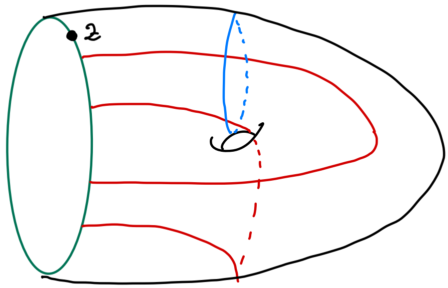
3. Strand Algebras

① CHORD DIAGRAMS



DGA

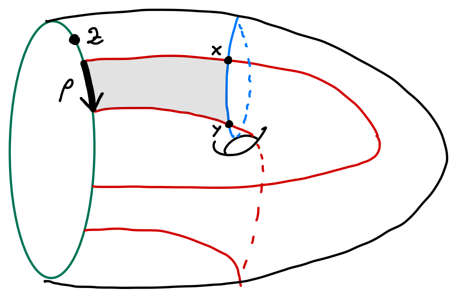
Consider the half-Heegaard diagram on the left.
The boundary is a circle w/ a basepoint z and pairs of matched circles.

We define a DGA associated to it with a two-fold objective:

1) in constructing a generator of $\widehat{CF}(\mathcal{H})$, we want to remember which curves are already occupied; ⇝ idempotents
[recall that each generator is a tuple of intersection points s.t. there is exactly one on each α (and on each β) curve]

2) remember how the partial domains meet the boundary, and if you can glue them to partial domains on the other side. ⇝ strands

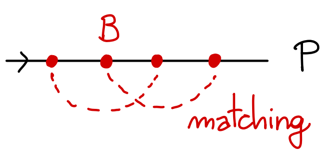
e.g.



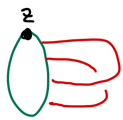
Def: A <u>CHORD DIAGRAM</u> $\mathcal{Z}$ consists of

- a cpt oriented 1-mfd P
- a finite set $B \subseteq P$, and
- a fixed-pt-free involution $\varphi : B \circlearrowleft$ (the <u>MATCHING</u>).

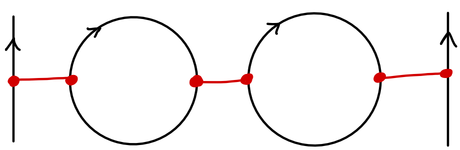
The contractible connected components of P are called <u>LINEAR BACKBONES</u>, the other ones <u>CIRCULAR BACKBONES</u>.

1) 

P = interval $\leftarrow$ Standard "punctured torus" chord diagram

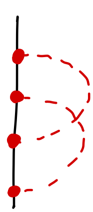
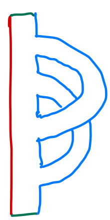
<u>RK</u>: It's obtained from 

(the boundary of the previous half Heegaard diagram) by cutting along z.

2) 

$\leftarrow$ An important ingredient in O-Sz's bordered HFK

RK: Chord diagram $\mathcal{Z}$ $\rightsquigarrow$ surface w/ partitioned boundary $F(\mathcal{Z})$ (sutured surface)

 $\rightsquigarrow$ 

$I \times P$
$\cup$
(1-handles)

[in this case $F(\mathcal{Z})$ is a punctured torus]

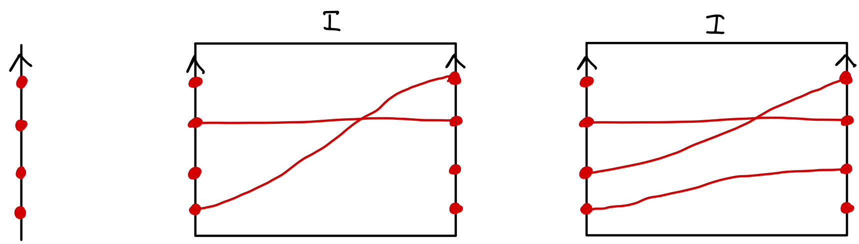
② PRE-STRAND ALGEBRA

Def: A <u>K-STRAND</u> $s = \{s_1, \dots, s_K\}$ on a chord diagram $\mathcal{Z}$ is a collection of smooth functions $s_i : I \longrightarrow P$ s.t.

- $s(0) = \{s_1(0), \dots, s_K(0)\}$ (resp. $s(1) = \{s_1(0), \dots, s_K(0)\}$) consists of K distinct points in B, and
- each s_i has <u>constant</u>, non-negative speed, i.e. $\frac{ds_i}{dt} = \alpha_i \geq 0$.

