Smooth 4-manifolds

① PLUMBING and X_{E_8}

Freedman $\Rightarrow$ $\exists!$ top 4-manifold X_{E_8} w/ $\pi_1 = 1$ and $Q_{M_{E_8}} \cong E_8$.

Today: X_{E_8} does not admit a smooth structure (Rokhlin).

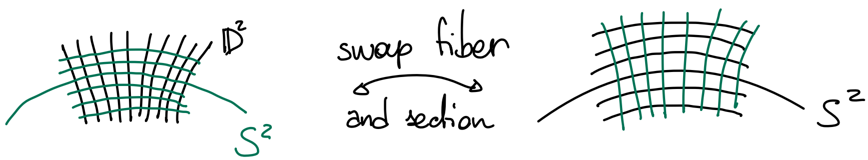
An explicit construction of X_{E_8}

Def: A weighted labelled graph is a finite graph Γ with

- each vertex $v \in V(\Gamma)$ has a weight $n_v \in \mathbb{Z}$;
- each edge $e \in E(\Gamma)$ has a label $+$ or $-$.

Def: The plumbed manifold P_Γ is obtained by the following:

- for each $v \in V(\Gamma)$, take the $\mathbb{D}^2$-bundle over S^2 with Euler number n_v [these are classified by $H^2(S^2;\mathbb{Z}) \cong \mathbb{Z}$];
- for each $e \in E(\Gamma)$, glue the disc bundles corresponding to the vertices of e identifying the disc bundles over $\mathbb{D}^2 = \nu(pt)$:

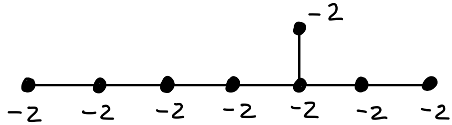


The map $\mathbb{D}^2 \times \mathbb{D}^2 \longrightarrow \mathbb{D}^2 \times \mathbb{D}^2$ is the swap composed with (id, id) or (r, r), depending on the sign of e.

reflection

Ex: Q_{P_Γ} is represented by the adjacency matrix of Γ (n_v on the diagonals, $\#(+\text{ labels}) - \#(-\text{ labels})$ off diag.).

Rk: If Γ is a tree, the label $+/-$ of the edges doesn't matter, because you can change the orientation of each S^2 so that all intersections are positive.

Start with $\Gamma =$ 

E_8 weighted graph

$\rightsquigarrow$ P_{E_8} 4-manifold with boundary.

det = 1

$H_1(\partial P_{E_8};\mathbb{Z}) = 0$, since it is presented by E_8

Ex: A closed connected Y^3 with $H_1(Y;\mathbb{Z}) = 0$ is a $\mathbb{Z}HS^3$ (INTEGER HOM. SPHERE), i.e. $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$.

Rk: ∂P_{E_8} is a $\mathbb{Z}HS^3$ called <u>Poincaré homology sphere</u>.

Thm (Freedman) Every $\mathbb{Z}HS^3$ bounds a contractible <u>topological</u> 4-manifold (a.k.a. a "fake ball").

Def: The <u>E_8 MANIFOLD</u> is the topological 4-mfd

$$X_{E_8} = P_{E_8} \cup (\text{fake ball})$$

Mayer-Vietoris $\Rightarrow Q_{X_{E_8}} \cong Q_{P_{E_8}} \cong E_8$.

Rokhlin's theorem implies that X_{E_8} is not smoothable.

② ROKHLIN'S THEOREM

Wu's formula: $\forall\, y \in H_2(X)$ we have

$$\langle w_2(TX),\, y\rangle \equiv Q_X(y,y) \pmod 2$$

Recall that $x \in H_2(X)$ is characteristic if $Q_X(x,y) \equiv Q_X(y,y) \pmod 2$. Thus, for closed oriented X,

$$PD(x) \equiv w_2(TX) \pmod 2 \quad \underset{\substack{\text{if } H_1(X) \text{ has} \\ \text{no 2-torsion}}}{\overset{\text{always}}{\rightleftarrows}} \quad x \text{ charact.}$$

[$H_1(X)$ with no 2 torsion $\overset{UCT}{\Rightarrow}$ every class in $H_2(X;\mathbb{Z}/2\mathbb{Z})$ has an integral lift to $H_2(X;\mathbb{Z})$, and hence it is represented by a closed oriented surface.]

