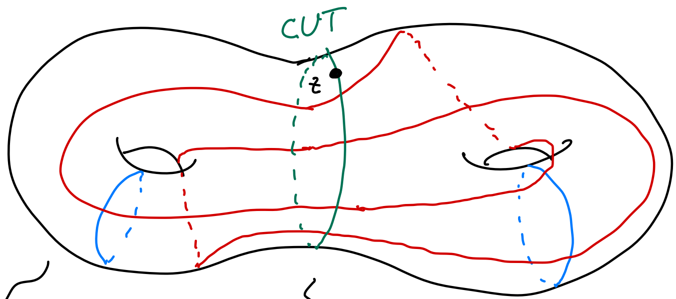
2. Bar construction

① MODERN TOPOLOGICAL MOTIVATION

1. Bordered Floer homology

A_∞-algebras and modules are helpful to compute HF using a cut-and-paste approach.



An A_∞-module over $\mathcal{A}_T$: $M_{\mathcal{A}_T}$

An A_∞-algebra $\mathcal{A}_T$ (in fact a DGA)

A type-D structure over $\mathcal{A}_T$: ${}^{\mathcal{A}_T}N$

<u>Thm</u> (Lipshitz-Ozsváth-Thurston) $\widehat{CF}(Y) \underset{q.i.}{\simeq} M_{\mathcal{A}_T} \boxtimes {}^{\mathcal{A}_T}N$

The next examples are special cases or directly inspired from bordered Floer homology.

2. Immersed curve invariant

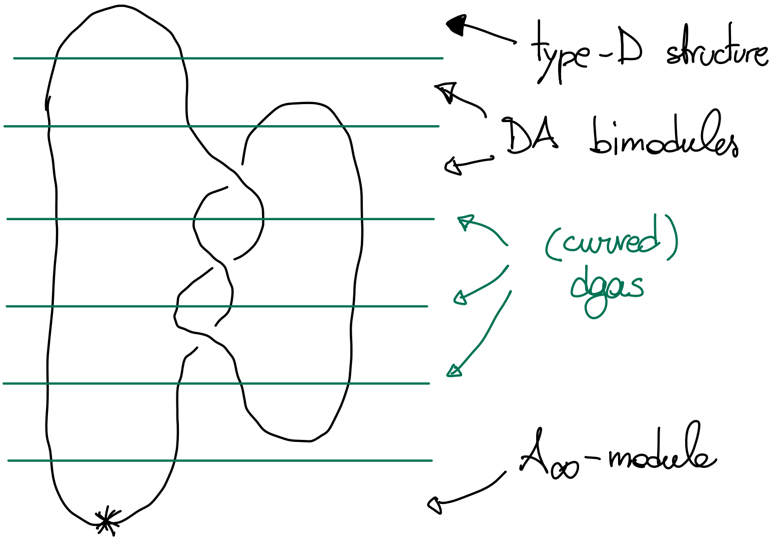
Let Y be a cpt oriented manifold with $\partial Y = T^2$ (e.g. Knot complements) + parameterisation of ∂Y.

LOT $\rightsquigarrow$ $\exists$ dga $\mathcal{A}_T$ associated to parameterisation

$\exists$ type-D structure N over $\mathcal{A}_T$ associated to Y.

Thm (Hanselman-Rasmussen-Watson) N can be interpreted as a collection of immersed closed curves with local systems on ∂Y. Moreover, this multi-curve does not depend on the choice of parameterisation of ∂Y.

3. Ozsváth-Szabó's bordered HFK



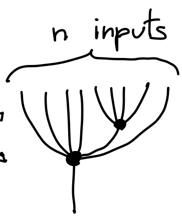
The tensor product of all these (bi)-modules recovers CFK (very efficient!!!)

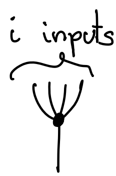
② BAR CONSTRUCTION

We now re-cast the definition of A_∞-algebra in a more compact (but also more abstract) terminology.

Recall: Let $\mathbb{K}$ be a commutative ring.

An $\underline{A_\infty\text{-ALGEBRA}}$ over $\mathbb{K}$ is a $\mathbb{K}$-bimodule A together with homogeneous $\mathbb{K}$-linear maps $\mu_i : A^{\otimes i} \longrightarrow A$ subject to the structure relations (R_n) $\forall n \geq 1$

(R_n) $\sum$ 

$= 0$, where 

denotes μ_i.