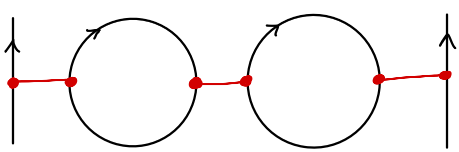
used to choose a canonical representative

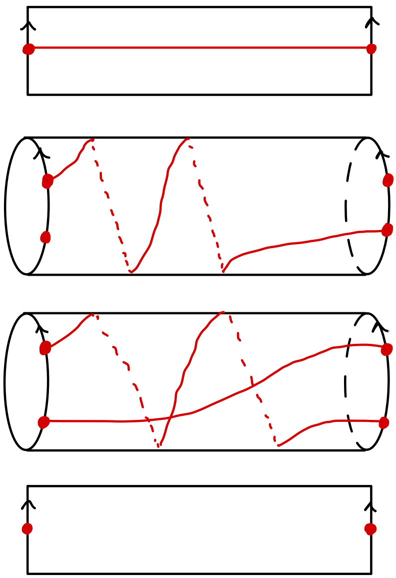
1) Examples of K-strands on the punctured torus chord diagram.



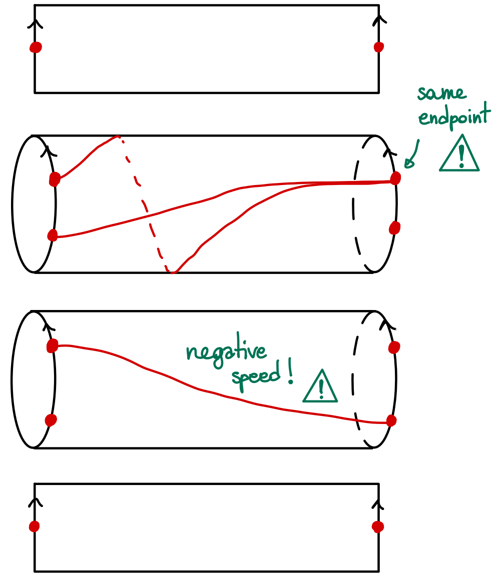
$\mathcal{Z}$ a 2-strand a 3-strand

2) Examples on the chord diagram





a 4-strand



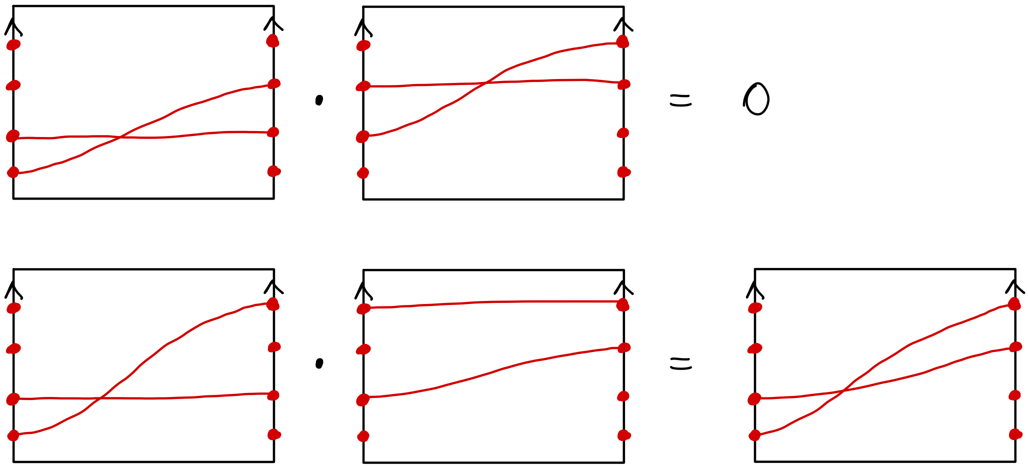
NOT a k-strand

Multiplication

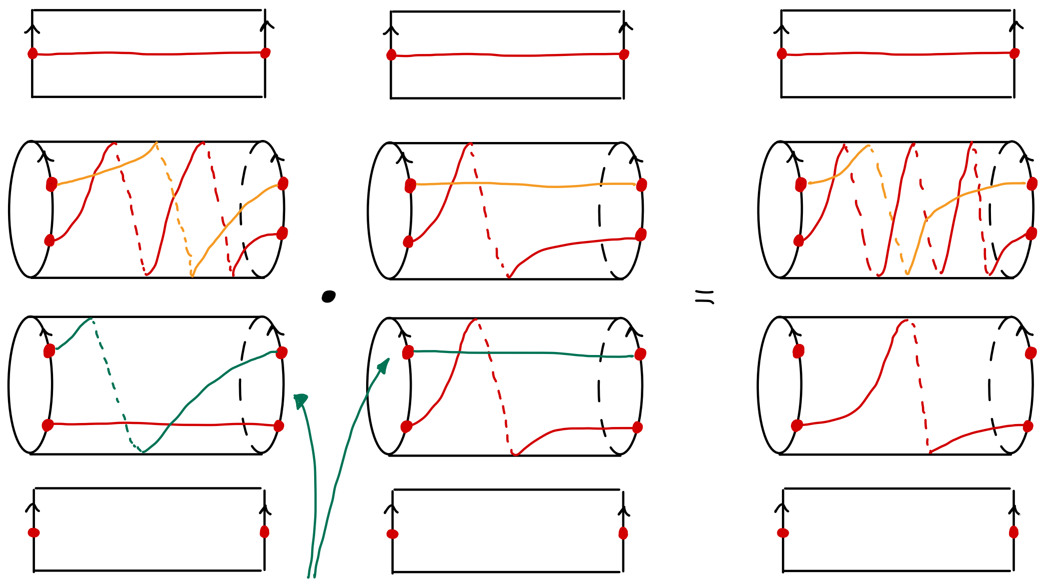
Def: Let $\widetilde{A}(\mathcal{Z}, K)$ be the $\mathbb{F}_2$-vector space freely generated by K-strands on $\mathcal{Z}$. Given two K-strands s and t, we define $s \cdot t$ as follows:

- if $s(1) \neq t(0)$, $s \cdot t = 0$ (the strands are not concatenable);
