Topological 4-manifolds

1 TOPOLOGICAL h-COBORDISM in dim 4

Freedman (1982) showed that the h-cobordism holds in TOP in dim 4 $\rightsquigarrow$ classification of $\pi_1 = 1$ closed 4-manifolds.

Thm (TOP h-COB dim 4)

Let $W^5 : M^4 \longrightarrow N^4$ TOP-cobordism with $\pi_1(W) = \pi_1(M) = \pi_1(N) = 1$ and $H_*(W, M; \mathbb{Z}) = 0$.

Then $W \cong_{TOP} I \times M$, and the homeo is id on $M \to \{0\} \times M$.

Sketch:

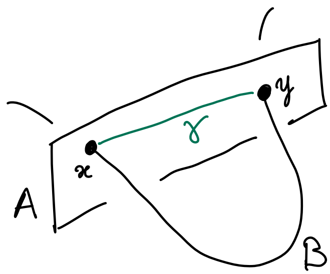
*) Take a TOP-handle dec. of W rel M (Quinn).

*) Rearrange handle by increasing index & trade all 0-, 1-, 4-, and 5-handles.

$\rightsquigarrow$ W consists just of 2- and 3-handles.

*) The handle chain complex is $0 \to C_3 \xrightarrow{\partial_3} C_2 \longrightarrow 0$, and since $H_*(W, M; \mathbb{Z}) = 0$, the map ∂_3 is an isomorphism $\rightsquigarrow$ after change of basis (achieved with handleslides) you get paired-up 2- and 3-handles.

*) h^2 and h^3 are algebraically complementary, but now the Whitney trick fails.



γ = path from x to y on A
$\cup$ path from y to x on B

M_2 (= the 4-mfd after the 2-handles & before the 3-handles) is still simply connected

$\rightsquigarrow$ $\exists$ <u>immersed</u> disc D in M_2 with boundary γ

<u>Problems to solve</u>

1) <u>W may intersect A and B</u>

We know that $\pi_1(M_2 - A) = \pi_1(M_2 - B) = 1$, by the same reasoning as in the std h-cobordism proof.

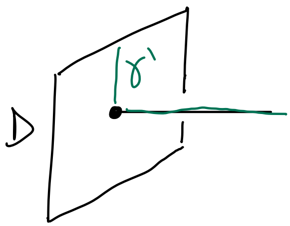
[Freedman] $\exists$ immersed spheres T_A and T_B that are geometrically dual to A and B, respectively $\rightsquigarrow$ Do some "Casson moves" and get a new collection of As and Bs such that

$\pi_1(M_2 - (As \cup Bs)) = 1$

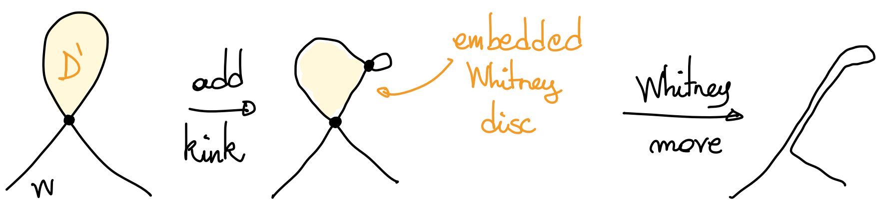
2) <u>D may have the wrong framing</u>

Take sums with an algebraically dual sphere T_D with $T_D \cdot T_D = \pm 1$ as necessary.

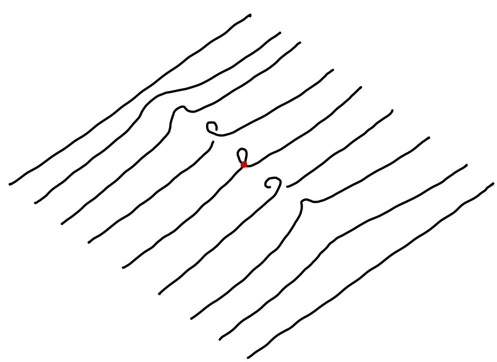
3) <u>D may have double points</u>



Near a double pt you have 2 branches
$\rightsquigarrow$ connect them with a loop γ'.
If $\gamma' = \partial D'$ in the complement of $(A \cup B \cup D)$,
embedded disc
then we can apply the Whitney trick & get rid of the double pt.