Spin 4-mfds: Let X be a $\pi_1 = 1$ smooth 4-mfd.

Q_X even $\Leftrightarrow 0$ is characteristic $\Leftrightarrow w_2(TX) = 0$ ($\Leftrightarrow X$ spin)

($\Rightarrow$ if $H_1(X)$ has no 2-torsion)

Lemma: X^4 closed oriented. Then $\exists$ a characteristic class.

[More generally, $\forall\, X^4$ oriented $w_2(TX)$ has an integral lift.]

uses only unimodularity

Pf (algebraic)

Let $Q : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ be the sym, bil., unimod inter. form.

Reduce it mod 2 to get

$$Q'' : \Lambda'' \times \Lambda'' \longrightarrow \mathbb{F}_2$$

Unimodularity over $\mathbb{F}_2 \Longrightarrow$ every $f \in \mathrm{Hom}_{\mathbb{F}_2}(\Lambda'', \mathbb{F}_2)$ is represented by some $Q''(\alpha_f, \bullet)$.

$q'' \in \mathrm{Hom}(\Lambda'', \mathbb{F}_2)$ defined by $q''(x) = Q''(x,x)$:

$$Q''(\alpha+\beta, \alpha+\beta) = Q''(\alpha,\alpha) + Q''(\beta,\beta) + \underbrace{\cancel{2Q(\alpha,\beta)}}_{0}$$

Thus, $\exists\, w \in \Lambda'' = \Lambda \otimes \mathbb{F}_2$ with

$$Q''(w, x) = Q''(x,x) \quad \forall x \in \Lambda'' \qquad \square$$

Thm (Van der Blij's lemma)

Let $Q : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ be a sym. bil. unimod. form, w charact. Then $w \cdot w \equiv \sigma(Q) \pmod 8$.

Cor: A closed, oriented, even 4-mfd has $\sigma(Q_X) \equiv 0 \pmod 8$.

Thm (Rokhlin) A closed oriented SPIN ($\Leftarrow$ even, $\pi_1 = 1$, C^∞) 4-mfd has $\sigma(Q_X) \equiv 0 \pmod{16}$.

<u>Application</u>: X_{E_8} is closed, oriented, even, and $\pi_1 = 1$ (Van Kampen).
Since $\sigma(X_{E_8}) = -8$, it is not smoothable.

Rokhlin's invariant

Given Y $\mathbb{Z}HS^3$, there exists a simply connected spin 4-mfd X with $\partial X = Y$.

<u>Def</u>: $\mu(Y) := \frac{\sigma(X)}{8} \pmod 2$

Rokhlin's thm implies that $\mu(Y)$ is well-defined.

<u>Idea of Rokhlin's theorem</u>

*) $\mathrm{Spin}^{(c)}$ structures on manifolds (tomorrow)

*) $\Omega_n^{\mathrm{Spin}^{(c)}} = \{(M^n, s) \mid s \in \mathrm{Spin}^{(c)}(M)\} / \mathrm{Spin}^{(c)}$ cobordism

*) $\Omega_4^{\mathrm{Spin}^c} \cong \Omega_4^{\mathrm{char}} = \{(X^4, \Sigma^2) \mid [\Sigma] \text{ char.}\} / \text{char. bordism}$ oriented

$[(X,\Sigma) \sim (X',\Sigma')$ if $\exists (W^5, Y^3)$ cobordism and $[Y] = PD(w_2(TW))]$ all oriented

*) $\vartheta : \Omega_4^{\mathrm{char}} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is an isom.

$(X, \Sigma) \longmapsto \left(\sigma(X), \frac{1}{8}(\Sigma\cdot\Sigma - \sigma(X))\right)$

The two canonical generators are $(\mathbb{CP}^2, \mathbb{CP}^1) \longmapsto (1, 0)$
and $(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}, 3\cdot\mathbb{CP}^1 \# \overline{\mathbb{CP}^1}) \longmapsto (0, 1)$

*) For $\Sigma \subseteq X$ characteristic, $\exists \; \mathrm{Arf}(X,\Sigma) \in \mathbb{Z}/2\mathbb{Z}$ invariant under char. cob. & which depends only on $[\Sigma] \in H_2(X)$.
By evaluating it on the generators above we get

$$0 = \mathrm{Arf}(X,0) \equiv \frac{1}{8}\left(\Sigma\cdot\Sigma - \sigma(X)\right) \pmod 2.$$

③ THE ARF INVARIANT

Algebraic Arf

Let H be a finite-dim. vector space $/\mathbb{F}_2$, and let $H \times H \longrightarrow \mathbb{F}_2$ be a unimodular symmetric bilinear form, denoted $(x,y) \mapsto x\cdot y$, and often called "intersection form".

