triangulations of the same top mfd can be combinatorially different (Hauptvermutung is false)

② HANDLES

CAT = TOP, PL, or C^∞

W^n CAT-mfd, $W_0 \subseteq W$ codim-0 closed submfd.

topologically, not in the sense of cpt + $\partial = \phi$

Def: A CAT-HANDLE DECOMPOSITION of W rel M is a filtration $W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots$ s.t.

1) $\bigcup_{i \geq 0} W_i = W$

2) W_i is a closed codim-0 submfd

3) $h_i := \overline{W_i - W_{i-1}}$ is a cpt submfd s.t.

$$(h_i, h_i \cap W_{i-1}) \underset{CAT}{\cong} (\mathbb{D}^k, \partial \mathbb{D}^k) \times \mathbb{D}^{n-k}$$

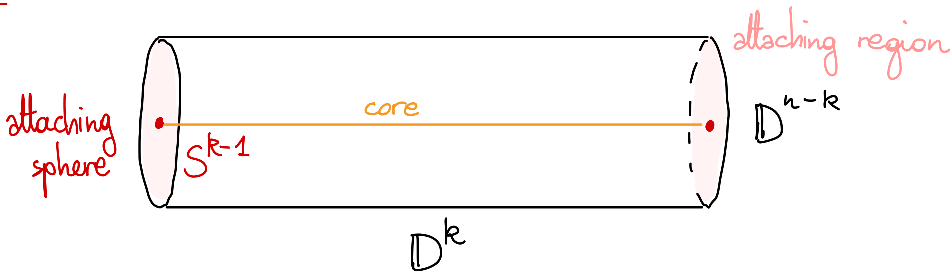
(k-handle)

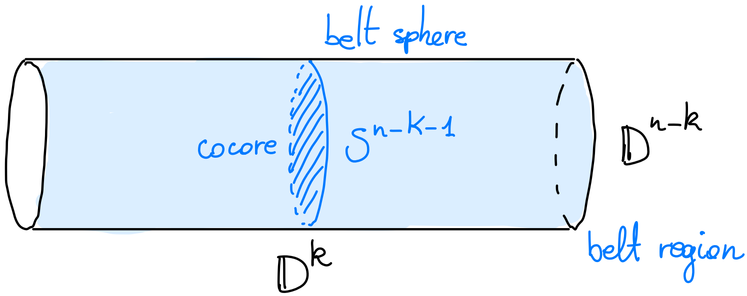
4) $\{h_i\}$ is locally finite

5) If $CAT = C^\infty$, smooth corners

(this can be done in an essentially unique way)

Notation h^k ← index





Handles are basically thickened cells.

Facts:

1) (Relative) PL/C^∞ handle decompositions exist for all PL/C^∞-mfds in all dimensions

[C^∞ uses Morse theory, see e.g. Milnor.
PL uses barycentric subdivisions, Hudson 1969]

2) (Relative) TOP handle decompositions exist for all mfds in dim ≥ 5.

$$\left[\begin{array}{ll} \dim \geq 6 & \text{High-dim top techniques} \\ \dim = 5 & \text{Quinn 1982} \end{array}\right]$$

3) A TOP handle decomp. of W^4 exists iff W is smoothable.

③ h-COBORDISM THEOREM

Def: A COBORDISM W from M^n to N^n closed (cpt no ∂) oriented n-mfds. is a cpt oriented $(n+1)$-dim. mfd w/ $\partial W = (-M) \cup N$.
If W exists, M and N are called CAT-cobordant.

Rk: Since $\partial W = (-M) \cup N = -(-N) \cup (-M)$,
W is also a cobordism from $-N$ to $-M$. This is called the UPSIDE DOWN cobordism; note that we haven't changed the orientation of W.

Rk: $I \times M$ is the IDENTITY COBORDISM from M to M.

Def: A cobordism W from M to N is an h-COBORDISM if $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences.

true except maybe for TOP in dim 4, where it's open

Rk: If W, M, N all have $\pi_1 = 1$ and are htopic to a CW cx,

then h-cob $\Leftrightarrow H_*(W, M; \mathbb{Z}) = 0 \Leftrightarrow H_*(W, N; \mathbb{Z}) = 0$.

