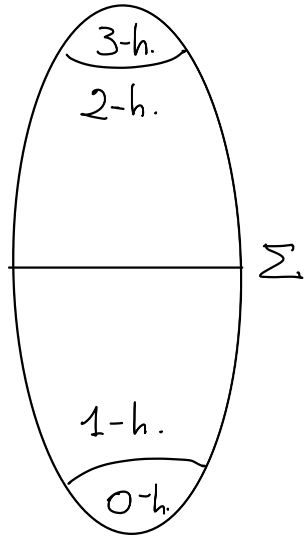
# Heegaard Floer homology and Donaldson's theorem

# ① HEEGAARD FLOER HOMOLOGY

$\left.\begin{array}{l} Y^3 = \text{closed oriented 3-mfd} \\ \mathfrak{s} \in \mathrm{Spin}^c(Y) \end{array}\right\} \rightsquigarrow HF^\circ(Y,\mathfrak{s}) \quad \circ \in \{+,-,\infty,\wedge\}$

a graded module over $\mathbb{F}[U]$

## Sketch of the definition (Ozsváth - Szabó)




Pick a handle decomposition of $Y$ with only one 0-handle and one 3-handle. Let $\Sigma$ be the surface after the 1-handles.

On $\Sigma$ we have two sets of pairwise disjoint simple closed curves:

- $\underline{\alpha} := \{\alpha_1, \dots, \alpha_g\}$ belt spheres of the 1-handles
- $\underline{\beta} := \{\beta_1, \dots, \beta_g\}$ attaching spheres of the 2-handles

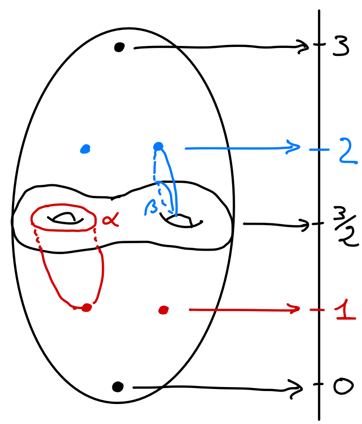
<u>Morse-theoretic perspective</u>:

$f: Y \longrightarrow \mathbb{R}$ self-indexing Morse function.

$\Sigma := f^{-1}(3/2)$

$\underline{\alpha} := \Sigma \cap W^s(\text{index-1 crit. pts})$

$\underline{\beta} := \Sigma \cap W^u(\text{index-2 crit. pts})$



## The module

Heegaard diagram

Input data: $\mathcal{H} = (\Sigma, \alpha, \beta, z)$, $z \in \Sigma - (\alpha \cup \beta)$ bpt.

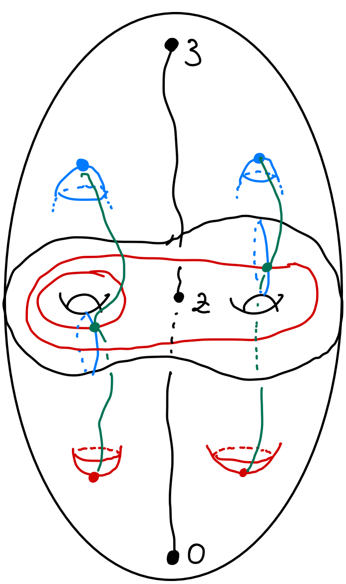
$G_{\mathcal{H}} := \{ (x_1, \dots, x_g) \mid x_i \in \alpha_i \cap \beta_{\sigma(i)} \text{ for some } \sigma \in S_g \}$

**Def:** $CF^-(\mathcal{H}) := \mathbb{F}[U] \langle G_{\mathcal{H}} \rangle$.

**Lemma:** The basept $z$ gives a map $s_z : G_{\mathcal{H}} \longrightarrow \mathrm{Spin}^c(Y)$.

**Fact:** $\mathrm{Spin}^c(Y) \cong \{ v \text{ non-vanishing vector field on } Y \} / \sim$,
where $v \sim w$ if they are isotopic in $Y - B^3$ (equivalently, on the 2-skeleton of $Y$).