Def: A <u>QUADRATIC ENHANCEMENT</u> of $\cdot$ is a map

$$q : H \longrightarrow \mathbb{F}_2$$

satisfying $q(x+y) = q(x) + q(y) + x\cdot y \quad \forall x,y \in H$.

<u>Rk</u>: If $\exists$ a quadratic enhancement, then $x\cdot x = 0 \quad \forall x \in H$.
Thus H has a symplectic basis $\{x_1, y_1, \dots, x_n, y_n\}$, i.e. one for which $x_i \cdot x_j = 0$ and $x_i \cdot y_j = \delta_{ij}$.
In particular, $\dim H = 2n$ is <u>even</u>.

Idea: Unimodularity $\Rightarrow \forall x \neq 0 \ \exists y \in H$ s.t. $x \cdot y = 1$.

Then $W = \langle x, y \rangle$ has int. form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (unimod.), so $\dim W = 2$ and $H = W \oplus W^{\perp}$. Restrict to $W^{\perp}$ and proceed by induction.

Examples: $H = \mathbb{F}_2 \langle x, y \rangle = \{0, x, y, x+y\}$ with $x \cdot y = 1$.

$\exists$ two quadratic enhancements up to isomorphism, called $H^{0,0}$ and $H^{1,1}$.

q	$H^{0,0}$	$H^{1,1}$
0	0	0
x	0	1
y	0	1
$x+y$	1	1

Rk: $H^{0,0} \oplus H^{0,0} \cong H^{1,1} \oplus H^{1,1}$ (by a change of basis)

Lemma: (H, q) quadratic enhancement. Then

$$(H,q) \cong \begin{cases} n \cdot H^{0,0} & \overset{\text{def.}}{\Longrightarrow} \ \mathrm{Arf}(H,q) = 0 \\ (n-1) \cdot H^{0,0} \oplus H^{1,1} & \overset{\text{def.}}{\Longrightarrow} \ \mathrm{Arf}(H,q) = 1 \end{cases}$$

Pf: Pick a sympl. basis of $H \rightsquigarrow H \cong H_1 \oplus \dots \oplus H_n$.

Each H_i is isomorphic to either $H^{0,0}$ or $H^{1,1}$.

Trade pairs of $H^{1,1}$ until you have at most one. $\square$

We still do not know that $\mathrm{Arf} = 0$ and $\mathrm{Arf} = 1$ are mutually exclusive. This is guaranteed by the following lemma:

Lemma (voting lemma): (H, q) quadr. enhancement. Then
$$\{|q^{-1}(0)|, |q^{-1}(1)|\} = \{2^{2n-1} - 2^{n-1}, 2^{2n-1} + 2^{n-1}\},$$
and $\operatorname{Arf} q$ is the value attained most of the times.

Pf: Induction on the length of $H = H_1 \oplus \cdots \oplus H_n$. $\square$

Arf invariant of knots

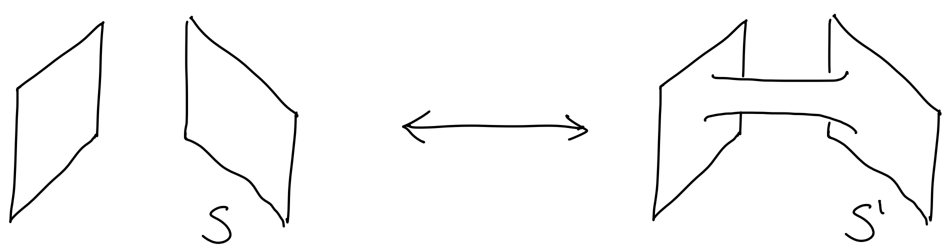
$K \subseteq S^3$ knot, S Seifert surface for K.