Def: Let A be a $\mathbb{K}$-bimodule. The $\underline{\text{BAR CONSTRUCTION}}$ is

$$TA := \mathbb{K} \oplus A \oplus A\otimes A \oplus A\otimes A\otimes A \oplus \cdots$$

There is an obvious algebra structure on TA, with

- multiplication $\otimes : A^{\otimes n} \otimes A^{\otimes m} \longrightarrow A^{\otimes(n+m)}$
- unit $\eta : \mathbb{K} \longrightarrow \mathbb{K} \subset TA$

  ↖ copy of $\mathbb{K}$ inside TA

However, we are more interested in TA as a $\underline{\text{coalgebra}}$.

Def: A (counital, coassociative) <u>COALGEBRA</u> over $\Bbbk$ is a tuple (C, Δ, ε) where:

- C is a $\Bbbk$-bimodule
- $\Delta : C \longrightarrow C \otimes C$ is the <u>comultiplication</u>
- $\varepsilon : C \longrightarrow \Bbbk$ is the <u>counit</u>

subject to the following relations:

- $(id_C \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta$ (coassociativity of Δ)
- $(id_C \otimes \varepsilon) \circ \Delta = id_C = (\varepsilon \otimes id_C) \circ \Delta$ (compatibility)

Def: the <u>DECONCATENATION</u> comultiplication on TA is

$$TA \longrightarrow TA \boxtimes TA$$

this symbol is to distinguish it from the $\otimes$ in TA

$$a_1 \otimes \cdots \otimes a_n \longmapsto \sum_{i=0}^{n} (a_1 \otimes \cdots \otimes a_i) \boxtimes (a_{i+1} \otimes \cdots \otimes a_n)$$

Prop: $(TA, \Delta, \varepsilon)$ is a coalgebra, where Δ is the deconcatenation comultiplication and the projection

$$\varepsilon : TA = \Bbbk \oplus A \oplus A^{\otimes 2} \oplus \cdots \longrightarrow \Bbbk$$

is the counit.

Def: $(B, \nabla, \eta, \Delta, \varepsilon)$ is a <u>BIALGEBRA</u> if
(multiplication: ∇; unit: η)

- (B, ∇, η) is a (unital associative) algebra
- (B, Δ, ε) is a (counital coassociative) coalgebra
- $\varepsilon \circ \eta = id_{\mathbb{K}}$
- $\Delta \circ \eta = \eta \otimes \eta$ } here we identify $\mathbb{K} = \mathbb{K} \otimes \mathbb{K}$
- $\varepsilon \circ \nabla = \varepsilon \otimes \varepsilon$ }
- $\Delta \circ \nabla = (\nabla \otimes \nabla) \circ (id_B \otimes \tau \otimes id_B) \circ (\Delta \otimes \Delta)$ on $B \otimes B$
(swap $\tau: B \otimes B \to B \otimes B$)

Def: A <u>HOPF ALGEBRA</u> is a bialgebra H with an <u>"antipodal" map</u> $S : H \to H$ satisfying

$$\nabla \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = \nabla \circ (id \otimes S) \circ \Delta$$

Remarks:

- •) $(TA, \otimes, \eta, \Delta, \varepsilon)$ is not a bialgebra (let alone a Hopf algebra).
(canonical unit & counit: η, ε; deconcatenation: Δ)
- •) $\exists$ a comultiplication on TA (called "standard") such that $(TA, \otimes, \eta, \Delta_{std}, \varepsilon)$ is a Hopf algebra

•) $\exists$ a non-std multiplication on TA such that $(TA, \bullet, \eta, \Delta, \varepsilon)$ is a Hopf algebra ("shuffle algebra")

Reduced bar construction

Def: An AUGMENTATION of a unital algebra (A, ∇, η) is a $\mathbb{k}$-linear map $\varepsilon : A \longrightarrow \mathbb{k}$ s.t. $\varepsilon \circ \eta = id_{\mathbb{k}}$

Def: A COAUGMENTATION of a counital coalgebra (A, Δ, ε) is a $\mathbb{k}$-linear map $\eta : \mathbb{k} \to A$ s.t. $\varepsilon \circ \eta = id_{\mathbb{k}}$

Rk: TA is naturally coaugmented.