<u>How to create a kink</u> :

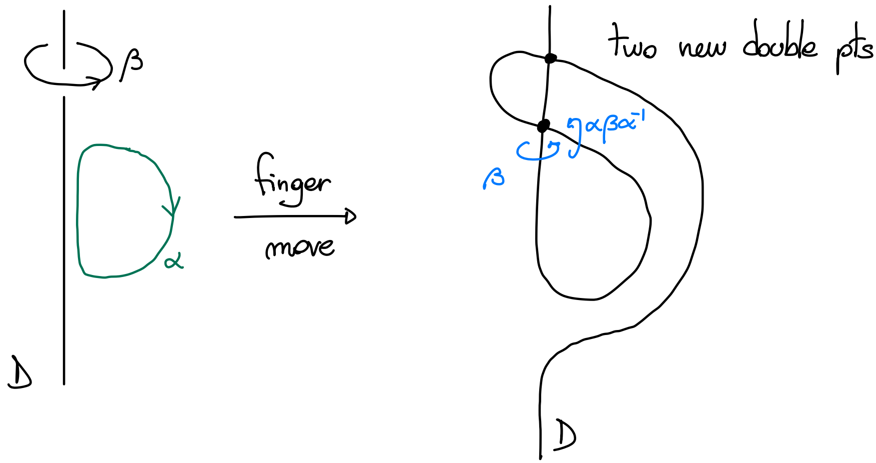


However, in finding D' we run in the same 3-problem as before.

1) <u>Make the complement of D simply connected</u> (so $\exists\, D'$ immersed)

<u>Fact</u>: the complement of D has perfect π_1 (i.e., generated by commutators of meridians of D).

These can be eliminated using finger moves.



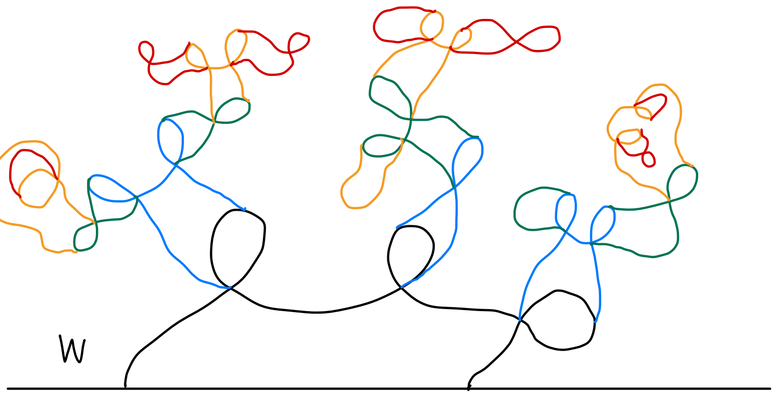
β and $\alpha\beta\alpha^{-1}$ commute in a nbd of the double pt

(e.g. $\exists$ torus $\underset{\beta}{S^1} \times \underset{\alpha\beta\alpha^{-1}}{S^1} \subseteq \nu(\text{double pt}) - D$, and its π_1 is abelian).

2) <u>Framing of D'</u>

Can be fixed with an algebraically dual sphere to D.

3) Double pts of D' : we push the problem to D'', etc...



By iterating the procedure construct an infinite object.

Thicken each step into an immersed $\mathbb{D}^2 \times \mathbb{R}^2$ and the final result is called a CASSON HANDLE C.

Thm (Casson '73) C is hty equivalent to $\mathbb{D}^2 \times \mathbb{R}^2$ rel ∂.

Thm (Freedman '81) C is homeomorphic to $\mathbb{D}^2 \times \mathbb{R}^2$ rel ∂.

