The Thom Conjecture and the mixed invariant

① ADJUNCTION FORMULA

Thm: Let $\Sigma \subseteq X$ smooth, cnct, cx curve in a cx surface.

Then $\boxed{\langle k_X, [\Sigma] \rangle + [\Sigma]^2 = 2g(\Sigma) - 2}$.

$c_1(\det T^*X)$, the canonical class

Pf: $TX|_\Sigma = T\Sigma \oplus N_X(\Sigma)$

$$c_1(TX|_\Sigma) = c_1(T\Sigma) + c_1(N_X(\Sigma))$$

$$\underbrace{\langle c_1(TX|_\Sigma), [\Sigma] \rangle}_{= c_1(\text{anti-canonical})} = \langle c_1(T\Sigma), [\Sigma] \rangle + \langle c_1(N_X(\Sigma)), [\Sigma] \rangle$$

Euler class of a bundle
⇝ gives intersection of two sections when paired with $[\Sigma]$.

[$c_1(\det E) = c_1(E)$ by splitting principle and $\det(L_1 \oplus \cdots \oplus L_n) = L_1 \otimes \cdots \otimes L_n$]

$$-\langle k_X, [\Sigma] \rangle = \chi(\Sigma) + [\Sigma]^2 \qquad \square$$

e.g. $X = \mathbb{CP}^2 \Rightarrow k_X = -3 \cdot h^*$ ← dual of std generator

Euler sequence
$0 \to \varepsilon \to \mathcal{O}(1)^{\oplus 3} \to T\mathbb{P}^2 \to 0$

Σ degree d curve $\Rightarrow [\Sigma] = d \cdot h$

$$-3d + d^2 = 2g - 2$$

Cor (genus-degree formula)

The genus of a smooth degree d curve in $\mathbb{CP}^2$ is $\dfrac{(d-1)(d-2)}{2}$.

i.e., in homology class $d \cdot h \in H_2(\mathbb{CP}^2)$.

② ADJUNCTION INEQUALITIES

Change of setting: complex surface $\rightsquigarrow$ smooth 4-mfd X

complex curve $\rightsquigarrow$ smooth real surface Σ

Remark 1: formula vs inequalities

In the cx setting we have a closed formula for $g(\Sigma)$ in terms of $[\Sigma]$.

In the smooth setting this is impossible: if Σ_g represents $\alpha \in H_2(X)$, the so does Σ_{g+1}.

The best you can hope for is an inequality.

We can hope to determine the MINIMUM GENUS of a surface repres. a given homology class.

Def: GENUS FUNCTION $G_X : H_2(X) \longrightarrow \mathbb{Z}_{\geq 0}$

$$G_X(\alpha) := \min \left\{ g(\Sigma) \;\middle|\; \begin{array}{l} \Sigma \xhookrightarrow{sm} X \\ [\Sigma] = \alpha \end{array} \right\}$$

Rk: This is well-defined (α is always repr. by a smooth surface).

Remark 2 : smooth vs locally flat

Locally flat: continuous embedding locally modelled over $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$, $(x,y) \mapsto (x,y,0,0)$

*) Smooth $\Rightarrow$ loc. flat

*) Loc. flat $\not\Rightarrow$ Smooth

e.g. the (primitive, char.) class $(5,3) \in H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$ is

- •) represented by a locally flat sphere (Lee–Wilczyński '90)
- •) not represented by a smooth sphere

[Ruberman '96 : $G_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}}(5,3) = 3$.]

Thom Conjecture: $G_{\mathbb{CP}^2}(d) = \dfrac{(d-1)\cdot(d-2)}{2}$

(i.e. sm. cx curves in $\mathbb{CP}^2$ are genus-minim. in their homology class)

Proved by Kronheimer–Mrowka '94.

Today we sketch a proof by Ozsváth–Szabó '03.

see also MS_3T (intermediate result)

Symplectic Thom Conjecture (proved by Ozsváth–Szabó '00)

Smoothly embedded symplectic surfaces in a closed symplectic 4-mfd are genus-minimising in their homology class.