$(H_1(S; \mathbb{F}_2), \cdot)$ is unimodular, symmetric, bilinear. Then
over $\mathbb{F}_2$
$$q_{S,K}(a) := lk(a^+, a) \pmod 2$$
is a quadratic enhancement. $[lk(a^+, b) - lk(b^+, a) = a \cdot b]$

Def: $\operatorname{Arf} K := \operatorname{Arf}(q_{S,K})$

Thm: $\operatorname{Arf} K$ is well-defined.

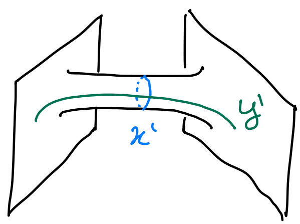
Pf: All Seifert surfaces for a link in S^3 are obtained by a sequence of surgeries on embedded arcs.



Take a symplectic basis $x_1, y_1, \dots, x_n, y_n$ for H, i.e. the intersection form is represented by a block diagonal matrix

$$\begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

Extend such a bases with x' and y' as indicated in the figure.



(By an isotopy of the endpoints of the surgery arc, we can assume that y' is disjoint from all other x_i and y_i.)

We get a new symplectic basis, and $lk((x')^+, x') = 0$, so the new H summand is $H^{0,0}$, so $\mathrm{Arf}\, K$ doesn't change. $\square$

Rk: Same works in Y $\mathbb{Z}HS^3$.

Rk: Arf can be extended to links by considering the induced q on $H_1(\Sigma; \mathbb{F}_2)/H_1(\partial\Sigma; \mathbb{F}_2)$.

Arf invariant of a surface Σ^2 in X^4

Let $H_1(X^4;\mathbb{Z})=0$ and $\Sigma \hookrightarrow X^4$ prop. embedded surface.

For $x \in H_1(\Sigma^2;\mathbb{Z})$, let C be an immersed curve representing x.

$C=\partial D$ for some surface $D \subseteq X$.

$\exists$ 2 framings of D at ∂D:

*) the framing coming from Σ

$(T\Sigma \cap N_{D|X})|_{\partial D}$ gives a section of $(N_{D|X})_{\partial D}$

*) the framing coming from D

D is a connected surface with ∂ $\Rightarrow$ it has trivial normal bundle (because it is orientable and D is hty equiv. to $\vee S^1$)

$\rightsquigarrow$ restricts to a trivialisation of $(N_{D|X})|_{\partial D}$.

Let $O(D)$ be the difference of these framings, which is an integer (# twists to make one section agree with the other).

Def: For $H_1(X;\mathbb{Z})=0$ and Σ characteristic, let

$$q_{X,\Sigma}(x) := D\cdot\Sigma + O(D) + d(C) \pmod 2$$

double pts of C in Σ

This is a quadratic enhancement of $(H_1(\Sigma;\mathbb{F}_2),\bullet)$.

[Note that if you tube D w/ a closed surface, the quantity $D\cdot\Sigma$ does not change mod 2 if Σ is characteristic.]

Def: For $H_1(X)=0$ and Σ characteristic prop. embedded, let

$$\mathrm{Arf}(X,\Sigma) := \mathrm{Arf}(q_{X,S}).$$

[The definition can be extended to $H_1(X)\neq 0$ as well.]

Theorem (Rokhlin, Freedman-Kirby, Klug)

Let X^4 be a topological 4-mfd with $\partial X = Y$ a $\mathbb{Z}HS^3$.

Let $\Sigma \subseteq X$ be a prop. emb. orient. surface with $K=\partial\Sigma$ Knot.

If Σ is characteristic, then

$$\boxed{\mathrm{Arf}(X,\Sigma)+\mathrm{Arf}(K) \equiv \frac{\sigma(X)-[\Sigma]^2}{8} + \mu(Y) + ks(X) \pmod 2}$$

4 OTHER SMOOTH EXCLUSIONS

Thm (Donaldson '82, '87) X^4 closed, oriented, smooth, definite.

Then Q_X is diagonalisable (i.e. $\cong \underbrace{n\cdot(1)}_{\text{pos. def.}}$ or $\underbrace{n\cdot(-1)}_{\text{neg. def.}}$)

Cor: $X_{E_8} \# X_{E_8}$ is not smoothable.

Note that Rokhlin is not enough, because $\sigma(X_{E_8} \# X_{E_8}) = -16$.