$\Rightarrow$ C is a genuine topological Whitney disc, and can be used to cancel intersections between A and B. $\square$

② THE INTERSECTION FORM

Def: X^4 cpt oriented topological 4-mfd. Its intersection form is

$$Q_X : H^2(X,\partial X;\mathbb{Z}) \times H^2(X,\partial X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

given by $Q_X(\alpha,\beta) = \langle \alpha \cup \beta, [X] \rangle$ fundamental class

Properties

*) By Poincaré duality, Q_X is defined on $H_2(X;\mathbb{Z})$

*) Factors through $H^2(X,\partial X;\mathbb{Z}) / \mathrm{Tors}$

*) Change of basis $[Q_X] = C^T \cdot [Q_X]' \cdot C$
for $C \in GL_n(\mathbb{Z})$ (so $\det C = \pm 1$)
$\Rightarrow \det Q_X$ is well-defined.

*) $Q_{-X} = -Q_X$.

*) If X is not orientable, you can still define it over $\mathbb{Z}/2\mathbb{Z}$.

Prop: Let X be a cpt oriented smooth / topological 4-mfd
Then every $\alpha \in H_2(X;\mathbb{Z})$ can be represented by a smoothly / locally flatly embedded closed surface.
near each pt it looks like $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ in charts
$(x,y) \mapsto (x,y,0,0)$

This proposition follows from general results about representing homology classes in low (co-)dimension.

Prop. also works for:

*) $H_2(X,\partial X)$ and properly embedded surfaces with boundary

*) non-orientable Σ and $\mathbb{Z}/2\mathbb{Z}$ coefficients

*) non-cpt X and non-cpt surfaces

<u>Sketch of a proof for C^∞, closed, oriented X</u>

$$\begin{array}{ccc} \{U(1)\text{-bundles}/X\}/_{\sim} & \xrightarrow[\sim]{c_1} & H^2(X;\mathbb{Z}) \\ & & \Psi \\ L_\alpha \longrightarrow X & \longmapsto & \alpha \end{array}$$

Pick the 0-section s_0 and a generic section. Then $[s_0 \pitchfork s] = PD(\alpha)$.

<u>Thm</u> (geometric interpretation of Q_X)

Let X be cpt, oriented, smooth, $\alpha,\beta \in H^2(X,\partial X)$.

If $[\Sigma_\alpha],[\Sigma_\beta] \in H_2(X)$ are the Poincaré duals, then

smooth surfaces

$$Q_X(\alpha,\beta) = \#(\Sigma_\alpha \pitchfork \Sigma_\beta)$$

algebraic count

Pf: α is represented by a 2-form η_α supported in a nbd of Σ_α. In coordinates, if $\Sigma_\alpha = \{x=y=0\}$, η_α can be chosen as

$$\eta_\alpha = f(x,y)\, dx \wedge dy$$

where $f(\cdot,\cdot)$ is a bump function near 0 w/ $\int_{\mathbb{R}^2} f(x,y) = 1$.
Analogously, choose a similar η_β for β.

$$\int_M \alpha \cup \beta = \sum_{p \in \Sigma_\alpha \cap \Sigma_\beta} \int_{\nu(p)} f(x,y) \cdot f(z,w) \cdot (\pm dx \wedge dy \wedge dz \wedge dw)$$

sign depends on sign of intersection

$$= \#(\Sigma_\alpha \pitchfork \Sigma_\beta) \qquad \square$$

③ EXAMPLES of INTERSECTION FORM

Q_{S^4}: $H_2(S^4) = 0$, so nothing interesting here.

$Q_{\mathbb{CP}^2} = (1)$, i.e. $H_2(\mathbb{CP}^2) = \mathbb{Z}$, and the matrix representing $Q_{\mathbb{CP}^2}$ is $[1]$.