$$\{\text{coaugm. counital coalgebras}\} \longleftrightarrow \{\text{non-counital coalgebras}\}$$

$$A \longmapsto \overline{A} = \operatorname{Ker} \varepsilon \cong A / \operatorname{Im} \eta$$

$$\mathbb{k} \oplus \overline{A} \longleftarrow \overline{A}$$

This in fact induces an equivalence of categories.

e.g. $\overline{TA} = A \oplus A^{\otimes 2} \oplus \cdots$ non-counital coalgebra

The comultiplication is given by strict deconcatenation:

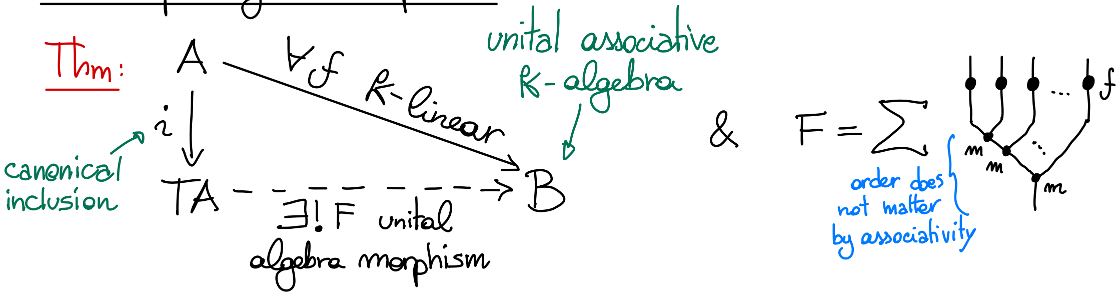
$$\overline{\Delta} : \overline{TA} \longrightarrow \overline{TA} \boxtimes \overline{TA}$$

$$a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \boxtimes (a_{i+1} \otimes \cdots \otimes a_n)$$

③ UNIVERSAL PROPERTIES of TA

Warm-up: algebra morphisms

Thm:



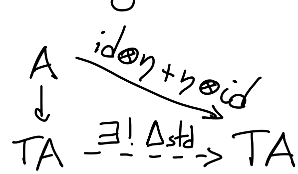
Sketch of proof: Uniqueness

- unitality forces definition of F on $\Bbbk \subset TA$
- commutativity forces def. of F on $A \subset TA$
- homomorphism forces def of F on $A^{\otimes 2}, A^{\otimes 3}, \dots$

  by $F(v \otimes w) = m(F(v), F(w))$

Check that F so defined is indeed a morphism. □

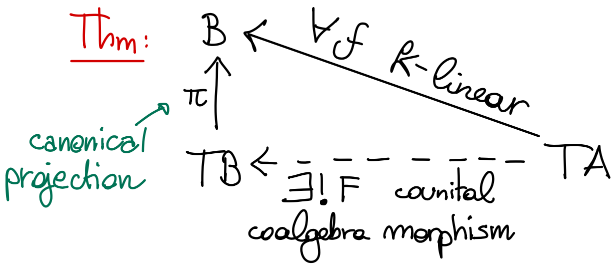
Application: define $\Delta_{std} : TA \longrightarrow TA \boxtimes TA$ by:

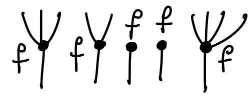


- $\Delta_{std}(1) = 1 \boxtimes 1$
- $\Delta_{std}(a) = a \boxtimes 1 + 1 \boxtimes a$
- $\Delta_{std}(v \otimes w) = \Delta_{std}(v) \otimes \Delta_{std}(w)$ (hom. of alg.)

Universal property for coalgebra morphisms between TC

Thm:



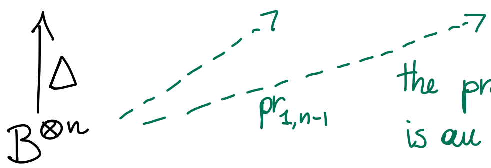
& $F = \sum$ 

Proof: Uniqueness:

- counitality forces F_0, projection of F onto $k \subseteq TB$.
- commutativity forces F_1, the projection of F onto $B \subseteq TB$.
- using $\Delta \circ F = (F \boxtimes F) \circ \Delta$, we inductively show that F_n, the projection on $B^{\otimes n} \subset TB$ is determined.

