1. A_∞ algebras

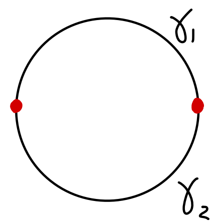
1. Historical Motivation

(X, x_0) = pointed topological space

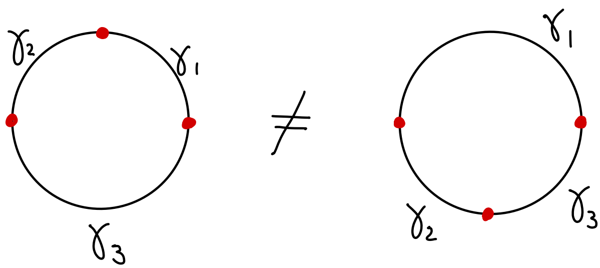
ΩX = loop space $= \{ \gamma : S^1 \xrightarrow{C^0} X \mid \gamma(1) = x_0 \}$

ΩX comes with an "operation", namely composition

$$m_2 : \Omega X \times \Omega X \longrightarrow \Omega X$$
$$(\gamma_1 , \gamma_2) \longmapsto \gamma_1 * \gamma_2$$



<u>RK</u>: This operation is not associative, but almost.

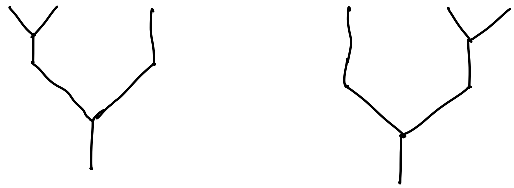


$(\gamma_1 * \gamma_2) * \gamma_3$ $\qquad\qquad$ $\gamma_1 * (\gamma_2 * \gamma_3)$

The concatenation points can be moved continuosly, so we get

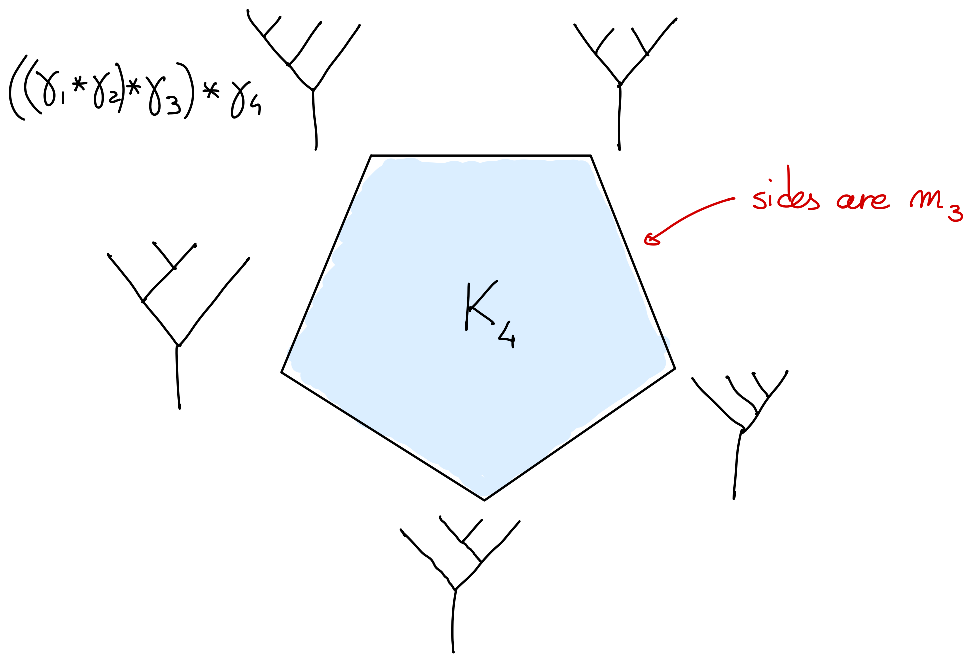
$$m_3 : I \times \Omega X \times \Omega X \times \Omega X \longrightarrow \Omega X$$

homotopy between $(\gamma_1 * \gamma_2) * \gamma_3$ and $\gamma_1 * (\gamma_2 * \gamma_3)$



<u>Binary tree representation</u>: when we take a sequence of non-associative multiplications, the order we do them is encoded by a binary tree with n leaves and 1 root.