$Q_{\overline{\mathbb{CP}^2}} = (-1) \rightsquigarrow Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$Q_{S^2\times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: H$$

Rk: $Q_{S^2\times S^2} \otimes \mathbb{R} \cong Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}} \otimes \mathbb{R}$, but over $\mathbb{Z}$ they are different. For example:

- $\forall \alpha \in H_2(S^2\times S^2)$, $Q_{S^2\times S^2}(\alpha,\alpha) \equiv 0 \pmod 2$
- the same is not true for $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

even

In terms of invariants, we say that $Q_{S^2\times S^2}$ and

odd

$Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}}$ have different parity, while they have the same rank and signature and they are both indefinite.

Thm: $S^2\times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ are the only two S^2-bundles over S^2.

Idea: All S^2-bundles over $\mathbb{D}^2$ are trivial $\rightsquigarrow$ it only depends on the gluing, which is an S^2-bundle over S^1.

The possible results are classified by

$$\pi_1(\mathrm{Diff}^+(S^2)) \cong \pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}.$$

blow up a pencil of lines in $\mathbb{CP}^2$ and get a $\mathbb{CP}^1$-fibration

$S^2\times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ are both S^2-bundles over S^2, but they are different because the inters. forms are different. $\square$

K3 surface

Def: A <u>K3 surface</u> is a smooth, simply-connected complex surface with $c_1 = 0$.

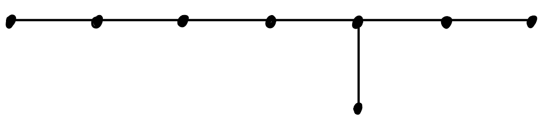
e.g. $\{x^4 + y^4 + z^4 + w^4 = 0\} \subseteq \mathbb{CP}^3$

Fact: All K3 surfaces are diffeomorphic (as real 4-mfds).

$$Q_{K3} = 2 \cdot \begin{bmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & & 1 \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & \\ & & & & 1 & & & -2 \end{bmatrix} \oplus 3 \cdot H$$

E_8 matrix

The matrix E_8 is the adjacency matrix of the E_8 graph:



<u>Remarks:</u>

•) $\det E_8 = +1$ (K3 is <u>closed</u>)

•) E_8 (and Q_{K3}) is even

•) E_8 is neg. definite $\Rightarrow \sigma(K3) = -16$, $rk(K3) = 22$.

Invariants of Q_X

- RANK: $\operatorname{rk} Q_X = \operatorname{rk}_{\mathbb{Z}}(\Lambda)$
- SIGNATURE: $\sigma(X) = \sigma(Q_X \otimes_{\mathbb{Z}} \mathbb{R})$

  We also define $b_2^{\pm}$ as the ranks of maximal $(\pm)$-definite subsp.
- DEFINITENESS: Q_X is $(\pm)$-definite or indefinite

  [for mfds w/ ∂ you can have semidefinite] $Q_X(x,x)>0$ $\forall x \neq 0$
- PARITY

  Q_X is EVEN if $Q_X(x,x) \equiv 0 \pmod 2$ $\forall x \in \Lambda$

  Q_X is ODD otherwise.

Rk: Q_X is even iff all diagonal entries of $[Q_X]$ are even.

4 PROPERTIES of Q_X

Unimodularity

Def: Let $\Lambda = \mathbb{Z}^n$, $Q : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ sym. bil. form.

Q is called UNIMODULAR if

$$\begin{aligned} L : \Lambda &\longrightarrow \Lambda^* = \operatorname{Hom}(\Lambda; \mathbb{Z}) \\ x &\longmapsto Q(x, \cdot) \end{aligned}$$

is an isomorphism.

Lemma: Given a basis b for Λ and its dual basis b^* for Λ^*, the matrices ${}_b[Q]_b$ and $[L]_b^{b^*}$ are the same.

Cor: Q unimodular $\Leftrightarrow$ $\det Q = \pm 1$.