## Construction of $s_z(\underline{x})$




Each intersection pt identifies a trajectory $\gamma_i$ from an index-1 to an index-2 crit. pt.

$z \in \Sigma$ identifies a trajectory $\gamma_z$ from the index-0 to the index-3 critical pt.

The vector field $\nabla f$ is non-singular away from these trajectories.

including $\gamma_z$ ?

Moreover, $\nabla f$ has degree $0$ on each sphere $\partial \nu(\gamma_i)$, because in each $\nu(\gamma_i)$ there are 2 crit. pts of opposite parity (hence $\deg \nabla f$ is $+1$ near one of them and $-1$ near the other).

Modify $\nabla f$ inside each $\nu(\gamma_i)$ to be non-singular.

Thus, we have a splitting $CF^-(\mathcal{H}) = \bigoplus_{s \in Spin^c(Y)} CF^-(\mathcal{H}, s)$.

For the differential we need a different perspective.

## Symmetric products

**Def:** $Sym^g \Sigma = \{ \{x_1, \dots, x_g\} \text{ unordered tuples} \}$

**Rk:** It is a manifold:

$$\begin{aligned} Sym^g \mathbb{C} &\xrightarrow{\sim} \mathbb{C}^g \\ \{x_1, \dots, x_g\} &\longmapsto (x - x_1) \cdot \ldots \cdot (x - x_g) \end{aligned}$$

**Def:**

$$\left.\begin{aligned} \mathbb{T}_\alpha &:= \alpha_1 \times \dots \times \alpha_g \subseteq Sym^g \Sigma \\ \mathbb{T}_\beta &:= \beta_1 \times \dots \times \beta_g \subseteq Sym^g \Sigma \end{aligned}\right\} \Rightarrow G_{\mathcal{H}} = \mathbb{T}_\alpha \cap \mathbb{T}_\beta$$

**Thm (Perutz)** Given $\mathcal{H}$ Heegaard diagram and a complex structure $j$ on $\Sigma$, there is a symplectic form $\omega$ on $Sym^g \Sigma$ compatible with the complex structure $Sym^g j$ such that $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ are Lagrangians.

<u>Recall</u>: $\omega \in \Omega^2(X^{2n})$ is <u>symplectic</u> if $d\omega = 0$ and $\omega^n$ is a volume form on $X^{2n}$.

$L^n \subseteq X^{2n}$ is <u>Lagrangian</u> if $\omega|_L \equiv 0$.

<u>Floer's idea</u>: Given $(X, \omega)$ and $L_0, L_1$ Lagrangians, we want to do Morse theory on

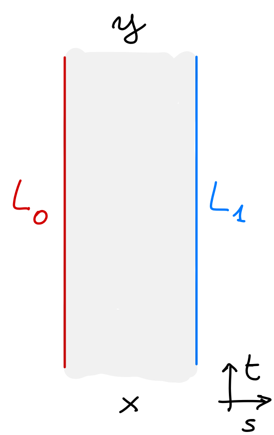
$$\mathcal{P} := \{ \gamma : I \longrightarrow X \mid \gamma(0) \in L_0,\ \gamma(1) \in L_1 \}$$

using as "Morse function" an <u>ACTION FUNCTIONAL</u> $\mathcal{A}$.

<u>Key pts</u>:

*) $\mathrm{Crit}(\mathcal{A}) = L_0 \cap L_1$ (const. paths)

*) trajectories b/w critical pts $x$ and $y$ are maps $u : I \times \mathbb{R} \longrightarrow X$ satisfying boundary conditions and Cauchy-Riemann




$$\left( \frac{\partial}{\partial t} + J \frac{\partial}{\partial s} \right)(u) = 0$$

actually need to perturb to $J_s$

$\rightsquigarrow$ Get a Morse-like complex.