<u>Case $n=4$</u>

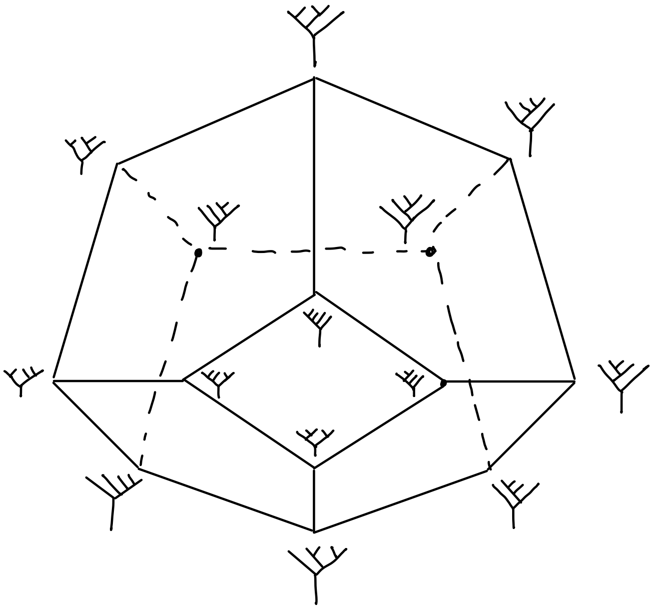


Since all the concatenation basepts can be homotoped simultaneously, we have a map

$$m_4 : \underset{\text{pentagon}}{K_4} \times (\Omega X)^{\times 4} \longrightarrow \Omega X$$

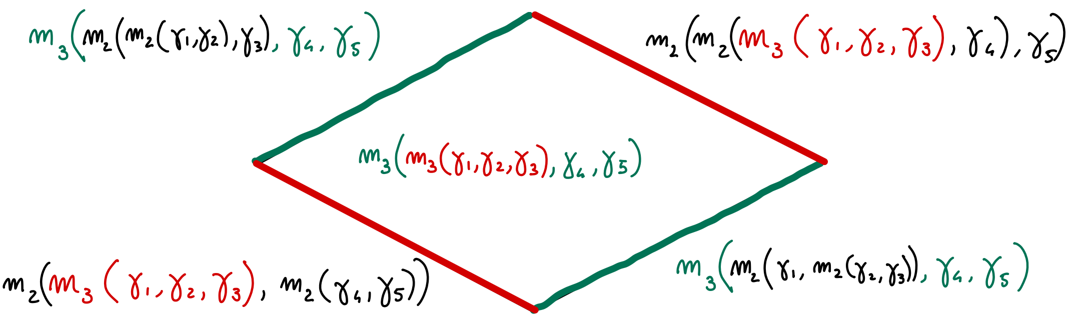
<u>Case $n=5$</u>

$K_5 =$



Q: What do the squares in the boundary of K_5 mean?

<u>Solution</u>: It's the composition of two m_3 operations



Thm (Stasheff '63) $\exists$ sequence of polytopes K_n of dimension $n-2$, called <u>ASSOCIAHEDRA</u>, and maps

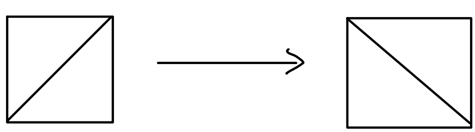
$$m_n : K_n \times (\Omega X)^n \longrightarrow \Omega X.$$

Moreover, a topological space Y is hty equivalent to ΩX (i.e. it can be "delooped") if and only if

- $\pi_0(Y)$ is a group $(\cong \pi_1(X))$, and
- $\exists$ maps $m_n : K_n \times (Y^{\times n}) \longrightarrow Y$ satisfying certain compatibility & unit conditions. (Y is an A_∞-space)

Thm (Stasheff '63) Y an A_∞-space. Then $C_*(Y)$ is an A_∞-algebra, where $\mu_1 =$ differential and $\mu_{i\geq 2}$ are induced by m_i.

Ex: K_n can be interpreted as a polytope whose vertices are triangulations of $(n+1)$-gon.
Higher operations are interpreted as changes of triangulations

e.g. m_3 : 

2. DEFINITION

more generally, commutative ring

grading by non-comm. grp.

Def: Let $\mathbb{K}$ be a field. A $\mathbb{Z}$-graded A_∞-ALGEBRA over $\mathbb{K}$ is a $\mathbb{Z}$-graded vector space

→ $\mathbb{K}$-module

$$A = \bigoplus_{p \in \mathbb{Z}} A_p$$

together with homogeneous $\mathbb{K}$-linear maps

$$\mu_i : \underbrace{A \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} A}_{i \text{ times}} \longrightarrow A$$

henceforth denoted by $A^{\otimes i}$; $\mathbb{K}$ need not be central in A

subject to the structure relations (R_n) $\forall n \geq 1$

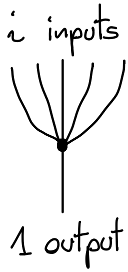
$$(R_n) \sum_{r=0}^{n-1} \sum_{s=1}^{n-r} \underline{(-1)^{r+s \cdot t}} \mu_u \left(id^{\otimes r} \otimes \mu_s \otimes id^{\otimes t} \right) = 0$$

where $t = n - r - s \in [0, n-1]$

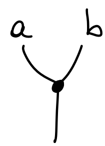
$$u = r + t + 1 = n - s + 1$$

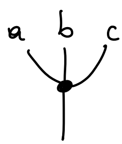
from this point onwards, char $\mathbb{K} = 2$, so I will skip all signs