(for $\Leftarrow$ here we'd need hty eq. to CW cx, but it is true a posteriori by Freedman)

Thm (h-cobordism, Smale 1960s) CAT = TOP, PL, C^∞.

Let W^{n+1} be a CAT-cobordism from M to N with

$\pi_1(W) = \pi_1(M) = \pi_1(N) = 1$ and $H_*(W, M; \mathbb{Z}) = 0$.

If $n \geq 5$, then $\exists$ CAT-isomorphism $W \cong I \times M$

which is the identity on $M \longrightarrow \{0\} \times M$. Thus, $M \cong_{CAT} N$.

4 PROOF of the h-cobordism thm

I will do the smooth case.

PL case is the same (Rourke-Sanderson)

TOP case requires checking that some techniques go through

(e.g. tangent bundles, transversality, handle decompositions, ...)

This is done by Kirby-Siebenmann.

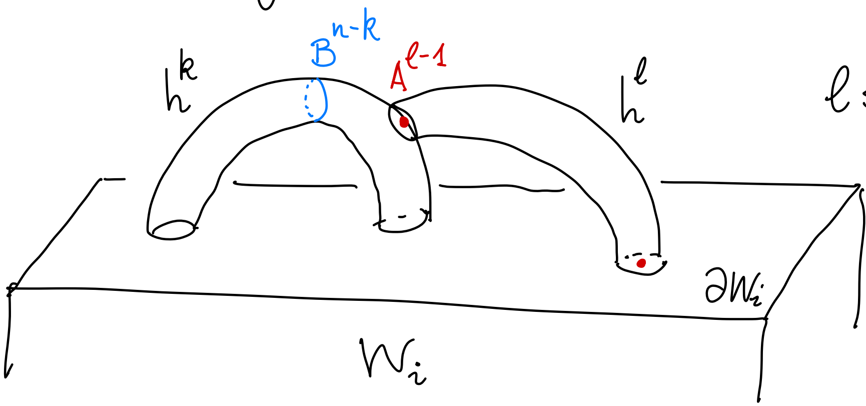
0. The starting point

Pick a handle decomposition for W rel M_0.

GOAL: modify it to remove all handles $\rightsquigarrow I \times M_0$.

1. Rearranging handles

By transversality we can assume that handles are attached with indices in increasing order



$\ell \leq k$

$$\operatorname{codim} A + \operatorname{codim} B = (n-\ell+1) + k > n = \dim(\partial W_i)$$

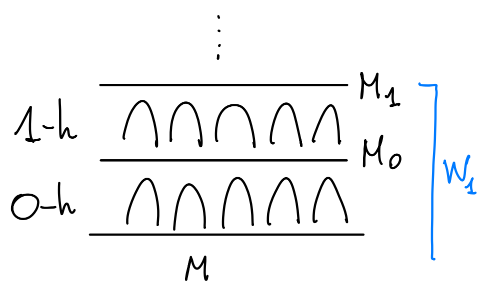
$$\Rightarrow \quad A \pitchfork B = \emptyset$$

You can isotope h^ℓ off of h^k.

Notation until the end of the proof

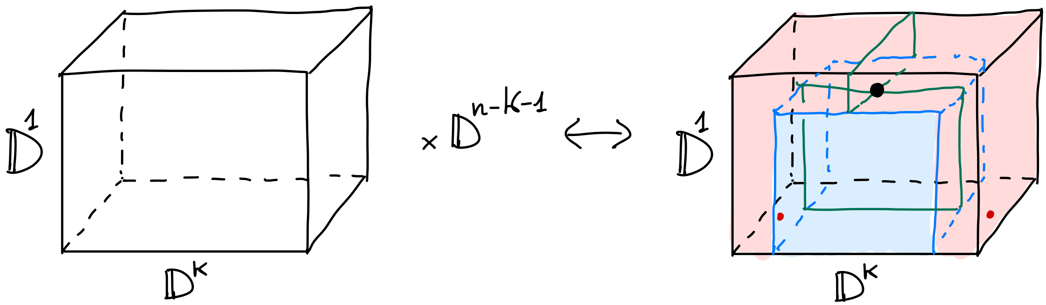
$W_i :=$ $(n+1)$-mfd after attaching all the i-handles