## The differential

$$\partial^- : CF^-(\mathcal{H}) \longrightarrow CF^-(\mathcal{H}) \quad , \quad x \in G_{\mathcal{H}}$$

count only J-hol. represent.

$$\partial^-(x) := \sum_{y \in G_{\mathcal{H}}} \sum_{\substack{\phi \in \pi_2(x,y) \\ \operatorname{ind}\phi = 1}} \# \overline{\mathcal{M}(\phi)} \cdot U^{n_z(\phi)} \cdot y \quad + \text{extend linearly}$$

top. discs connecting $x$ and $y$

replaces $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$

$n_z(\phi) :=$ intersection # between $\phi$ and $V_z := \{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$

(all intersections are positive if these are J-hol. submfds)

## Variants

*) $CF^-(\mathcal{H})$ is naturally an $\mathbb{F}[U]$-module ($\mathbb{F}_2$)

*) $CF^\infty(\mathcal{H}) := CF^-(\mathcal{H}) \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ , an $\mathbb{F}[U,U^{-1}]$-mod.

*) $CF^+(\mathcal{H}) := \dfrac{CF^\infty(\mathcal{H})}{U \cdot CF^-(\mathcal{H})}$ an $\mathbb{F}[U]$-module

*) $\widehat{CF}(\mathcal{H}) := \dfrac{CF^-(\mathcal{H})}{U \cdot CF^-(\mathcal{H})}$ an $\mathbb{F}$-vector space ($U$ acts as $0$)

Rk: There are induced differentials on each variant, which respect the spin$^c$ decomposition and square to $0$. Thus, we get

$$HF^\circ(\mathcal{H}) = \bigoplus HF^\circ(\mathcal{H}, s).$$

# Fundamental properties

*) $HF^\circ(\mathcal{H}, s)$ does not depend on the Heegaard diagram

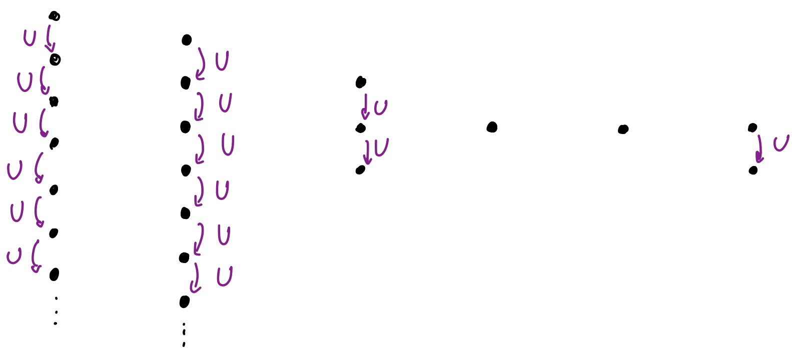
$\rightsquigarrow HF^\circ(Y, s)$ an invariant of $(Y, s)$

*) $HF^\circ(Y, s)$ carries an action of $\Lambda^*\Big(\overbrace{H_1(Y)/\text{Tors}}^{H^1(Y)}\Big)$ [homological action]

*) $s \in \text{Spin}^c(Y)$ torsion (i.e. $c_1(s)$ is torsion) $\Rightarrow$ Absolute $\mathbb{Q}$-grading $(\deg U = -2)$

*) Structure: $HF^-(Y, s)$ for $s$ torsion

Direct sum of "towers" $\mathbb{F}[U]$ and "finite pieces" $\mathbb{F}[U]/(U^k)$.




**Fact**: If the triple cup product on $Y$ vanishes, then

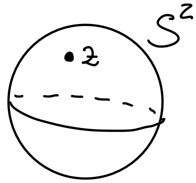
$\exists!$ tower in $\text{Ker}(\Lambda\text{-action})$, called BOTTOM TOWER.