Pf: $\{x_1, \dots, x_n\}$ basis of Λ

$\{x_1^*, \dots, x_n^*\}$ dual basis for Λ^* (i.e. $x_i^*(x_j) = \delta_{ij}$)

Then $L(x_i) = \sum Q(x_i, x_j)\, x_j^*$, so the matrix $[L]$ is the same as $[Q]$ in these bases. $\square$

Lemma: If Q_X is the int. form of X and $Y = \partial X$, then

$$\begin{tikzcd} H_2(X)/\mathrm{Tors} \arrow[r] \arrow[d, dashed] & \mathrm{Hom}(H_2(X);\mathbb{Z}) \arrow[d, phantom, "\text{\rotatebox{90}{$\cong$}}\ \mathrm{UCT}"] \\ H_2(X,Y;\mathbb{Z})/\mathrm{Tors} \arrow[r, phantom, "\underset{PD}{\cong}"] & H^2(X;\mathbb{Z})/\mathrm{Tors} \end{tikzcd}$$

is the natural map induced by $H_2(X) \xrightarrow{i_*} H_2(X,Y)$.

Proof: $x \mapsto \alpha_x \in H^2(X)/\mathrm{Tors}$ defined by $\alpha_x(y) = Q_X(x,y)$.

Then, $\alpha_x(y) \overset{!}{=} (PD(x))(y)$, and $\alpha_x(y) = Q_X(x,y) = \#(\Sigma_x \pitchfork \Sigma_y) = \int_{\Sigma_y} PD(x)$

surfaces repr. x and y $\square$

Thm: If $H_1(X;\mathbb{Z})=0$, then $[Q_X]$ presents $H_1(Y)$.

In any case, $H_1(Y)=0 \Rightarrow Q_X$ unimodular.

includes the case of X closed

Pf: Consider the LES

$$\to H_2(X) \xrightarrow{\pi_*} H_2(X,Y) \longrightarrow H_1(Y) \longrightarrow H_1(X) \to$$

If $H_1(X)=0$, then

$$\mathrm{Tors}(H_2(X,Y)) \underset{PD}{\cong} \mathrm{Tors}(H^2(X)) \underset{UCT}{\cong} \mathrm{Tors}(H_1(X)) = 0,$$

so we get an exact sequence

$$H_2(X)/\mathrm{Tors} \xrightarrow{\pi_*} H_2(X,Y)/\mathrm{Tors} \longrightarrow H_1(Y) \longrightarrow 0$$

$\pi_* = L_X$ after removing torsion, and is represent. by $[Q_X]$.

If $H_1(Y)=0$ (with no assumption on $H_1(X)$), then by exactness L_X is surjective, hence injective:

$$0 \to \mathrm{Ker} f \to \mathbb{Z}^n \longrightarrow \mathbb{Z}^n \longrightarrow 0$$

free, so SES splits

$$\Rightarrow \mathbb{Z}^n \cong \mathbb{Z}^n \oplus \mathrm{Ker} f \Rightarrow \mathrm{Ker} f = 0 \qquad \square$$

Rk: $\mathrm{Ann}(Q_X) = \ker(L_X) = i_*(H_2(Y)) \subseteq \dfrac{H_2(X)}{\mathrm{Tors}}$.

Connected sum

Thm: Given cpt top. mfds X_1 and X_2, $Q_{X_1 \# X_2} \cong Q_{X_1} \oplus Q_{X_2}$.

Pf: Removing a B^4 and gluing along an S^3 does not change the 2^{nd} homology (& intersection form).

Rk: In TOP the converse (for $\pi_1 = 1$) manifolds holds: if $\pi_1(X) = 1$ and $Q_X \cong Q_1 \oplus Q_2$, then $\exists$ TOP mfds X_1 and X_2 s.t. $Q_{X_i} \cong Q_i$ and $X \cong_{TOP} X_1 \# X_2$.

Rk: The converse does not hold in C^∞.

For example Q_{K3} splits an H, but $K3 \not\cong X_1 \# X_2$ with $b_2^+(X_i) \geq 1$. [Use mixed invariant.]

Thm (Freedman-Taylor)

Let X^4 smooth cpt, $\pi_1(X) = 1$ and $Q_X \cong Q_1 \oplus Q_2$. Then $\exists$ smooth X_1 and X_2 s.t. $Q_{X_i} \cong Q_i$ and $X \cong_{C^\infty} X_1 \cup_Y X_2$, where Y is a $\mathbb{Z}HS^3$.