$M_i := \partial W_i - M$



Interlude: handle moves

1) Geometric pair cancellation



disc attached along

$D^k \times \{-1\} \times D^{n-k-1}$

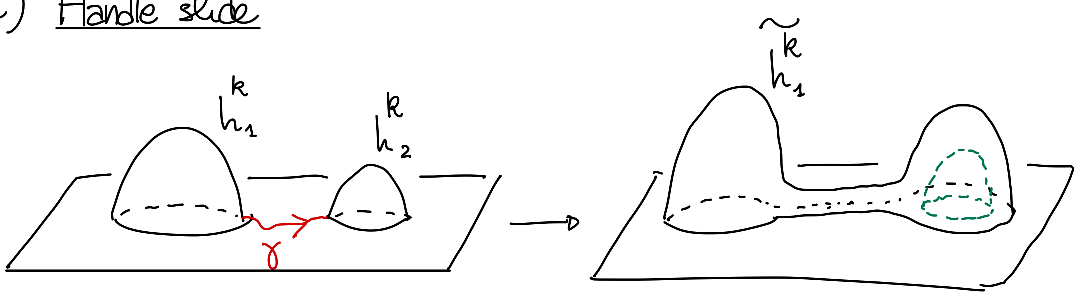
h^k and h^{k+1} geom. compl.

$(B_{h^k} \cap A_{h^{k+1}} = 1$ pt$)$
in M_k

Attaching this does not change W_* up to a deformation

$\rightsquigarrow$ Can introduce / cancel geom. complementary pairs.

2) Handle slide



Replace attaching sphere with # of attaching spheres.

$[\tilde{h}^k_1] = [h^k_1] + [h^k_1]$ in $H_*(W_k, W_{k-1})$

2. Handle trading

*) WLOG W has <u>no 0-handles</u>

Connectedness is completely determined by 0- & 1-handles.

$\leadsto$ Every 0-handle is connected to another 0-handle or M_0 by a 1-handle, necessarily geom. compl.

$\leadsto$ Cancel them.

*) WLOG W has <u>no 1-handles</u> if $n \geq 4$

Simply-connectedness is completely determined by 1- & 2-handles.

A 1-handle gives a loop γ, which can be pushed in M_2 ($n \geq 3$).

$\gamma = \partial D^2$, with $D^2 \hookrightarrow W^{n+1}$ if $n \geq 4$ (Whitney emb.)

D^2 can be pushed below 3-handles (belt spheres have dim $n-3$, which miss 2-dim. objects in M_i).

Similarly, push D^2 above 2-handles $\Rightarrow D^2 \subseteq M_2$.

Thicken it & get a 2/3 cancelling pair. Use 2-handle to cancel the geom. compl. 1-handle given by γ.

<u>Cor</u>: If $n \geq 4$, cancel all 0-, 1-, n-, (n+1)-handles

3. Handle homology

Def: HANDLE CHAIN COMPLEX

$$C_k = \mathbb{Z}\langle k\text{-handles}\rangle, \quad \partial_k : C_k \to C_{k-1}$$

$$\partial_k h^k = \sum_{h^{k-1}} \underbrace{\langle \partial_k h^k, h^{k-1}\rangle}_{\substack{\shortparallel \\ \#(\underset{\text{att. sphere}}{A^{k-1}} \cap \underset{\text{belt sphere}}{B^{n+1-k}})}} h^{k-1}$$

algebraic inters. number in M_{k-1}

Thm: $H_*(C_k) \cong$ singular homology of (W, M)

Pf: Same as for cellular homology

Application to h-cob. theorem

$$C_2 \xleftarrow{\partial_3} C_3 \leftarrow \cdots\cdots \leftarrow C_{n-2} \xleftarrow{\partial_{n-1}} C_{n-1}$$

Since $H_*(W, M; \mathbb{Z}) = 0$, after a change of basis (achieved w/ sequence of handleslides) we have

$$[\partial_k] = \left[\begin{array}{c|c} 0 & I \\ \hline 0 & 0 \end{array}\right]$$

Each k-handle is paired with either a $(k-1)$- or $(k+1)$-handle and they are algebraically complementary $(\#(A \cap B) = 1)$.