**Def**: $d_b(Y, s) :=$ grading of the (homogeneous) generator of the bottom tower.

(bottom-most correction term)

## Examples

1) $Y = S^3$



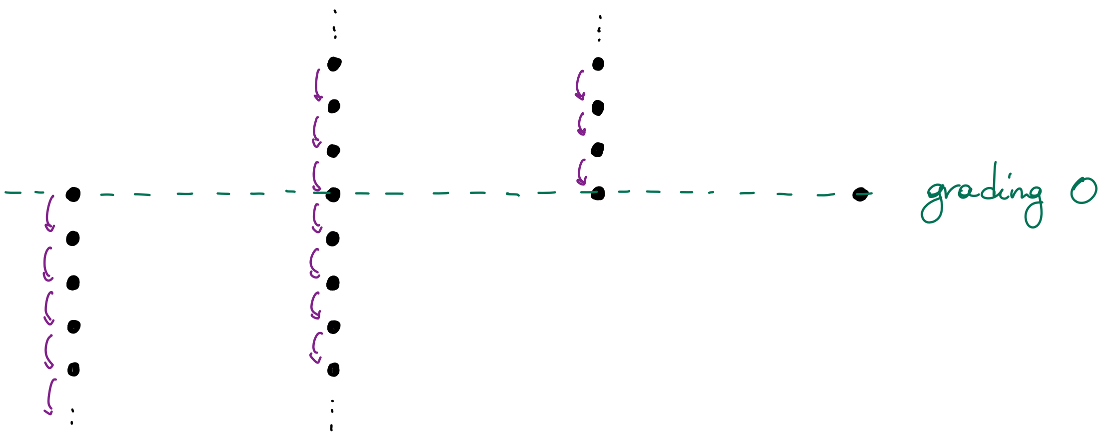

$Sym^0(S^2) = pt$

$\mathbb{T}_\alpha, \mathbb{T}_\beta = pt$

in the unique spin$^c$ str.

$\exists$ unique generator $x \in G_{\mathcal{H}}$, and $\partial^-(x) = 0$ necessarily.

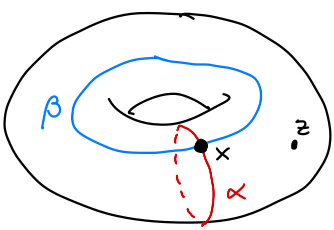
$HF^-(S^3)$ $\quad HF^\infty(S^3)$ $\quad HF^+(S^3)$ $\quad \widehat{HF}(S^3)$




$\mathbb{F}[U]$ $\quad \mathbb{F}[U, U^{-1}]$ $\quad \dfrac{\mathbb{F}[U, U^{-1}]}{U \cdot \mathbb{F}[U]}$ $\quad \mathbb{F}$

2) $Y = S^3$

$\Sigma = T^2$



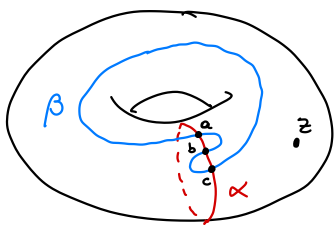

$Sym^1 \Sigma = \mathbb{T}$

$\mathbb{T}_\alpha = \alpha \qquad \mathbb{T}_\beta = \beta$

$CF^-(\mathcal{H}) = \mathbb{F}[U]\langle x \rangle$

3) $Y = S^3$

$\Sigma = \mathbb{T}^2$

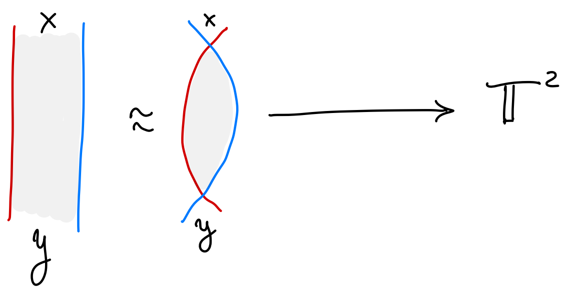



$\mathrm{Sym}^1 \Sigma = \mathbb{T}$

$\mathbb{T}_\alpha = \alpha \quad \mathbb{T}_\beta = \beta$

$CF^-(\mathcal{H}) = \mathbb{F}[U]\langle a, b, c\rangle.$

$\partial^-$ is complicated: topologically, there are two discs




Each of them has a single J-hol. repr., up to translation.