Pf: $\underbrace{Q_{E_8} \oplus Q_{E_8}}_{\text{even}}$ is neg. def., but it is $\not\cong \underbrace{16 \cdot (-1)}_{\text{odd}}$. $\square$

In the <u>indefinite case</u>, we know that if X is smooth, closed, indefinite,

$$Q_X \cong \begin{cases} a^- \cdot (-1) \oplus a^+ (1) & \text{if odd} \\ b E_8 \oplus c H & \text{if even} \end{cases}$$

Rokhlin $\Rightarrow$ $b = 2 \cdot d$, for some $d \in \mathbb{Z}$.

Rk: If $c \geqslant 3d$, $bE_8 \oplus cH$ is realised by $b \cdot K3 \# (c-3b)(S^2 \times S^2)$.

Conj (11/8) If $c < 3d$, the intersection form is not realisable by a smooth, closed 4-mfd.

Equivalently, $\forall X^4$ smooth closed <u>spin</u>, $b_2(X) \geqslant \frac{11}{8} \cdot |\sigma(X)|$.

Thm (Furuta's 10/8 thm '01, Hopkins-Lin-Shi-Xu '19)

If $2d \cdot E_8 \oplus cH$ is realisable by a smooth closed X^4, then

$$c \geqslant \begin{cases} 2d+2 & \text{if } d \equiv 1,2,5,6 \pmod 8 \\ 2d+3 & \text{if } d \equiv 3,4,7 \pmod 8 \\ 2d+4 & \text{if } d \equiv 0 \pmod 8 \end{cases}$$

Cor: X^4 closed, smooth, $\pi_1 = 1$, even, not homeo to $S^4, S^2\times S^2, K3$.

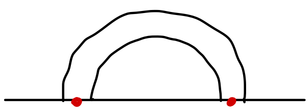
Then $\boxed{b_2(X) \geq \frac{10}{8}\cdot|\sigma(X)| + 4}$.

⑤ KIRBY DIAGRAMS

Goal: represent a smooth 4-mfd via an efficient handle decomp.

0-handle: WLOG $\exists!$ 0-handle $\mathbb{B}^4$, with $\partial\mathbb{B}^4 = S^3$. ← we work here

1-handles:



Attaching sphere is an $S^0 \subseteq S^3$.

Attaching region is $S^0\times\mathbb{B}^3$.

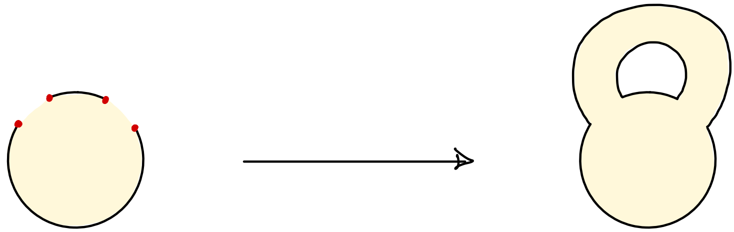
$\pi_{k-1}(O(n-k))$

In order to glue the handle we need to identify $S^0\times\mathbb{B}^3 \subseteq S^3$ with $(\partial\mathbb{D}^1)\times\mathbb{D}^3 \subseteq$ 1-handle. The choices up to diffeomorphism are given by $\pi_{k-1}(O(n-k)) \simeq [\underbrace{S^{k-1}}_{\text{attaching sphere}}, \underbrace{\mathrm{Diff}(\mathbb{D}^{n-k})}_{\text{normal bundle fiber}}]$.

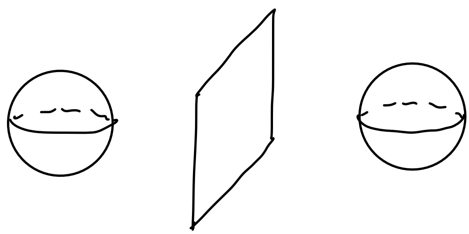
$k=1 \Rightarrow \pi_0(O(3)) = \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\text{framings}}$, depending on the orientation.

If X oriented, then every 1-handle is oriented $\rightsquigarrow$ unique way to attach it

After attaching the 1-handle we obtain $S^1 \times S^2$, which we can view as the result of gluing the boundaries of $S^3 \setminus (\mathbb{B}^3 \sqcup \mathbb{B}^3)$.



Convention: we glue $S^2 \leftrightarrow S^2$ via reflection in median plane.