If we can make them geom. complem., then we can cancel them all and win.

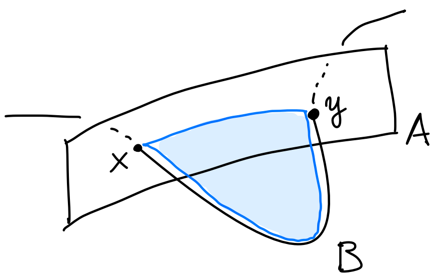
up to this pt $n \geq 4$ works

4. The Whitney trick $(n \geq 5)$

A^k = attaching sphere of h^{k+1}

B^{n-k} = belt sphere of h^k

complem. dimensions in M^n_K



Suppose $x, y \in A \cap B$ and we wish to cancel intersections (of opposite signs).

γ = path in A from x to y + path in B from y to x.

Lemma: M_K is simply connected if $n \geq 4$.

Pf: For all $k < n-1$ this follows from the fact that we are doing surgery on attaching spheres of codim > 2:

$M_K \to$ remove $\partial(A^K) \to$ attach belt region $\to M_{K+1}$

For $K = n-1$, turn the decomp. upside down.

Now $M_{n-1} = N_2$, which is simply connected as long as $2 < n-1$, i.e. $n \geq 4$. $\square$

Lemma: $M_k - (A \cup B)$ is simply connected if $\boxed{n \geq 5}$

Pf: $\underline{2 < k < n-1}$

Then $\operatorname{codim}_{M_k} A$, $\operatorname{codim}_{M_k} B > 2$

$\Rightarrow$ removing A and B does not affect π_1.

$\underline{k = 2}$

⊛ $\operatorname{codim} A > 2$, so that's fine. (need $n \geq 5$)

As for B, we have that $M_2 - B \cong M_1 - (\text{att. sphere of } h^2)$

if h^2 is the unique 2-handle

codim ≥ 4, so it's fine

$\underline{k = n-1}$: turn it upside down $\square$ Lemma

Recall: γ obtained by concatenating paths between x and y

γ is homotopically trivial in $M_k - (A \cup B)$ by lemma.

⊛ By Whitney's embedding thm $(n \geq 2 \cdot \dim D + 1)$,

we can assume $D^2 \subseteq M_k$ embedded. (need $n \geq 5$).

rank $k-1$ bundle

$E_\gamma :=$ the subbundle of $N_{D|M_k}$ restricted to γ that is:

- •) tangent to A along $\partial_A D$
- •) normal to B along $\partial_B D$

normal bundle of the disc is trivial

E_γ defines a path in $Gr_{k-1}(\mathbb{R}^{n-2})$

It extends to all of D iff this path is null-homotopic.

⊛ If $n \geq 5$, $\pi_1(Gr_{k-1}(\mathbb{R}^{n-2})) = \mathbb{Z}/2\mathbb{Z}$

x, y opposite signs $\Leftrightarrow 0$ if path preserves orient.

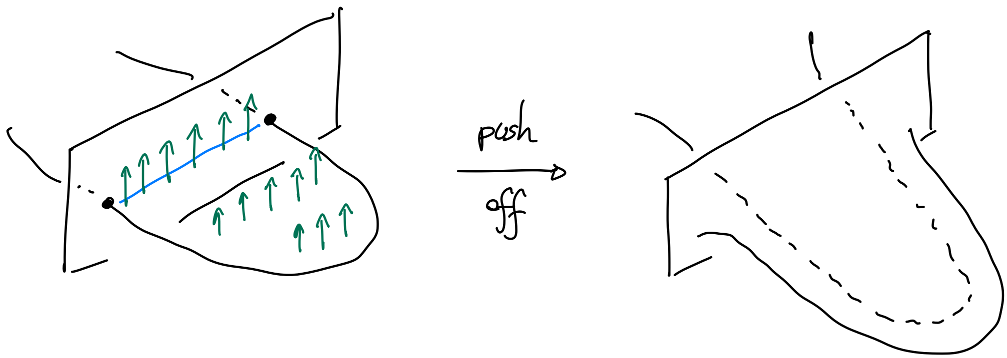
x, y same signs $\Leftrightarrow 1$ if path reverses orient.