A 3-mfd w/ the same $\mathbb{Z}$-homology as S^3

Signature

Thm (Novikov's additivity)

Let X_1, X_2 cpt oriented mfd, with $\partial X_1 \cong \overline{\partial X_2}$.

Then $\sigma(X_1 \cup_\partial X_2) = \sigma(X_1) + \sigma(X_2)$.

Pf: Use $\mathbb{Q}$ homology throughout, and let $Y = \partial X_1$, $X = X_1 \cup_\partial X_2$.

Mayer-Vietoris gives us a first exact sequence

$$\longrightarrow H_2(Y) \xrightarrow{(i_1,-i_2)} H_2(X_1) \oplus H_2(X_2) \xrightarrow{j} H_2(X) \xrightarrow{\partial} H_1(Y) \longrightarrow \quad (*)$$

If $K_\ell := \operatorname{Ker}\left(H_2(Y) \xrightarrow{i_\ell} H_2(X_\ell)\right)$, then the sequence below is exact too (exercise):

$$0 \longrightarrow K_1 + K_2 \longrightarrow H_2(Y) \xrightarrow{(i_1,i_2)} \frac{H_2(X_1) \oplus H_2(X_2)}{\operatorname{im}(i_1,-i_2)} \longrightarrow$$

$$\longrightarrow \frac{H_2(X_1)}{\operatorname{im}(i_1)} \oplus \frac{H_2(X_2)}{\operatorname{im}(i_2)} \longrightarrow 0 \qquad (**)$$

Lemma: There is a subspace $C \subseteq H_2(X)$ such that

$$H_2(X) \cong \left(\frac{H_2(Y)}{K_1+K_2} \oplus C \right) \overset{\perp}{\oplus} \frac{H_2(X_1)}{\mathrm{im}(i_1)} \overset{\perp}{\oplus} \frac{H_2(X_2)}{\mathrm{im}(i_2)}$$

orthogonal direct sums

The maps $\frac{H_2(X_j)}{\mathrm{im}(i_j)} \hookrightarrow H_2(X)$ and $\frac{H_2(Y)}{K_1+K_2} \hookrightarrow H_2(X)$ are induced by the inclusions, up to multipl. by a nonzero scalar.

Under the above identification, we have that

$$\mathrm{Ker}\left(\partial : H_2(X) \longrightarrow H_1(Y)\right) \cong \frac{H_2(Y)}{K_1+K_2} \oplus \frac{H_2(X_1)}{\mathrm{im}(i_1)} \oplus \frac{H_2(X_2)}{\mathrm{im}(i_2)}$$

and therefore $\partial : C \xrightarrow{\sim} \mathrm{im}\, \partial$ is an isomorphism.

[Note that by the lemma $K_1 + K_2 = \mathrm{Ker}\left(i : H_2(Y) \to H_2(X)\right)$.]

Proof lemma

From (**) we have

$$\frac{H_2(X_1) \oplus H_2(X_2)}{\mathrm{im}(i_1, -i_2)} \cong \frac{H_2(Y)}{K_1+K_2} \oplus \frac{H_2(X_1)}{\mathrm{im}(i_1)} \oplus \frac{H_2(X_2)}{\mathrm{im}(i_2)}$$

Since $\mathrm{Ann}(Q_{X_j}) = \mathrm{im}(i_j)$, the intersection form induced on

$$W := \frac{H_2(X_1)}{\mathrm{im}(i_1)} \oplus \frac{H_2(X_2)}{\mathrm{im}(i_2)}$$

is non-singular, so we have a splitting $H_2(X) \cong W \oplus W^{\perp}$.