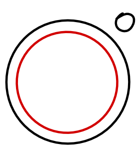


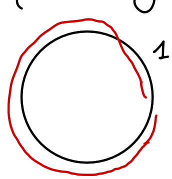
2-handles

These are attached along a union of S^1's, i.e. a link in $\#^{k_1}(S^1 \times S^2)$.

Framing: $\pi_1(O(2)) \cong \mathbb{Z} \rightsquigarrow$ framings are a $\mathbb{Z}$-torsor, and can be represented by a pushoff of S^1 in S^3.

If $Y = S^3$ (i.e., no 1-handles), then $\exists$ canonical framing (longitude $\leftrightarrow 0$), so we can identify $\{\text{framings}\} \leftrightarrow \mathbb{Z}$.

e.g. 

vs 

Rk: For framed knot (K, f) in S^3, the number associated to K is exactly $lk(K, f)$, taken with same orient.

Recall: $lk(A,B) = \#(F_A \pitchfork B)$, where F_A is a Seifert surface for A.

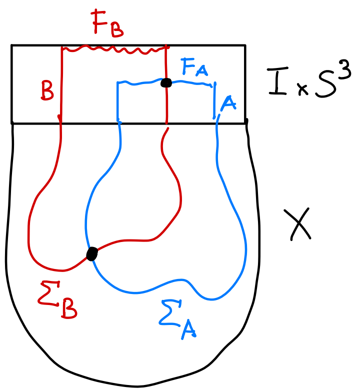
[The longitude is defined by $lk(K,f) = 0$, and each twist increases/decreases the framing by 1.]

Rk: $\Sigma_A, \Sigma_B \subseteq X$ w/ boundary $A, B \subseteq S^3 = \partial X$. Then

$$\#(\Sigma_A \pitchfork \Sigma_B) - lk(A,B) = Q_X([\Sigma_A],[\Sigma_B])$$

mirror

Pf:



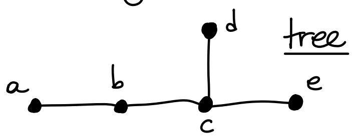
Choose Seif. surfaces for $\overline{A}$ and $\overline{B}$ in a collar of S^3, as shown. Glue to Σ_A and Σ_B to get closed surfaces $\widehat{\Sigma_A}$, $\widehat{\Sigma_B}$.

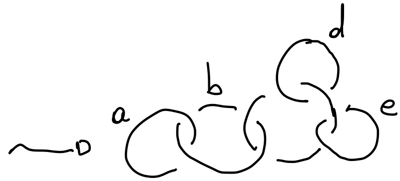
Then:

$$
\begin{aligned}
Q_X([\Sigma_A],[\Sigma_B]) &= \#(\Sigma_A \pitchfork \Sigma_B) + \#(F_A \pitchfork F_B) \\
&\overset{!}{=} \#(\Sigma_A \pitchfork \Sigma_B) + \#(F_A \pitchfork \overline{B}) \\
&\overset{!}{=} \#(\Sigma_A \pitchfork \Sigma_B) + lk(\overline{A}, \overline{B}) \qquad \square
\end{aligned}
$$

In particular, if $X = B^4$, $\#(\Sigma_A \pitchfork \Sigma_B) = lk(A,B)$.

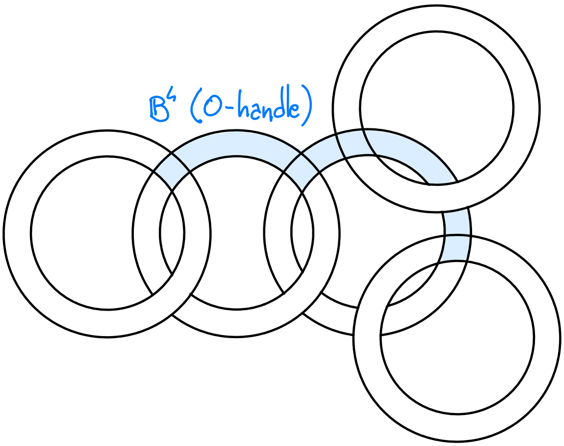
Plumbings revised





is a Kirby diagram for X_Γ

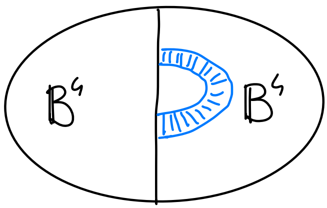
Each of these unknots bounds a disc D_v in $\mathbb{B}^4$.
The discs intersect according to the tree.