3 d-INVARIANTS of CIRCLE BUNDLES

$Y_{g,e}$ = circle bundle over Σ_g with Euler number e

$$\partial X_{g,e} = Y_{g,e}$$

$X_{g,e}$ ← disc bundle, $[\Sigma]^2 = e$

Properties: *) the triple cup product vanishes iff $e \neq 0$

*) $\mathrm{Spin}^c(X_{g,e}) = \{ s_k \mid k \in \mathbb{Z} \}$

where s_k is determined by

$$\langle c_1(s_k), [\Sigma] \rangle = e + 2k$$

*) Under change of orientation $X_{g,e} \rightsquigarrow \overline{X_{g,e}} = X_{g,-e}$

the labelling of spin^c structures changes as follows:

$$\begin{array}{ccc} \mathrm{Spin}^c(X_{g,e}) & \xrightarrow{\sim} & \mathrm{Spin}^c(X_{g,-e}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ s_k & \longmapsto & s_{k+e} \end{array}$$

This is because

$$\langle c_1(s_k^{g,e}), [\Sigma] \rangle = e + 2k = -e + 2(k+e) \stackrel{!}{=} \langle c_1(s_{k+e}^{g,-e}), [\Sigma] \rangle$$

*) If $e < \min\{0, 2-2g\}$, then

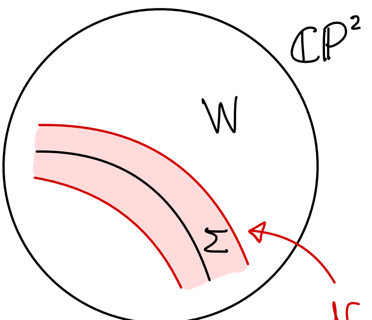
$$d_b(Y_{g,e}, \mathfrak{s}_g|_Y) = \frac{1}{4} + \frac{g^2}{e} + \frac{e}{4}$$

torsion

④ PROOF of the THOM CONJECTURE

Assume $d \neq 0$ and by contradiction suppose $\exists\, \Sigma \subseteq \mathbb{CP}^2$

$$g(\Sigma) \leqslant \frac{d^2-3d}{2} \quad \xrightarrow[\text{(stab.)}]{\text{WLOG}} \quad g(\Sigma) = \frac{d^2-3d}{2}.$$



The complement of Σ in $\mathbb{CP}^2$ is a 4-mfd w/ $b_2 = 0$ and $\partial W = -Y_{g,d^2} = Y_{g,-d^2}$.

Let $\mathfrak{s} \in Spin^c(\mathbb{CP}^2)$ w/ $c_1(\mathfrak{s}) = -3 \cdot H$. Identify $\mathfrak{s}|_{X_{g,d^2}}$:

$$\left.\begin{aligned} \langle c_1(\mathfrak{s}), [\Sigma] \rangle &= -3d \\ \langle c_1(\mathfrak{s}_k), [\Sigma] \rangle &= d^2 + 2k \end{aligned}\right\} \quad \mathfrak{s} = \mathfrak{s}_k \quad \text{if} \quad k = \frac{-d^2-3d}{2}.$$

$\rightsquigarrow$ Restricts also to $\mathfrak{s}_k$ on $\partial X_{g,d^2} = Y_{g,d^2}$.

However, $\partial W = -Y_{g,d^2} = Y_{g,-d^2} \rightsquigarrow$ need to change orient.

$$\Rightarrow \quad s|_{Y_{g,-d^2}} = s_{K+d^2} = s_g .$$

using $g = \frac{d^2-3d}{2}$.

$$\Rightarrow \quad d_b(Y_{g,-d^2}, s_g) = \frac{1}{4} - \frac{g^2}{d^2} - \frac{d^2}{4}$$

$$= -2-g$$

However, $\partial W = Y_{g,-d^2}$ and $b_2(W) = 0$:

$$\underbrace{c_1(s)^2}_{0} + \underbrace{b_2^-(W)}_{0} \leq \underbrace{4 \cdot d_b(Y_{g,-d^2}, s)}_{-8-4g} + \underbrace{2 \cdot b_1(Y)}_{4g} \quad ⚡$$

$\square$

Rk: This proof works also for X smooth $\mathbb{Z}H\mathbb{CP}^2$.