Consider the summands of $TB \boxtimes TB$ with n outputs

$$(TB \boxtimes TB)_n = (k \boxtimes B^{\otimes n}) \oplus (B \boxtimes B^{\otimes n-1}) \oplus \cdots \oplus (B^{\otimes n} \otimes id)$$

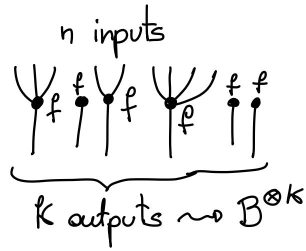


the projection of Δ on any of these is an isom $\Rightarrow pr_{1,n-1} \circ \Delta \circ F_n$ determines F_n

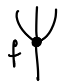
If $n \geq 2$, $pr_{1,n-1}(\Delta \circ F_n) = pr_{1,n-1}((F \boxtimes F) \circ \Delta) = (F_1 \boxtimes F_{n-1}) \circ \Delta$

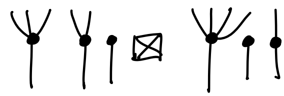
these are already determined inductively

Existence: define

$$F := \sum$$ 

Check that F so defined satisfies the desired conditions.

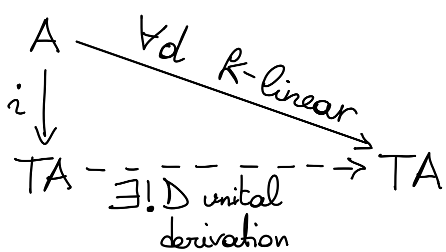
•) $\pi \circ F = \sum$  $= f$

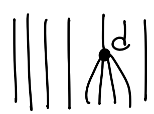
•) $\Delta \circ F = \sum$  $= (F \boxtimes F) \circ \Delta$ $\square$

Universal properties of derivation and coderivation

Def: A <u>DERIVATION</u> on a (unital associative) algebra B is a map $D : B \longrightarrow B$ satisfying:

- $D \circ \eta \equiv 0$ (derivative of constant vanishes)
- $D \circ \nabla = \nabla(id \otimes D + D \otimes id)$ (Leibnitz)

Thm: 

& $D = \sum$ 

Proof: Exercise (follow the pf of algebra morphisms). $\square$

Def: A <u>CODERIVATION</u> on a (counit., coass.) coalgebra C is a map $M : C \longrightarrow C$ satisfying

- $\varepsilon \circ M \equiv 0$ (counitality)
- $\Delta \circ M = (id \otimes M + M \otimes id) \circ \Delta$ (coLeibnitz)

Thm:

$$\begin{array}{ccc} A & \xleftarrow{\ \forall \mu\ k\text{-linear}\ } & TA \\ {\scriptstyle \pi}\uparrow & & \\ TA & \xleftarrow[\exists!\, M \text{ counital coderivation}]{} & TA \end{array}$$

& $M = \sum |||| \curlyvee |||$

Proof: Exercise (follow the pf of coalgebra morphisms). $\square$

RK: There are universal properties for $\overline{TA}$ as well, where you drop all the (co)unitality assumptions (also from the definitions of derivation and coderivation).

Def: A coderivation D (resp. M) is <u>coaugmented</u> if $\varepsilon \circ D \equiv 0$ (resp. $M \circ \eta \equiv 0$).

④ A_∞ - STRUCTURES

Def: Let A be a $\Bbbk$-bimodule. An <u>A_∞-STRUCTURE</u> on A is a counital coaugmented coderivation $M : TA \longrightarrow TA$ that is a differential, i.e. $M \circ M \equiv 0$.

Remarks:

1. One can replace the part underlined in blue with "(non-counital) coderivation $\overline{M} : \overline{TA} \rightarrow \overline{TA}$ s.t. $\overline{M} \circ \overline{M} \equiv 0$".
2. Dropping the condition that M be coaugmented recovers the definition of <u>CURVED A_∞-STRUCTURE</u>.

Prop: Let A be a $\Bbbk$-bimodule. The following are equivalent:

•) an <u>A_∞-algebra</u> $(A, \{\mu_i\})$ on A

•) an <u>A_∞-structure</u> $M: TA \circlearrowleft$

If you package the μ_i into $\mu : TA \longrightarrow A$ (with $\mu_0 \equiv 0$), then μ and M determine each other by $\mu = \pi \circ M$.