RK: If $n = 4$, $k = 2$, $\pi_1(Gr_1(\mathbb{R}^2)) = \mathbb{Z}$,

so E_γ extends to E_D iff this path induces the 0 element.

The obstruction $n \in \mathbb{Z}$ is called FRAMING.

After extending E to the disc, we can define an isotopy of A supported in a neighbourhood of D that removes the two intersections



THE END $\square$

Rk: Remarkably, the proof of the h-cobordism theorem works also in TOP in dim 4 [Freedman 1982].
It is FALSE in PL/C^∞ [Donaldson 1987].

5 CONSEQUENCES

1. The generalised Poincaré conjectures

CAT = TOP, PL, or C^∞

<u>CAT h-cob (n)</u> = "CAT h-cobordism thm for $W^{n+1}: M^n \to N^n$"

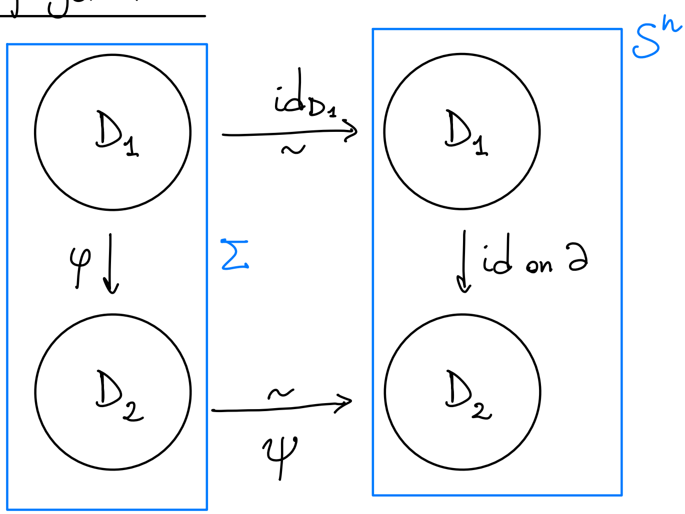
<u>CAT PC (n)</u> = " $\Sigma^n \underset{\text{htpy eq.}}{\sim} S^n \Rightarrow \Sigma \underset{CAT}{\simeq} S^n$ "

(Poincarè conjecture)

Def: A <u>CAT TWISTED SPHERE</u> is a mfd $\Sigma^n = D^n_1 \cup_\varphi D^n_2$, where $\varphi: S^{n-1} \circlearrowleft$ is a CAT-isom.

Thm (Alexander trick): If CAT = TOP or PL, then every twisted sphere is CAT-isomorphic to the std S^n.

<u>Pf for TOP</u>:



$$\psi: D_2 \longrightarrow D_2$$
$$x \longmapsto \|x\| \cdot \varphi^{-1}\left(\frac{x}{\|x\|}\right)$$
$$0 \longmapsto 0$$

ψ is a homeo with a singular pt at 0 $\square$

Rk: In PL you need to do it simplex by simplex (can still get singular points).

Rk: In C^∞ the Alexander trick is false.

Milnor showed that in dim 7 there are exactly 28 C^∞-inequivalent twisted spheres.

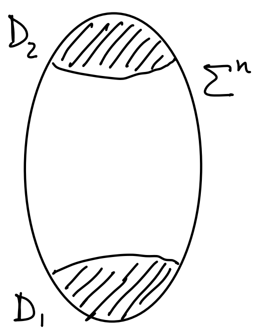
Thm (TOP/PL Poincaré conjecture)

The TOP/PL Poincaré conjecture holds in every $n \neq 4$.