$\Rightarrow \partial^-(a) = b, \quad \partial^-(b) = 0, \quad \partial^-(c) = b$

$$HF^-(S^3) = \frac{\mathrm{Ker}\,\partial^-}{\mathrm{Im}\,\partial^-} = \frac{\mathbb{F}[U]\langle b, a+c\rangle}{\mathbb{F}[U]\langle b\rangle} \cong \mathbb{F}[U]$$

Ex: Try to move the basepoint z inside one of the discs.

## Cobordism maps

A spin$^c$ cobordism $(Z, s) : (Y_0, t_0) \longrightarrow (Y_1, t_1)$ induces a map $F_{Z,s}$ in $HF^-$ (O-Sz, Juhász-Thurston-Zemke)

If $t_0$ and $t_1$ are torsion, $F_{Z,s}$ is <u>graded</u> and

$$\deg F_{Z,s} = \frac{c_1(s)^2 - 2\chi(Z) - 3\sigma(Z)}{4}$$

<u>Non-vanishing theorem</u> In the above setting, suppose that:

$\left.\begin{array}{l} *)\ t_0 \text{ and } t_1 \text{ are torsion} \\ *)\ Y_0 \text{ and } Y_1 \text{ have vanishing triple cup product} \end{array}\right\} \Rightarrow$ bottom towers are defined

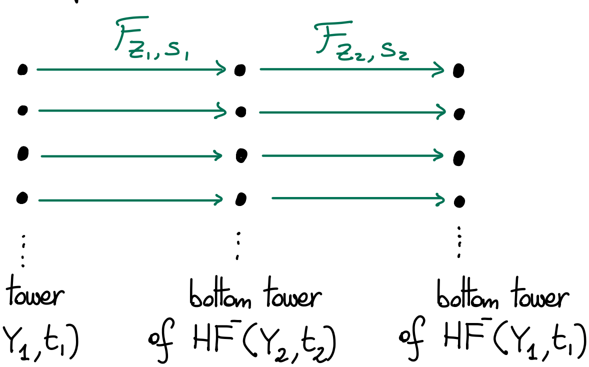
*) $H_1(Y_0; \mathbb{Q}) \xrightarrow{\sim} H_1(Z; \mathbb{Q})$ is an isomorphism.

*) $Z$ is <u>negative semidefinite</u>.

Then $F_{Z,s}$ is non-vanishing between the bottom towers.

<u>Cor</u>: $d_b(Y, t)$ is invariant under spin$^c$ $\mathbb{Q}H$-cobordism.

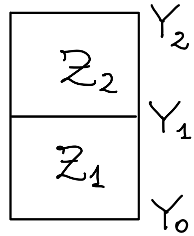
<u>Idea</u>: Given $\mathbb{Q}H$-cob. $Z_1$ and $Z_2$, you get non-vanishing degree-0 maps. $\square$



Functoriality properties: $*)$ $F_{I\times Y, s} = id_{HF^\circ(Y,s)}$

$*)$ $F_{Z_2 \cup Z_1} = F_{Z_2} \circ F_{Z_1}$,

where $F_Z = \sum_{s \in Spin^c Z} F_{Z,s}$




## ② DONALDSON'S THEOREM

### d-invariants and 4-manifolds

**Thm** $Y^3$ w/ vanishing triple cup product, $t \in Spin^c(Y)$ torsion.

Suppose that $(Y, t) = \partial(W, s)$, such that

•) $H^1(W;\mathbb{Q}) \longrightarrow H^1(Y;\mathbb{Q})$ is trivial, and

•) $W$ is negative semidefinite.