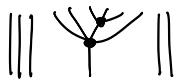
Proof:

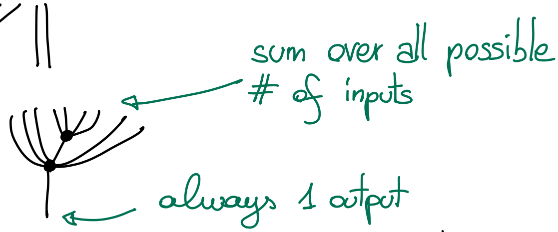
1. By the universal property, $\mu = \pi \circ M$ guarantees that μ and M determine each other. Moreover,

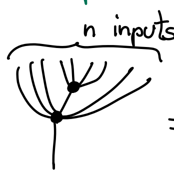
$$\mu_0 \equiv 0 \iff M \text{ is coaugmented.}$$

2. $M \circ M$ and 0 are both coderivation.

   By universal property, $M \circ M \equiv 0 \iff \pi \circ (M \circ M) \equiv 0$

3. $M \circ M = \sum$ 

   $\pi \circ (M \circ M) \equiv \sum$ 

   $\pi \circ (M \circ M) \equiv 0 \iff \forall n \; \sum$ 

$= 0$

   this is (R_n) $\square$

Remark: If A is a $\mathbb{Z}$-graded A_∞-algebra, then the map $\mu : TA \longrightarrow A$ obtained by collecting the μ_i is not graded. Fix it by considering $T(A[1]) \xrightarrow{\mu} A[1]$, so now it induces a graded $M : T(A[1]) \circlearrowleft$

← grading shift, sometimes denoted by SA

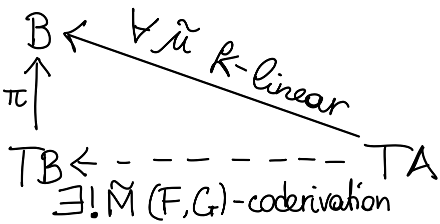
④ MORPHISMS

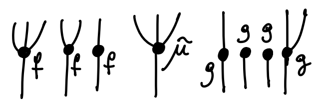
Def: Let $F, G : C_1 \longrightarrow C_2$ be a counital morph. of coalgebras.
An (F,G)-CODERIVATION is a map $\tilde{M} : C_1 \longrightarrow C_2$ s.t.:

- $\varepsilon \circ \tilde{M} \equiv 0$ (counital)
- $\Delta \circ \tilde{M} = (F \otimes \tilde{M} + \tilde{M} \otimes G) \circ \Delta$ (twisted coLeibnitz)

When $C_1 = C_2$ and $F = G = id$, we recover derivations.

Thm: Let $F, G : TA \longrightarrow TB$ counital morphisms of coalgebras.

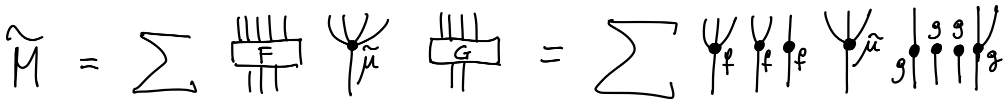


& $\tilde{M} = \sum$ 

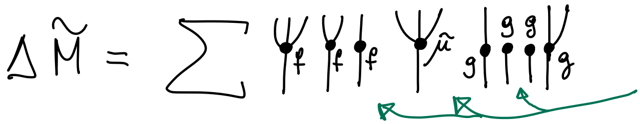
Rk: The universal property of coderivations is a corollary of this theorem (with $A = B$ and $F = G = id_{TA}$).

Proof: Exercise (follow the pf of coalgebra morphisms).

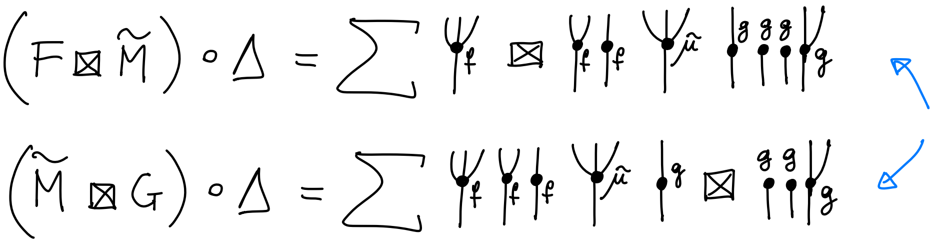
We only check that the formula



is indeed an (F,G)-coderivation.