Using ⊛ we have that the Kernel of the map

$$\partial : \underbrace{H_2(X)}_{\overset{\shortparallel}{W \oplus W^{\perp}}} \twoheadrightarrow \mathrm{im}(\partial) \subseteq H_1(Y)$$

is exactly

$$\mathrm{Ker}\,\partial \cong \overbrace{\frac{H_2(Y)}{K_1+K_2}}^{\subseteq W^{\perp}} \oplus \underbrace{\frac{H_2(X_1)}{\mathrm{im}(i_1)} \oplus \frac{H_2(X_2)}{\mathrm{im}(i_2)}}_{= W}$$

We leave it to the reader to check that the maps

$$\frac{H_2(X_j)}{\mathrm{im}(i_j)} \hookrightarrow H_2(X) \quad \text{and} \quad \frac{H_2(Y)}{K_1+K_2} \hookrightarrow H_2(X)$$

are induced by the inclusion and $2 \cdot i$ (twice the inclusion), respectively.

Thus, if we choose a complement C of $\dfrac{H_2(Y)}{K_1+K_2}$ in $W^{\perp}$, we have that $\partial : C \xrightarrow{\sim} \mathrm{im}\,\partial$ is an isomorphism. $\square$ Lemma

Claim: The restriction of the pairing

$$Q_X : \frac{H_2(Y)}{K_1+K_2} \times C \longrightarrow \mathbb{Z}$$

is non-singular. In particular, $\dim\left(\frac{H_2(Y)}{K_1+K_2}\right) = \dim C$.

Pf: Let $x \in H_2(Y) \setminus (K_1+K_2)$. By the lemma $i_*(x) \neq 0$, so $\exists\, \bar{y} \in H_2(X)$ dual to it, i.e. $Q_X(i_*(x), \bar{y}) = 1$.

Since $i_*(x) \in W^\perp$, we can assume that $\bar{y} \in W^\perp$ too.

Moreover, since Q_X restricts to 0 on $i_*(H_2(Y))$, we can choose $\bar{y} \in C$.

because one element can be pushed off into a collar of Y

Vice versa, given $\bar{y} \in C$, let x be a dual of $\partial\bar{y} \in H_1(Y)$, defined by $x \cdot \partial\bar{y} = 1$ in Y.

Then $Q_X(i_*(x), \bar{y}) = x \cdot \partial\bar{y} = 1$, because the pairing is happening in Y. $\square$ Claim

Pick basis $\{x_i\}$ for $\frac{H_2(Y)}{K_1+K_2}$ and a dual basis $\{\bar{y}_i\}$ for C.

Then $Q_X|_{\mathrm{span}\{x_i, \bar{y}_i\}} = \begin{bmatrix} 0 & 1 \\ 1 & * \end{bmatrix}$, because $x_i \subseteq Y$, so it is 0-framed (can be pushed off in a collar of Y).

Thus, we get contributions to $\sigma(X)$ only from

$$\frac{H_2(X_1)}{\mathrm{im}(i_1)} \quad \text{and} \quad \frac{H_2(X_2)}{\mathrm{im}(i_2)}$$

<u>Rk</u>: $\sigma\left(\frac{H_2(X_\ell)}{\mathrm{im}(i_\ell)}\right) = \sigma(H_2(X_\ell))$, because

$$\mathrm{im}(i_\ell) = \mathrm{Ann}(Q_{X_\ell}).$$

Thus, $\sigma(X) = \sigma(X_1) + \sigma(X_2)$ $\square$

<u>Rk</u>: The additivity theorem fails if X_1 and X_2 are glued along <u>part</u> of their boundaries (otherwise we would get $\sigma(X) = 0$ $\forall X^4$ smooth just by attaching handles, which have $\sigma = 0$).

<u>Rk</u>: On the other hand, the additivity thm still works if we glue X_1 and X_2 along some boundary components.

Signature and cobordisms

Thm: Two closed oriented 4-mfds X and X' are cobordant if and only if $\sigma(X) = \sigma(X')$.

(5) CLASSIFICATION

Indefinite unimodular forms

Meyer's lemma: Let $Q : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ bil. sym. unimodular form.
If Q is indefinite, then $\exists\, x_0 \in \Lambda$ s.t. $Q(x_0, x_0) = 0$.
[The difficult part here is going from $\mathbb{R}$ to $\mathbb{Q}$.]