The union of the nbdhs of the discs gives the 0-handle, and the said discs are just the cores.

The remaining part of X_Γ are 2-handles, attached along the boundary of the discs, i.e. the unknots of the Kirby diagram. The framings of the 2-handles are the self-intersections of the spheres (by remark above), which are exactly the weights of the tree.

1-handles revised



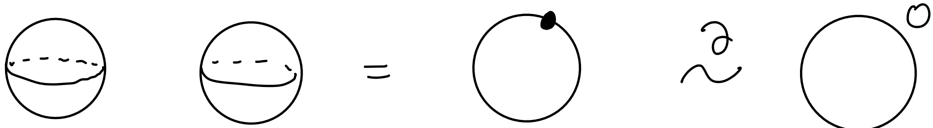
$S^4 = B^4 \cup B^4$ necessarily because $Q_{S^4} = 0$

$\exists$ 0-framed 2-handle sitting in B^4 attached along the unknot.

$B^4 \cup (2\text{-handle}) = 0\text{-trace on unknot}$

$B^4 - (2\text{-handle}) \cong S^1 \times B^3 \cong B^4 \cup (1\text{-handle})$

Thus, $B^4 \cup (1\text{-h})$ can be represented as a 0-framed 2-handle removed from B^4.



Notation: dotted handle

Rk: One can remove a disc from B^4 with boundary any slice knot. There is a description for such a manifold.

Surgery (3D)

$K \subseteq S^3$ knot $\rightsquigarrow$ μ = meridian, λ = longitude are a basis of $H_1(\underbrace{\partial \nu K}_{\mathbb{T}^2}) = \mathbb{Z} \oplus \mathbb{Z}$

$$S^3_{p/q}(K) := \Big(S^3 - \underbrace{\nu(K)}_{\cong S^1 \times \mathbb{D}^2}\Big) \cup_\varphi (S^1 \times \mathbb{D}^2)$$

where $\varphi(\partial \mathbb{D}^2) = p\mu + q\lambda$.

Fact/Ex: $\underbrace{S^3_{p/q}(K)}_{(p/q)\text{-SURGERY over } K}$ is well-defined up to diffeo.

Surgery trace (4D)

$X_n(K) := \mathbb{B}^4 \cup (n\text{-framed 2-handle along } K)$

Ex: $\partial(X_n(K)) = S^3_{n/1}(K)$.

Hint: When you attach a 2-handle the ∂ is modified by removing a copy of $S^1 \times \mathbb{D}^2$ (att. tube) and gluing in a copy of $S^1 \times \mathbb{D}^2$ (belt tube).

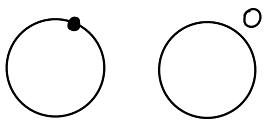
Back to Kirby diagrams: 3-handles and 4-handles.

These are easy thanks to the following theorem.

Thm (Laudenbach–Poenaru) Every diffeomorphism of $\#^n(S^1\times S^2)$ extends to a unique diffeo of $\natural^n(S^1\times B^3)$.

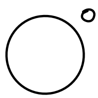
The 3-handles and the 4-handle together give a $\natural^n(S^1\times B^3)$. By LP there is a unique way to glue it to the previous part of the Kirby diagram if X is a closed 4-mfd.

e.g.

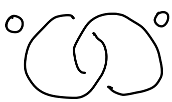


$\cup\ (3\text{-}h)\cup(3\text{-}h)\cup(4\text{-}h)$

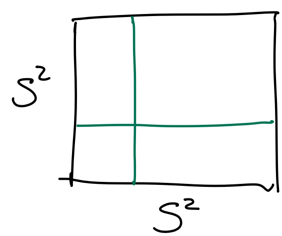
Examples:

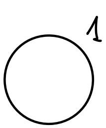


$$\underbrace{\bigcirc^{0}}_{S^2\times D^2}\ \cup\ \underbrace{(3\text{-}h)\cup(4\text{-}h)}_{S^1\times D^3} \cong S^4$$