Rk: The proof fails for $\#^n \mathbb{CP}^2$. In fact,

$$G_{\#^2\mathbb{CP}^2}(a,b) \neq G_{\mathbb{CP}^2}(a) + G_{\mathbb{CP}^2}(b)$$

⑤ THE MIXED INVARIANT

Prop (vanishing theorem): Let $W : Y_0 \longrightarrow Y_1$ w/ $b_2^+(W) \geq 1$.
Then $F_{W,s} : HF^\infty(Y_0, t_0) \longrightarrow HF^\infty(Y_1, t_1)$ is zero.

Application: the image of the map in HF^- is contained in HF_{red}.

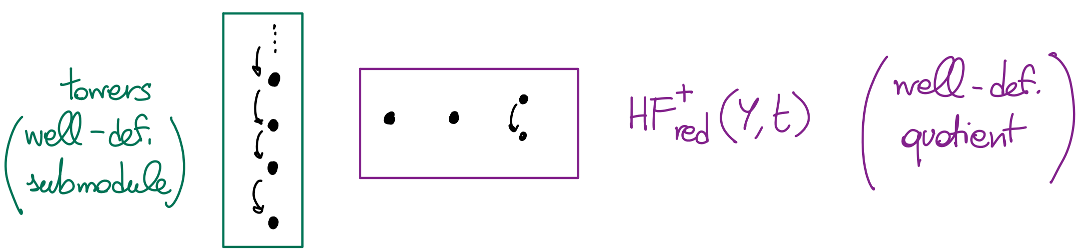
this is also the $\mathbb{F}[U]$-torsion submodule

$$
\begin{tikzcd}
 & HF^-_{red}(Y_1, t_1) := \mathrm{Ker}(\iota) \arrow[d, hook] \\
HF^-(Y_0, t_0) \arrow[ur, dashed, "\exists \text{ factorisation}"] \arrow[r, "F^-_{W,s}"] \arrow[d] & HF^-(Y_1, t_1) \arrow[d, "\iota"'] \\
HF^\infty(Y_0, t_0) \arrow[r, "0"] & HF^\infty(Y_1, t_1)
\end{tikzcd}
$$

induced by inverting U at the complex level (i.e., quotient by the $\mathbb{F}[U]$-torsion)

Analogously, $\exists$ similar factorisation for HF^+.

Quick recap of the structure of HF^+:



maps to the towers

$$\begin{tikzcd}
HF^{\infty}(Y_0,t_0) \arrow[r, "0"] \arrow[d] & HF^{\infty}(Y_1,t_1) \arrow[d] \\
HF^{+}(Y_0,t_0) \arrow[r] \arrow[d, two heads] & HF^{+}(Y_1,t_1) \\
HF^{+}_{red}(Y_0,t_0) \arrow[ur, dashed, "\exists \text{ factorisation}"'] &
\end{tikzcd}$$

Prop: $\exists$ isomorphism $HF^{+}_{red}(Y,t) \cong HF^{-}_{red}(Y,t)$

Proof: The SES

$$0 \longrightarrow CF^{-} \xrightarrow{\iota} U^{-1}\cdot CF^{-} \xrightarrow{\pi} \frac{U^{-1}\cdot CF^{-}}{CF^{-}} \longrightarrow 0$$

induces the Tate-Swan LES

up to a grading shift (sometimes ι is mult. by U)

$$\longrightarrow HF^{-} \xrightarrow{\iota_*} HF^{\infty} \xrightarrow{\pi_*} HF^{+} \xrightarrow{\delta} HF^{-} \longrightarrow$$

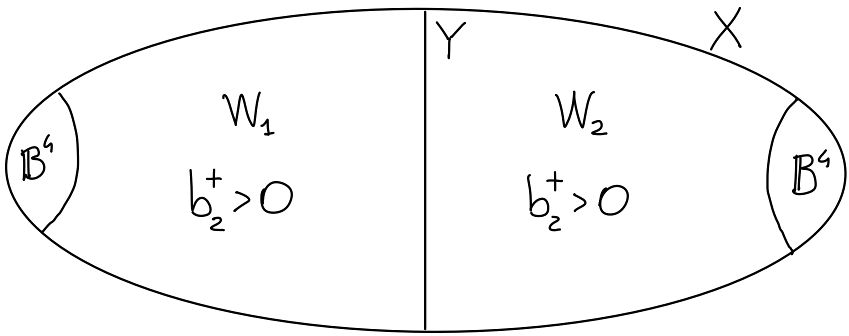
which by exactness induces isomorphisms

$$\begin{tikzcd}
\operatorname{coker} \pi_* \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\shortparallel"] & \operatorname{im} \delta \arrow[r, phantom, "\cong"] & \operatorname{Ker} \iota_* \arrow[d, phantom, "\shortparallel"] \\
HF^{+}_{red} \arrow[rr, "\sim", "\delta"'] & & HF^{-}_{red}
\end{tikzcd}$$