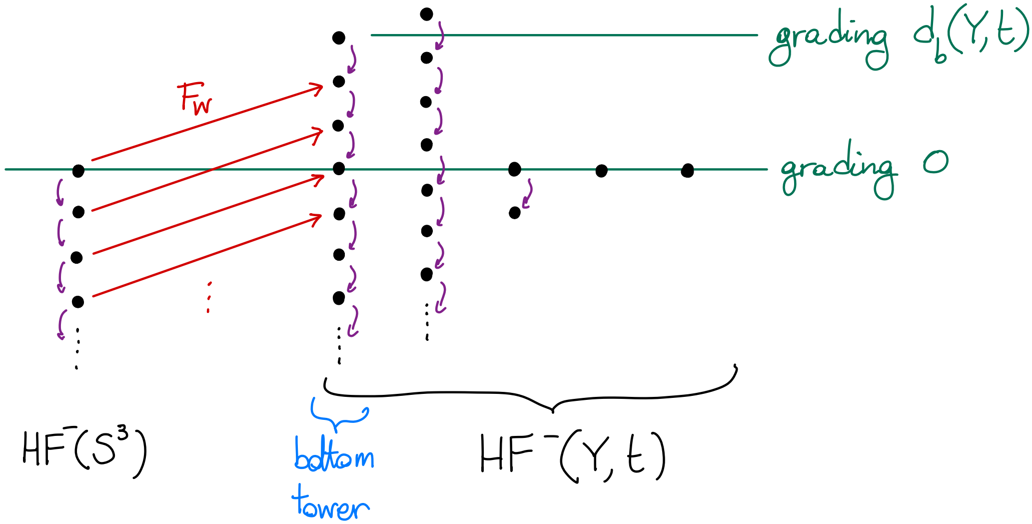
Then

$$\boxed{c_1(s)^2 + b_2^-(W) \leq 4 \cdot d_b(Y,t) + 2 \cdot b_1(Y)}$$

Idea: WLOG $b_1(W) = 0$, by doing surgery (replace $S^1 \times B^3$ with $D^2 \times S^2$) on non-trivial loops.

Let $Z := W - B^4$, which is a spin$^c$ cobordism $(S^3, t_0) \longrightarrow (Y, t)$.

FACT: The cobordism induces a non-trivial map between the bottom towers.




By nonvanishing, we must have $d_b(Y,t) \geq \deg F_W$.

$$\begin{aligned}\chi(W) &= 1 + b_2(W) && (b_1 = 0 \Rightarrow b_3 = 0 \text{ by PD}) \\ &= 1 + b_2^-(W) + b_2^0(W) \\ &= 1 + b_2^-(W) + b_1(Y) && (Q_W \text{ presents } H_1(Y;\mathbb{Z}))\end{aligned}$$

Thus, $\chi(Z) = b_2^-(W) + b_1(Y)$.

Moreover, $\sigma(Z) = \sigma(W) = -b_2^-(W)$.

Putting things together, we compute

$$\deg F_Z = \frac{c_1(s)^2 - 2\chi(Z) - 3\sigma(Z)}{4} = \frac{c_1(s^2) + b_2^-(W) - 2b_1(Y)}{4}$$

Now the fact that $d_b(Y,t) \geq \deg F_Z$ gives the result. $\square$

# Elkies lemma

**Thm (Elkies)** Let $Q : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ be a bilinear symmetric unimodular form on $\Lambda \cong \mathbb{Z}^n$ such that $\forall w \in \Lambda$ characteristic

$$Q(w,w) \geqslant rk(\Lambda)$$

Then $(\Lambda, Q)$ is the diagonal lattice, i.e., $Q \cong \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$.