$\cup\ (4\text{-}h) = S^2\times S^2$



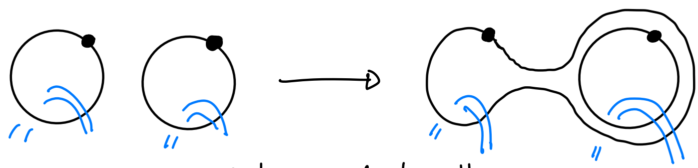


$\cup\ (4\text{-}h) = \mathbb{CP}^2$

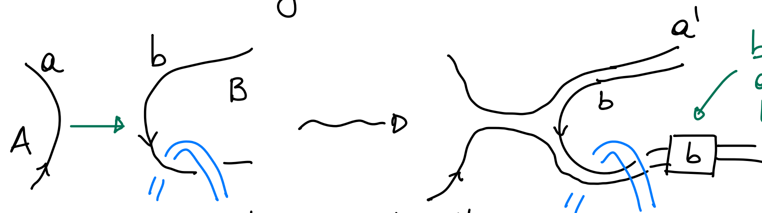
contains a sphere with self-intersection 1, namely $\mathbb{CP}^1$.

Kirby moves

1) Handle slides



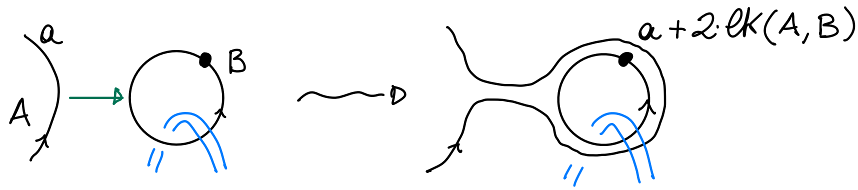
sliding 1-handles



b twists away from longitude

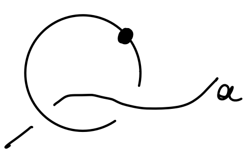
sliding 2-handles

[$a' = a + b + 2\,lk(A,B)$, since the new handle is $[A]+[B]$]



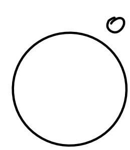
isotoping 2-handle in $S^1 \times B^3$

2) Cancellations



$\xleftrightarrow[\text{pair}]{1/2\text{-handle}} \emptyset$

(only one 2-handle going through)



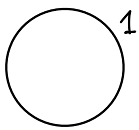
$\xleftrightarrow[\text{pair}]{2/3\text{-handle}} \emptyset$

(isolated 2-handle)

Thm: X^4 connected closed oriented smooth.

Every two Kirby diagrams are related by a sequence of Kirby moves.

⑤ THE FAKE $\mathbb{CP}^2$

 $\cup\ (4\text{-h.}) = \mathbb{CP}^2$

Poincaré sphere with opp. orient.

 $= X_1(RHT)$ has ∂ a $\mathbb{Z}HS^3$ [in fact it is $\overline{P}$]

Freedman $\Rightarrow$ $*\mathbb{CP}^2 = (X_1(RHT)) \cup C$ is a TOP 4-mfd.

contractible piece

Rk: $\mathbb{CP}^2$ and $*\mathbb{CP}^2$ have both $\pi_1 = 1$ and $Q_X = (1)$.

Are they homeomorphic? That depends on ks!

$\mathbb{CP}^2$ smooth $\Rightarrow$ $ks(\mathbb{CP}^2) = 0$

orient. reversal

0-handle

$K = LHT$ bounds a disc D in $\underbrace{(*\mathbb{CP}^2 \setminus B^4)}_{\text{call this } X}$, namely the core of the 2-handle.

Note that $H_1(D;\mathbb{F}_2)=0$, so $\mathrm{Arf}(X,D)=0$.

By Freedman-Kirby-Klug we have

$$\underbrace{\mathrm{Arf}(X,D)}_{\substack{\| \\ 0}} + \underbrace{\mathrm{Arf}(K)}_{\substack{\| \\ 1 \\ \text{for trefoil} \\ \text{knot}}} \equiv \underbrace{\frac{\sigma(X)-[D]^2}{8}}_{\frac{1-1}{8}=0} + \underbrace{\mu(S^3)}_{\substack{\| \\ 0}} + ks(X) \pmod 2$$

$\Longrightarrow ks(*\mathbb{CP}^2) = ks(X) = 1$

Thus, $*\mathbb{CP}^2 \neq_{TOP} \mathbb{CP}^2$, and in fact it does not admit any smooth structure.