Proof: $n = 0, 1, 2$: easy

$n = 3$: Perelman

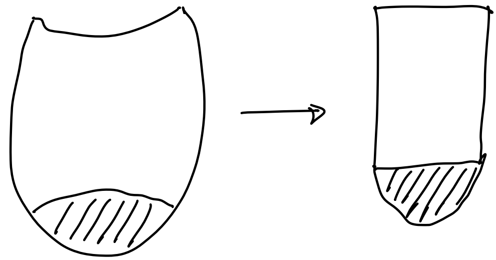
$\boxed{n \geq 6}$ We prove that every hty sphere Σ^n is a twisted sphere, then invoke Alexander's trick.



$\Sigma - (D_1 \cup D_2)$ is an h-cob. $S^{n-1} \to S^{n-1}$

$\underset{\text{h-cob.}}{\Longrightarrow}$ it is CAT-iso to product cob. + CAT-iso is id on the bottom.

Glue back D_1 with the identity map



$\Rightarrow \Sigma - D_2$ is a std disc

$\Longrightarrow \Sigma = (\Sigma - D_2) \cup D_2$ is a twisted sphere. $\square$

$\boxed{n=5}$ Σ admits a smooth structure [smoothing theory]

[Kervaire-Milnor, Wall] Every htpy S^5 bounds smooth, contractible X^6.

$X - D^6$ is an h-cob from S^5 to $\Sigma \underset{\text{h-cob}}{\Longrightarrow} S^5 \cong_{CAT} \Sigma$. $\square$

Thm: TOP PC (4) holds.

Rk: By contrast, PL PC (4) is open, and equivalent to C^∞ PC (4).

Sketch of proof: Let Σ be a TOP-mfd which is a htpy S^4.

Fact: Cone (Σ) is a TOP 5-mfd.

Apply Freedman's TOP h-cob theorem to Cone(Σ) - B^5 and get $\Sigma \cong_{TOP} S^4$. $\square$

Thm: If $n \neq 4$, every C^∞ hty S^n is a twisted sphere.

Pf: Same as TOP/PL PC, but without using the Alexander trick. $\square$

A panoramic view

$$\text{CAT h-cob}(n) \Rightarrow \begin{array}{c}\text{Every CAT}\\ \text{hty } S^{n+1}\\ \text{is twisted}\end{array} \xRightarrow[\text{CAT} = \text{TOP or PL}]{} \text{CAT PC}(n+1)$$

	0	1	2	3	4	5	6	7
h-cob	✓	✓	✓	TOP ✓ C^∞ ?	TOP ✓ C^∞ ✗	✓	✓	✓
Hty S^n ⇒ twisted	✓	✓	✓	✓	TOP ✓ C^∞ ?	✓ ⇑	✓	✓
PC	✓	✓	✓	✓	TOP ✓ C^∞ ?	✓	✓	TOP ✓ PL ✓ C^∞ ✗

all equivalent

$\text{PL} \sim C^\infty$

Rk: The following statements are equivalent:

1) C^∞ h-cob (3)

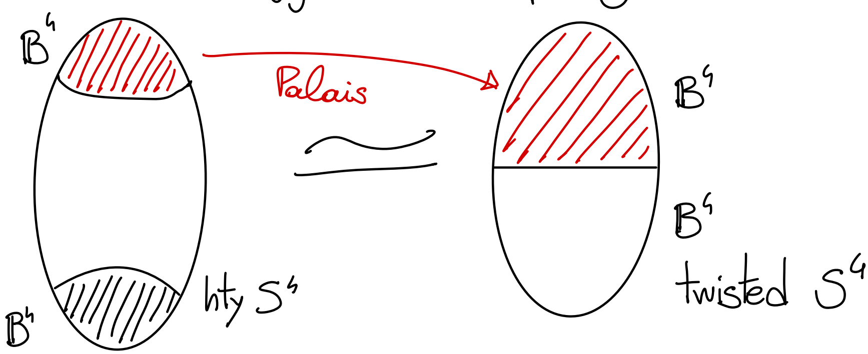
2) C^∞ hty $S^4 \Rightarrow$ twisted S^4

3) C^∞ PC (4)

Pf: $\underline{1 \Rightarrow 2}$: always true

$\underline{2 \Rightarrow 1}$: h-cob in dim 3 $\xrightarrow{\text{Perelman}}$ $M = N = S^3$.