$\square$

Construction of the mixed invariant

Let X^4 closed, smooth, with $b_2^+ \geq 2$.



Fix $s \in Spin^c(X)$ and consider maps in HF°.

$$\begin{tikzcd}
 & HF^+(Y,s) \arrow[r, "F_{W_2,s}"] \arrow[d, two heads] & HF^+(S^3) \\
 & HF^+_{red}(Y,s) \arrow[ur, dashed] \arrow[d, "\cong"] & \\
 & HF^-_{red}(Y,s) \arrow[d, hook] & \\
HF^-(S^3) \arrow[r, "F_{W_1,s}"'] \arrow[ur, dashed] \arrow[uuurr, bend left, "\Phi_{X,s}"] & HF^-(Y,s) &
\end{tikzcd}$$

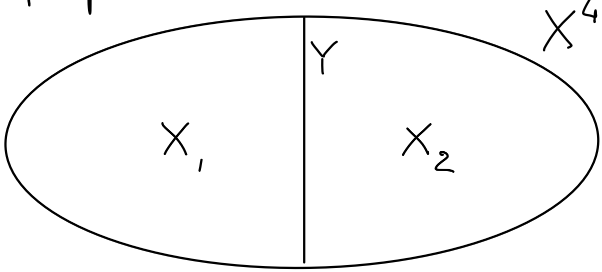
called the MIXED INVARIANT

$\rightsquigarrow$ Get a map $\Phi_{X,s} : HF^-(S^3) \longrightarrow HF^+(S^3)$.

Thm: $\Phi_{X,s}$ does not depend on the chosen cut Y.

A TQFT perspective

$\partial X_1 = Y$



$\partial X_2 = -Y$

$X_1 \rightsquigarrow \xi_1 \in HF^-(Y)$ defined by $\xi_1 := F^-_{W_1}(1)$

$HF^-(-Y) \ni \xi_2 \leftsquigarrow X_2$ defined by $\xi_2 := F^-_{W_2}(1)$

[Here $W_1 := X_1 - B^4$, seen as a cobordism $S^3 \longrightarrow Y$, and
$W_2 := X_2 - B^4$, seen as a cobordism $-S^3 \longrightarrow -Y$. $S^3 \cong -S^3$]

In the typical TQFT framework there should be a pairing

$$(\cdot,\cdot)_Y : HF^-(Y) \otimes HF^-(-Y) \longrightarrow \mathbb{F}$$

and we would then define $HF^-(X) := (\xi_1, \xi_2)_Y$.

What we actually have is that $\xi_i \in HF^-_{\underline{\underline{red}}}(\pm Y_i)$, and the above pairing $(\cdot,\cdot)_Y$ can be define on the <u>reduced</u> homologies.

To define it, start from a certain "tautological" pairing

$$\langle \cdot , \cdot \rangle_Y : HF^+_{(red)}(Y) \otimes HF^-_{(red)}(-Y) \longrightarrow \mathbb{F},$$

which satisfies the following duality property.

Let W^4 with $\partial W = Y_1 \amalg (-Y_0)$. We can see W as a cobordism in two different ways:

- $W : Y_0 \longrightarrow Y_1 \quad \rightsquigarrow \quad \mathcal{F}_W^+ : HF^+_{(red)}(Y_0) \to HF^+_{(red)}(Y_1)$
- $W : -Y_1 \to -Y_0 \quad \rightsquigarrow \quad \mathcal{F}_W^- : HF^-_{(red)}(-Y_1) \to HF^-_{(red)}(-Y_0)$

Then:

$$\langle \cdot , \mathcal{F}_W^-(\cdot) \rangle_{Y_0} = \langle \mathcal{F}_W^+(\cdot), \cdot \rangle_{Y_1} \qquad \circledast$$

Using $\delta : HF^+_{red}(Y) \xrightarrow{\sim} HF^-_{red}(Y)$, we define

$$(\cdot , \cdot)_Y : HF^-(Y) \otimes HF^-(-Y) \longrightarrow \mathbb{F}$$

by $(\xi, \eta)_Y := \langle \delta^{-1}(\xi), \eta \rangle_Y$.