& place a $\boxtimes$ anywhere between outputs



the sum is $\Delta \circ \widetilde{M}$

Thus, $\widetilde{M}$ so defined gives an (F,G)-coderivation. $\square$

Def: An A_∞-MORPHISM between A_∞-algebras A and B is a counital morphism of coaugmented coalgebras $F : TA \longrightarrow TB$ that is a chain map, i.e.

$$F \circ M_A = M_B \circ F.$$

Unpacking this definition

1. By the universal property, F is determined by maps

$$f_i : A^{\otimes i} \longrightarrow B \qquad \text{for } i \geq 0$$

Moreover, F coaugmented $\iff f_0 \equiv 0$.

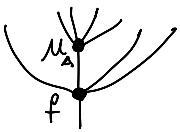
2. Both $F \circ M_A$ and $M_B \circ F$ are (F,F)-coderivations, so

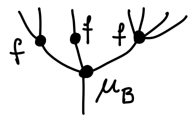
$$F \circ M_A = M_B \circ F \iff \pi \circ F \circ M_A = \pi \circ M_B \circ F$$

3.

$$\pi \circ F \circ M_A \qquad = \qquad \pi \circ M_B \circ F$$

|| $\quad\Updownarrow$ iff $\quad$ ||

$\sum$ 

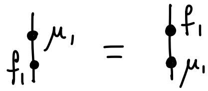
$=$ $\sum$ 

By splitting the last relation by number of inputs we get:

$$(Rn) \quad \sum_{\substack{i \in [0, n-1] \\ j \in [1, n-i]}} f_{n-j+1}\left(a_1 \otimes \cdots \otimes a_i \otimes \mu_j^A(a_{i+1} \otimes \cdots \otimes a_{i+j}) \otimes \cdots \otimes a_n\right)$$

$$= \sum_{i_1 + \cdots + i_j = n} \mu_j^B\left(f_{i_1}(a_1 \otimes \cdots \otimes a_{i_1}) \otimes \cdots \otimes f_{i_j}(a_{i_1 + \cdots i_{j-1}+1} \otimes \cdots \otimes a_n)\right)$$

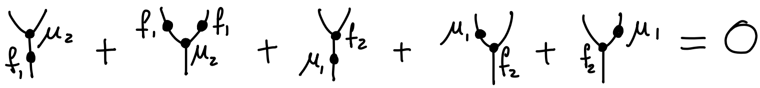
First relations

(R1) 

recall that μ_1 is a differential

$f_1 \circ \mu_1^A = \mu_1^B \circ f_1$, i.e. $f_1 : A \longrightarrow B$ is a <u>chain map</u>

$\Longrightarrow$ f_1 descends to $\overline{f_1} : H_*(A) \longrightarrow H_*(B)$

(R2) 

$$f_1 \circ \mu_2 + \mu_2 \circ (f_1 \otimes f_1) = \underbrace{\mu_1 \circ f_2 + f_2 \circ (\mu_1 \otimes id + id \otimes \mu_1)}_{d(f_2) \text{ in } Hom(A^{\otimes 2}, B)}$$

f_1 respects μ_2 up to homotopy, so

$\overline{f_1} : H_*(A) \longrightarrow H_*(B)$ is a homomorphism of algebras.

<u>Def:</u> A morphism of A_∞-algebras $\{ f_i : A^{\otimes i} \longrightarrow B \}$ is an <u>A_∞-QUASI-ISOMORPHISM</u> if f_1 is a quasi-isom., i.e. $\overline{f_1} : H_*(A) \xrightarrow{\sim} H_*(B)$ is an isomorphism.

Internal Homs: a different perspective on morphisms

<u>Def:</u> Given A, B $\Bbbk$-bimodules, consider the $\Bbbk$-bimodule

$$Hom_{\Bbbk}(A, B) = \{ \text{linear maps} \}$$

If A and B are chain complexes, so is $\mathrm{Hom}_k(A,B)$, and

$$df := d_B \circ f + f \circ d_A$$

RK: Chain maps $f: A \to B$ are cycles in this complex.

$f, g: A \longrightarrow B$ are homologous $\iff \exists$ homotopy H s.t. $f - g = d_B \circ H + H \circ d_A$

Special cases

1) $\mu_1 : A \longrightarrow A \subseteq TA$ induces a unital derivation

$M_1 : TA \longrightarrow TA$ by universal property.