Pictorial interpretation

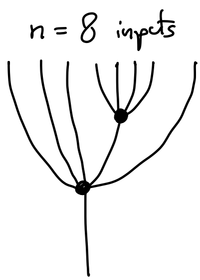
$\mu_i =$ 

e.g. $\mu_1(a) \rightsquigarrow$ 

$\mu_2(a,b) \rightsquigarrow$ 

$\mu_3(a,b,c) \rightsquigarrow$ 

(Rn) can be interpreted graphically: consider all 2-node planar trees with n inputs and 1 output, e.g.



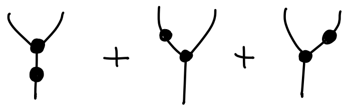
$\rightsquigarrow m_5(id^{\otimes 3} \otimes m_4 \otimes id)$

The signed sum of all these terms vanishes.

Unpacking the first relations

(R1)  $= 0$

$\mu_1 \circ \mu_1 = 0$, i.e. μ_1 is a <u>differential</u>

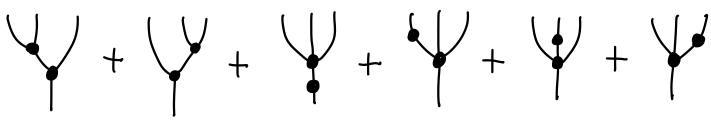
(R2)  $= 0$

$$\mu_1 \circ \mu_2 + \mu_2(\mu_1 \otimes id) + \mu_2(id \otimes \mu_1) = 0$$

Writing $\mu_2(a,b) = a \cdot b$ and $\mu_1 = d$, this is

$d(a \cdot b) = (da) \cdot b + a \cdot (db)$ [recall $2 = 0$]

(i.e <u>Leibniz rule</u>)

(R3)  $= 0$

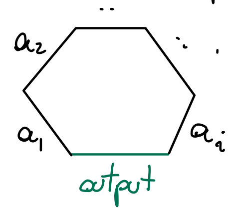
$$\mu_2(\mu_2 \otimes id) + \mu_2(id \otimes \mu_2) + \underbrace{\mu_1 \circ \mu_3}_{\uparrow \text{ call it } H} + \mu_3\big(\underbrace{\mu_1 \otimes id \otimes id + id \otimes \mu_1 \otimes id + id \otimes id \otimes \mu_1}_{\text{differential } d \text{ on } A^{\otimes 3}}\big) = 0$$

$$\Rightarrow \mu_2(\mu_2 \otimes id) + \mu_2(id \otimes \mu_2) = d \circ H + H \circ d$$

i.e. the product is associative up to homotopy.

A less popular pictorial representation

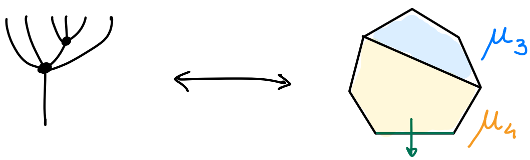
i sides are inputs



μ_i can be represented with an $(i+1)$-gon

(Rn) Summing over all possible diagonals (including sides) gives 0.

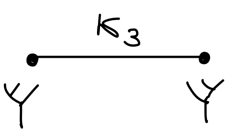
e.g.



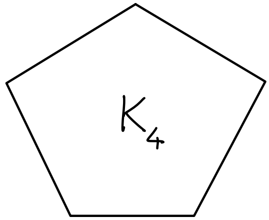
Examples

1) (Stasheff '63) $C_*(\Omega X)$ is an A_∞-algebra with $\mu_1 =$ differential, μ_2 induced by loop composition.

The associahedra encode the A_∞-relations



$$\mu_2(\mu_2 \otimes id) - \mu_2(id \otimes \mu_2) = \overbrace{\mu_1 \circ \mu_3 + \mu_3 \circ \left(\mu_1 \otimes id^{\otimes 2} + id \otimes \mu_1 \otimes id + id^{\otimes 2} \otimes \mu_1\right)}^{d\mu_3}$$



$$\left.\begin{array}{l}\mu_2(\mu_3 \otimes id) + \mu_2(id \otimes \mu_2) + \\ \mu_3(\mu_2 \otimes id \otimes id) + \mu_3(id \otimes \mu_2 \otimes id) + \mu_3(id \otimes id \otimes \mu_2)\end{array}\right\} \text{5 sides}$$

$$=$$

$$\mu_1 \circ \mu_4 + \mu_4 \circ \mu_1 \quad =: d\mu_4$$

EX: Reconstruct (R6) from the K_5 associahedron.