Cap off with two copies of $\mathbb{B}^4$.



Thus, the h-cob is $\mathbb{B}^4 - \mathbb{B}^4 \approx I \times S^3$.

$\underline{3 \Rightarrow 2}$ obvious

$\underline{2 \Rightarrow 3}$ follows from a thm of Cerf, who proved that there are no non-trivial C^∞ twisted S^4

$(\pi_0(\mathrm{Diff}(S^3)) = 1)$ $\square$

Rk: C^∞ PC (4) $\Longleftrightarrow$ Every contractible smooth X^4 with $\partial X = S^3$ is $\cong_{C^\infty} \mathbb{B}^4$

Rk: C^∞ PC (7) is false (Milnor 1956)

Def: Θ_n = twisted spheres / diffeo

Θ_n measures the failure of C^∞ PC(n) for $n \neq 4$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Θ_n	1	1	1	1	1	1	1	$\mathbb{Z}_{28}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^{\oplus 4}$	$\mathbb{Z}_6$	$\mathbb{Z}_{992}$	1	$\mathbb{Z}_3$	$\mathbb{Z}_2$

2. Characterisation of discs

Thm: X^n cpt, CAT-mfd, $n \geq 6$, with $\pi_1(X) = \pi_1(\partial X) = 1$. Then the following are equivalent:

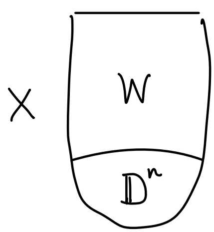
1) $X \simeq_{CAT} \mathbb{D}^n$

2) $X \underset{\text{hty}}{\sim} pt$

3) $H_*(X) = H_*(pt)$

Pf: $1 \Rightarrow 2 \Rightarrow 3$

$\underline{3 \Rightarrow 1}$ Let $W = \overline{X - \mathbb{D}^n}$.



By excision $H_*(W, S^{n-1}) = H_*(X, \mathbb{D}^n) = 0$.

$\Longrightarrow$ W is an h-cobordism

$\underset{\text{h-cob}}{\Longrightarrow}$ $W \underset{CAT}{\cong} I \times S^{n-1} \Rightarrow X \underset{CAT}{\cong} \mathbb{D}^n$. $\square$

Thm: X^5 cpt, $\pi_1 = 1$, $H_*(X) \cong H_*(pt)$. Then

$\partial X \cong_{CAT} S^4 \Longrightarrow X \cong_{CAT} \mathbb{D}^5$ [CAT $= C^\infty$ or TOP]

Pf: $\underline{C^\infty}$: Cap off with $\mathbb{D}^5$ and use CAT PC(5) and Palais.

$\underline{\text{TOP}}$: Remove $\mathbb{D}^5$ and use Freedman's top h-cob.

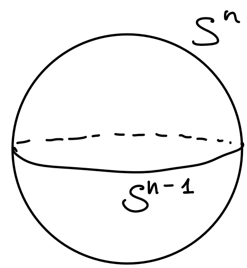
Then re-glue in $\mathbb{D}^5$.

3. Schoenflies problem

Thm (TOP Schoenflies, Brown 1960)

$S^{n-1} \hookrightarrow S^n$ TOP bicollared embedding.

Then, up to homeo, (S^n, S^{n-1}) is std.



Thm (C^∞ Schoenflies, $n \geq 5$)

$S^{n-1} \hookrightarrow S^n$ C^∞ embedding, $n \geq 5$.

Then, up to C^∞ ambient isotopy, (S^n, S^{n-1}) is std.

Pf: Σ is bicollared $\Rightarrow$ $S^n - \Sigma$ has 2 connected cpts, each of which simply connected, with $\partial = S^{n-1}$, and $H_* = H_*(pt)$.

By the characterisation of discs $\Sigma = \partial D^n$

$\underset{\text{Palais}}{\Longrightarrow}$ (S^n, S^{n-1}) std. $\square$

Rk: $S^{n-1} \hookrightarrow S^n$ in the end has image in the equator, but maybe the diffeo with the equator is non-standard.

Rk: C^∞ Schoenflies $n \leq 3$ is true, $n = 4$ is open.