Thus, our TQFT would give

$$\begin{aligned} HF^-(X) &= (\xi_1, \xi_2)_Y = \left(\mathcal{F}^-_{W_1}(1), \mathcal{F}^-_{W_2}(1)\right)_Y \\ &= \langle \delta^{-1} \circ \mathcal{F}^-_{W_1}(1), \mathcal{F}^-_{W_2}(1) \rangle_Y \\ &\overset{\circledast}{=} \langle \mathcal{F}^+_{W_2} \circ \delta^{-1} \circ \mathcal{F}^-_{W_1}(1), 1 \rangle_{S^3} \end{aligned}$$

Thus, $HF^-(X)$ is controlled by the map $\mathcal{F}^+_{W_2} \circ \delta^{-1} \circ \mathcal{F}^-_{W_1}$, which is exactly the definition of the mixed invariant Φ_X.

⑥ EXOTIC MANIFOLDS

Thm (O-Sz.) X^4 closed, $\pi_1 = 1$, C^∞ cx surface with $b_2^+ \geq 2$.
Then $\Phi_{X,k} = \pm 1$, where k is the canonical spin^c structure.
$c_1(k)$ is the canonical class

Recall: K3 is any $\pi_1 = 1$, C^∞ cx surface with $c_1(k_X) = 0$.
(All such cx surfaces are diffeomorphic to each other).
e.g. $\{x^4 + y^4 + z^4 + w^4 = 0\} \subseteq \mathbb{CP}^3$ is a K3 surface.
$Q_{K3} = 2E_8 \oplus 3H$

Thm: $K3 \# \overline{\mathbb{CP}^2}$ and $3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2}$ are an exotic pair (i.e., they are homeom. but not diffeom. to each oth.)

Pf: Homeomorphic

Both are closed, $\pi_1 = 1$. Their int. forms are indefinite, so they are the same iff they have same rk, σ, parity:

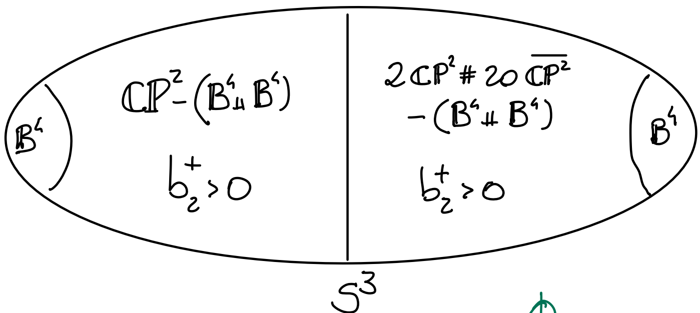
	$(2E_8 \oplus 3H) \oplus (-1)$	$3\cdot(+1) \oplus 20(-1)$
rk	$2\cdot 8 + 3\cdot 2 + 1 = 23$	$3 + 20 = 23$
σ	$-16 - 1 = -17$	$3 - 20 = -17$
parity	odd (b/c of $\overline{\mathbb{CP}^2}$)	odd

$\Rightarrow$ By Freedman the mfds are homeom. to each other.

<u>Not diffeomorphic</u>

$$OS_z \Rightarrow \Phi_{K3\#\overline{\mathbb{CP}^2},k} \neq 0$$

However $\Phi_{3\cdot\mathbb{CP}^2\#20\cdot\overline{\mathbb{CP}^2},s} = 0 \ \forall s$, since the mixed invariant is a map that factors through $HF_{red}(S^3) = 0$.



$$\begin{tikzcd} HF^-(S^3) \arrow[r] \arrow[rr, bend left, "\Phi_{X,s}"] & HF_{red}(S^3) \arrow[r] \arrow[d, equal] & HF^+(S^3) \\ & 0 & \end{tikzcd}$$

$\square$