2) B = ordinary algebra over $\mathbb{k}$

Hochschild cochain complex

$$\mathrm{Hom}(\mathbb{k}, B) \longrightarrow \mathrm{Hom}(B, B) \longrightarrow \mathrm{Hom}(B^{\otimes 2}, B) \longrightarrow \dots$$

$$(df)(b_0, \dots, b_N) := b_0 \cdot f(b_1, \dots, b_N) + \sum_{i=1}^{N} f(b_0, \dots, b_{i-1} b_i, \dots, b_N) + f(b_0, \dots, b_{N-1}) \cdot b_N$$

$\rightsquigarrow$ Hochschild cohomology of B.

Let ε = formal variable of degree $2-N$, for some $N \neq 2$.

$A := B[\varepsilon]/(\varepsilon^2)$, $c : B^{\otimes N} \longrightarrow B$ linear

Define maps $\mu_i : A^{\otimes i} \longrightarrow A$ by setting

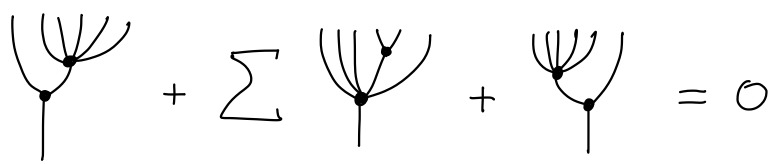
- μ_2 induced by multiplication on B
- $\mu_N := \varepsilon \cdot c$
- All other $\mu_i \equiv 0$.

EX: $(A, (\mu_i))$ is an A_∞-algebra iff c is a Hochschild cocycle.

Hint: the only non-trivial relation is $(R(N+1))$;

if there is at least an ε in the inputs, all summands vanish;

inputs w/ no ε give exactly the cocycle condition



3) Def: A DIFFERENTIAL GRADED ALGEBRA (DGA) is an A_∞-algebra with $\mu_i \equiv 0$ $\forall i \geq 3$.

Spoiler: strands algebras (an important family of examples) are dgas.

Homology

Prop: Given a ($\mathbb{Z}$-graded) A_∞-algebra A, there is a canonical algebra structure on $H_*(A)$.

ordinary, not A_∞

Pf: $\mu_2 : A \otimes A \longrightarrow A$ satisfies

(R2) $\mu_2 \circ \mu_1 + \mu_1 \circ \mu_2 = 0$ (μ_2 is a chain map)

$\Rightarrow$ μ_2 descends to $H_*(A \otimes A) \longrightarrow H_*(A)$, hence by composition $H_*(A) \otimes H_*(A) \to H_*(A \otimes A) \longrightarrow H_*(A)$ we get

$$\overline{\mu_2} : H_*(A) \otimes H_*(A) \longrightarrow H_*(A)$$

Then, we consider

(R3) $\mu_2(\mu_2(a,b),c) - \mu_2(a,\mu_2(b,c)) = d \circ H + H \circ d$

Recall that

- $d_A = \mu_1$,
- $d_{A^{\otimes 3}} = \mu_1 \otimes id \otimes id + id \otimes \mu_1 \otimes id + id \otimes id \otimes \mu_1$,
- $H = \mu_3$.

Passing to homology, we get

$$\overline{\mu_2}\left(\overline{\mu_2} \otimes id\right) + \overline{\mu_2}\left(id \otimes \overline{\mu_2}\right) = 0 ,$$

i.e. $\overline{\mu_2}$ is associative. $\square$

<u>RK</u>: Next time we will see that $H_*(A)$ can be given an A_∞-structure extending $\mu_1^H = 0$ and $\mu_2^H = \overline{\mu_2}$.

Note $\mu_1^H = 0 \Rightarrow \mu_2^H$ is associative, no matter what μ_3^H is.

(So there can be non-trivial A_∞-algebras that are associative.)

<u>Units</u>

<u>Def</u>: A <u>STRICT UNIT</u> of A is <u>an element $1 \in A$</u> s.t. of degree 0

•) $\mu_2(a, 1) = a = \mu_2(1, a) \quad \forall a \in A$

•) $\forall i \neq 2$, $\mu_i(a_1, \dots, a_i) = 0$ if $\exists j : a_j = 1$

Def: A <u>HOMOLOGICAL UNIT</u> is $1 \in H_*(A)$ s.t.

$$\overline{\mu_2}(\alpha, 1) = \alpha = \overline{\mu_2}(1, \alpha) \quad \forall \alpha \in H_*(A).$$

Thm: If A is homologically unital, then $\exists$ an A_∞-quasi-isom. A_∞-algebra B (with $\mu_1^B \equiv 0$) that is <u>strictly unital</u>.