When restricted to $A^{\otimes i}$, this is

$$\mu_1^{A^{\otimes i}} := \sum id_A \otimes \cdots \otimes id_A \otimes \mu_1 \otimes id_A \otimes \cdots \otimes id_A$$

For $f \in \mathrm{Hom}(A^{\otimes i}, B)$, we have

$$df = \mu_1^B \circ f + f \circ \left(\sum id \otimes \cdots \otimes \mu_i^A \otimes \cdots \otimes id\right)$$

2) Let A, B be A_∞-algebras. Recall that

$$\mathrm{Hom}_k(\overline{TA}, \overline{TB}) \overset{\sim}{\longleftrightarrow} \left\{ TA \longrightarrow TB \;\middle|\; \begin{array}{l} k\text{-linear} \\ \text{counital} \\ \text{coaugmented} \end{array} \right\}$$

$$dF := M_A \circ F + F \circ M_B$$

Thus, A_∞-morphisms from A to B correspond to cycles in $\mathrm{Hom}_k(\overline{TA}, \overline{TB})$.

⑤ HOMOTOPIES

Def: Let $M_A : TA \circlearrowleft$ and $M_B : TB \circlearrowleft$ A_∞-structures (counital coaug. coderiv. w/ $M^2 = 0$), and let $F, G : TA \longrightarrow TB$ be A_∞-morphisms (counital coaugm. morphisms of coalg. w/ $FM = MF$).

An A_∞-HOMOTOPY is a counital coaugmented (F,G)-coderivation $H : TA \longrightarrow TB$ such that

$$F - G = \underbrace{M_B \circ H + H \circ M_A}_{= \; dH \text{ in } \mathrm{Hom}_k(\overline{TA}, \overline{TB})} \qquad \circledast$$

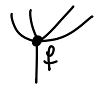
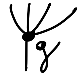
Unpacking the definition

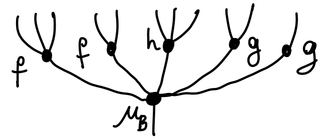
1. By universal property, H is determined by a collection of maps

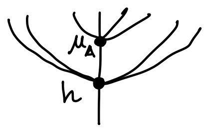
$$h_i : A^{\otimes i} \longrightarrow B$$

and $H = \sum$  .

2. Both sides of $\circledast$ are (F,G)-coderivations, so they agree if and only if $\pi(F - G) = \pi(M_B \circ H + H \circ M_A)$, i.e.

3. $\pi(F-G) = \pi F - \pi G = \sum$  $- \sum$ 

$\pi(M_B \circ H) = \sum$ 

$\pi(H \circ M_A) = \sum$ 

Thus, the relation (Rn) with n inputs is

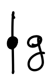
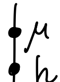
$$f_n(a_1 \otimes \cdots \otimes a_n) - g_n(a_1 \otimes \cdots \otimes a_n) =$$

$$\sum_{\substack{i \in [0, n-1] \\ j \in [1, n-i]}} h_{n-j+1}(a_1 \otimes \cdots \otimes a_i \otimes \mu_j(a_{i+1} \otimes \cdots \otimes a_{i+j}) \otimes \cdots \otimes a_n) +$$

$$\sum_{i_1 + \cdots + i_j = n} \mu_j(f_{i_1}(a_1 \otimes \cdots \otimes a_{i_1}) \otimes \cdots \otimes g_{i_j}(a_{i_1 + \cdots i_{j-1}+1} \otimes \cdots \otimes a_n))$$

↑ $h_{i_\ell}(a_{i_1 + \cdots + i_{\ell-1}} \otimes \cdots \otimes a_{i_\ell})$

First relation

(R1)  $-$  $=$  $+$ 

$$f_1 - g_1 = \mu_1 \circ h_1 - h_1 \circ \mu_1$$

⇝ h_1 is a homotopy between f_1 and g_1.

Thus, A_∞-homotopic morphisms induce the same map in H_*.

Thm (Lefèvre-Hasegawa)

1. A_∞-homotopy is an equivalence relation.
2. An A_∞-quasi-isom. always has an A_∞-homotopy inverse.

NEXT:

- ☐ A_∞-categories
- ☐ Homological perturbation (SKIP?)
- ☐ Strands algebras from bordered Floer