Meyer's lemma is used to prove the following classif. result.

Thm (Serre) Q and Q' sym bil. unimodular form.
Suppose that Q and Q' are indefinite. Then

$$Q \cong Q' \iff \begin{cases} \operatorname{rk} Q = \operatorname{rk} Q \\ \sigma(Q) = \sigma(Q') \\ \text{same parity (both even or both odd)} \end{cases}$$

Odd case:

$$\boxed{\begin{array}{l} Q \ \underset{\text{unimod.}}{\overset{\text{odd}}{\text{indef.}}} \Rightarrow \ Q \cong a^+ \cdot (1) \oplus a^- \cdot (-1) \\ \text{where} \quad a^\pm = \dfrac{\operatorname{rk} Q \pm \sigma(Q)}{2} \end{array}}$$

[Note that $\operatorname{rk} Q \equiv \sigma(Q) \pmod 2$ if $\operatorname{Ann} Q = 0$.]

Even case: there is an extra algebraic restriction here:

$$\sigma(Q) \equiv 0 \pmod 8.$$

This follows from the more general Van der Blij's lemma.

Def: Given a symmetric bilinear unimodular form (Λ, Q), an element $w \in \Lambda$ is called:

-) CHARACTERISTIC if $\forall y \in \Lambda$
$$Q(w,y) \equiv Q(y,y) \pmod 2$$
-) ORDINARY otherwise

Thm (Van der Blij's lemma)

(Λ, Q) sym. bil. unimodular, $w \in \Lambda$ characteristic. Then

$$Q(w,w) \equiv \sigma(Q) \pmod 8$$

Cor: Q even unimodular $\Rightarrow \sigma(Q) \equiv 0 \pmod 8$.

Using Van der Blij's lemma, we have the following classif.

$$\boxed{Q \ \begin{matrix}\text{even}\\ \text{indef.}\\ \text{unimod.}\end{matrix} \Rightarrow Q \cong b\cdot E_8 \oplus c\cdot H \qquad \text{where} \quad b = -\frac{\sigma(Q)}{8}, \quad c = \frac{\operatorname{rk} Q - |\sigma(Q)|}{2}}$$

(Here if $b<0$ change all the signs of the matrix E_8.)

Definite unimodular forms

Too many!

If Q is even, we know that $\operatorname{rk} Q = \pm\sigma(Q) \equiv 0 \pmod 8$.

$\underline{\operatorname{rk} Q = 8}$: only E_8 (up to sign, here and below)

$\underline{\operatorname{rk} Q = 16}$: $E_8 \oplus E_8$, E_{16}

$\underline{\operatorname{rk} Q = 24}$: 24 different forms

$\underline{\operatorname{rk} Q = 32}$: > 80 millions

$\underline{\operatorname{rk} Q = 40}$: $> 10^{51}$

For odd definite forms it's much worse.

Topological 4-manifolds

Fact (Kirby-Siebenmann): given a closed, cnct, topological 4-mfd X, $\exists\ ks(X) \in \mathbb{Z}/2\mathbb{Z}$ s.t.:

- $ks(X_1 \# X_2) = ks(X_1) + ks(X_2)$
- X admits a smooth structure $\Rightarrow ks(X) = 0$.

In general, for a topological n-mfd, $ks(X) \in H^4(X, \mathbb{Z}/2\mathbb{Z})$ is the primary obstruction to endowing X with a PL structure. For $n \geq 5$ this obstruction is complete, and the PL structures (if $\exists$) are classified by $H^3(X; \mathbb{Z}/2\mathbb{Z})$.
For $n = 4$ it is only an obstruction.

Thm (Freedman '82) Let Q be bil. sym. unimod. form.

- Q even $\Rightarrow \exists!\ X$ TOP-mfd w/ $Q_X \cong Q$.
- Q odd $\Rightarrow \exists$ exactly 2 TOP-mfd w/ $Q_X \cong Q$, distinguished by their ks invariant.