**Idea:** Study the formal THETA SERIES of $(\Lambda, Q)$:

$$\theta_\Lambda(t) := \sum_{v \in \Lambda} e^{i\pi \cdot Q(v,v) \cdot t}$$

and compare it with that of $\theta_{\mathbb{Z}^n} = (\theta_{\mathbb{Z}})^n$.

standard lattice

in general, $\theta_{\Lambda_1 \oplus \Lambda_2} = \theta_{\Lambda_1} \cdot \theta_{\Lambda_2}$

**Facts:**

1) $\theta_\Lambda(t)$ is convergent on $H = \{ \operatorname{Im} z > 0 \} \subseteq \mathbb{C}$.
2) $\theta_\Lambda(t+2) = \theta_\Lambda(t)$ (using periodicity of $e^{i\pi t}$)
3) Poisson inversion formula

the volume of $\mathbb{R}^n/\Lambda$

$$\theta_\Lambda(i \cdot t) = t^{-\frac{n}{2}} \cdot v^{-1} \cdot \theta_{\Lambda^*}\left((it)^{-1}\right)$$

(Follows from the Poisson formula for Fourier transforms)

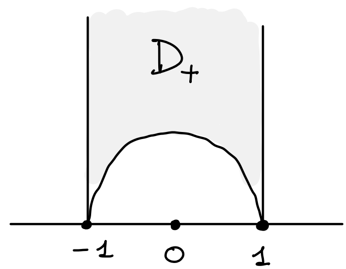
If $\Lambda$ is unimodular, $\Lambda = \Lambda^*$, so there is a symmetry

$$\left(t/i\right)^{n/2} \cdot \vartheta_\Lambda(t) = \vartheta_\Lambda(-t^{-1})$$

4) $\vartheta_{\mathbb{Z}}(t)$ has no zeroes in $H$.

Putting these facts together, we obtain that

$$R(t) := \frac{\vartheta_\Lambda(t)}{(\vartheta_{\underline{\mathbb{Z}}}(t))^n}$$




is holomorphic on $D_+ \subseteq \mathbb{C}$.

Moreover $\lim\limits_{t \to i\infty} R(t) = 1$, and the only potential poles are $\pm 1$.

we are extending $Q$ to $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$

Define $\vartheta'_\Lambda(t) = \sum\limits_{v \in \Lambda + \frac{w}{2}} e^{i\pi \cdot Q(v,v) \cdot t}$, for $w \in \Lambda$ characteristic.

<u>Fact 5)</u> $\vartheta_\Lambda(t+1) = \left(t/i\right)^{-n/2} \cdot \vartheta'_\Lambda\left(-\frac{1}{t}\right)$

(From Poisson inversion, using $(-1)^{Q(v,v)} = (-1)^{Q(w,v)}$ )

We now argue that $R(t)$ is bounded as $t \longrightarrow \pm 1$.

By fact 5), we need to check $\lim\limits_{t \to i\infty} \vartheta'_\Lambda(t) / (\vartheta'_{\mathbb{Z}}(t))^n$.

$\vartheta'_{\mathbb{Z}}(t)$ counts odd integers, so

$$\vartheta'_{\mathbb{Z}}(t) = 2\left(e^{\frac{i\pi t}{4}} + e^{9\cdot\frac{i\pi t}{4}} + \dots\right)$$
$$= 2e^{\frac{i\pi t}{4}}\cdot\left(1 + \underbrace{e^{2\pi i t}}_{\to 0} + \underbrace{e^{6\pi i t}}_{\to 0} + \dots\right)$$

as $t \to i\infty$

Thus, $\left(\vartheta'_{\mathbb{Z}}(t)\right)^n \sim C_1 \cdot e^{\frac{i\pi n t}{4}}$ as $t \to i\infty$

Van der Blij's lemma $\left(Q(w,w) \equiv n \pmod 8\right)$ implies

$$\vartheta'_{\Lambda}(t) = \sum_{m \equiv n \,(\mathrm{mod}\, 8)} a_m \cdot e^{\frac{i\pi m t}{4}}$$

lengths of char. vectors

The hypothesis that no char. vector is shorter than $n$ implies

$$\vartheta'_{\Lambda}(t) = e^{\frac{i\pi n t}{4}} \cdot \left(a_n + \sum_{\substack{m > n \\ m \equiv n \,(\mathrm{mod}\, 8)}} a_m \cdot \underbrace{e^{\frac{(m-n)\pi i t}{4}}}_{\to 0}\right)$$

as $t \to i\infty$

Thus, $\vartheta'_{\Lambda}(t) / \left(\vartheta'_{\mathbb{Z}}(t)\right)^n$ is bounded at $i\infty$

$\underset{\Longrightarrow}{\text{Fact 5)}}$ $R(t)$ bounded near $\pm 1$, so $R(t)$ has no poles.

$\Longrightarrow$ (Liouville thm) $R(t) \equiv$ constant, which must be 1 because $\lim_{t \to i\infty} R(t) = 1$

$\Rightarrow \Theta_\Lambda \equiv \vartheta_{\mathbb{Z}^n}$

$\Longrightarrow$ $\Lambda$ and $\mathbb{Z}^n$ have the same number of vectors of length 1,
Let $\pm v_1, \dots, \pm v_n$ be the unit vectors of $\Lambda$.
By Cauchy-Schwarz $|Q(v_i, v_j)| < 1$ if $i \neq j$, but since $\Lambda$ is integral this implies $Q(v_i, v_j) = \delta_{ij}$.
Thus, $v_1, \dots, v_n$ is an orthonormal set in $\Lambda$, which spans a $\mathbb{Z}^n \subseteq \Lambda$.

Rk: For a sublattice $\Lambda' \subseteq \Lambda$, $[\Lambda : \Lambda'] = \frac{\det \Lambda'}{\det \Lambda}$.

Thus, if $\Lambda$ contains a copy of $\mathbb{Z}^n$, it is $\mathbb{Z}^n$. $\square$

# Donaldson's theorem

Let $X^4$ closed oriented smooth 4-mfd, with $Q_X$ neg. definite.
Then $Q_X \cong n\cdot(-1)$, where $n = b_2(X)$.

Proof: By removing a $B^4$ we get a negative definite 4-mfd $W$ with $\partial W = S^3$. Thus, for every spin$^c$ structure $s \in \mathrm{Spin}^c(W)$,

$$\boxed{(c_1(s))^2 + b_2(W) \leq 0.}$$

Claim: Every characteristic class of $\Big(\underbrace{H_2(X)/\mathrm{Tors}}_{=:\Lambda}, Q_X\Big)$ is represented by the Chern class of a spin$^c$ structure.

Step 1: Recall the set

$H^2(X)/\mathrm{Tors}$

$$\mathrm{Char}(\Lambda, Q_X) = \{ w \in \Lambda^* \mid \langle w, x\rangle \equiv Q_X(x,x) \bmod 2\}$$

which is a torsor over $H^2(X)/\mathrm{Tors}$ by mult. by 2 and addition.

[To check that the action is transitive, given $w$ and $w'$ as above we have $\langle w-w', x\rangle \equiv 0 \pmod 2$ $\forall x \in \Lambda$, so $\frac{w-w'}{2} \in \Lambda^*$.]

Define the set of integral lifts of $w_2(X)$:

$$IL(w_2) := \{ w \in H^2(X;\mathbb{Z}) \mid [w]_2 = w_2(X)\}.$$

This is a torsor over $H^2(X;\mathbb{Z})$, by mult. by 2 and addition.

The Wu formula gives a map $\mathbb{IL}(w_2) \longrightarrow \mathrm{Char}(\Lambda, Q_X)$, which is modelled over $H^2(X) \longrightarrow H^2(X)/\mathrm{Tors}$, hence surjective.

<u>Step 2</u>: Every integral lift of $w_2$ is the Chern class of a spin$^c$ structure, by exactness of

$$\check{H}^1(X; C^\infty \mathrm{Spin}^c(n)) \longrightarrow \check{H}^1(X; C^\infty SO(n)) \oplus \check{H}^1(X; C^\infty U(1)) \xrightarrow{w_2 + c_1} H^2(X; \mathbb{Z}/2\mathbb{Z}).$$

$\square$ Claim.

Thus, for every char. vector $w \in \mathrm{Char}(Q_X)$, $w \cdot w \leq -n$. By Elkies lemma, the lattice is the (negative) diagonal lattice. $\square$