- if the concatenation (after smoothing) contains a bigon, then we set $s \cdot t = 0$;
- in all other cases, $s \cdot t$ is the concatenation, properly rescaled.

1)



2)



if you add the green strands you get a bigon

To get a nonzero element you need to concatenate the strand that winds the most with the strand that winds the most.

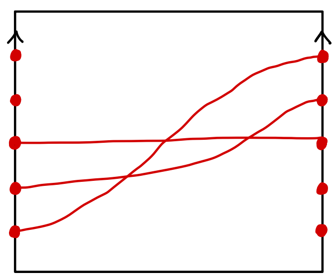
Differential

Def: for s a k-strand, we define ∂s as the of all k-strands obtained by resolving a crossing of s <u>without producing a bigon</u>, properly rescaled.

Extend linearly to a map

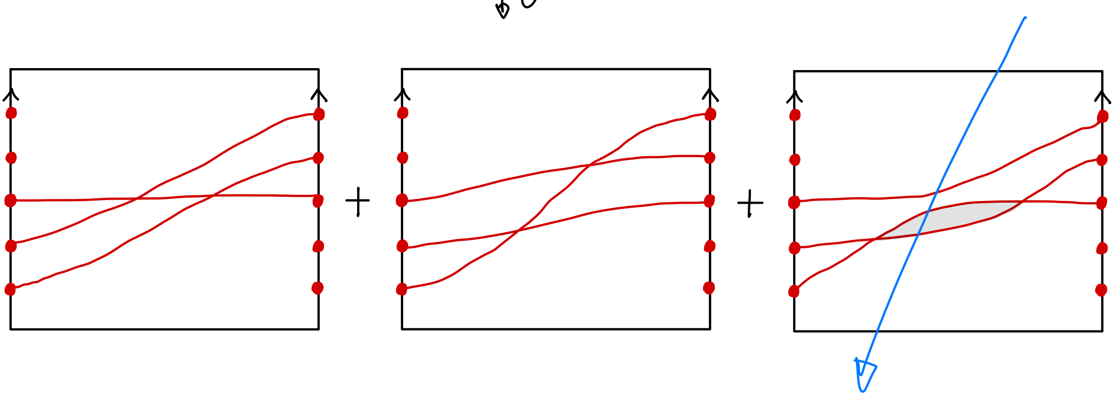
$$\partial : \widetilde{\mathcal{A}}(\mathcal{Z}, k) \longrightarrow \widetilde{\mathcal{A}}(\mathcal{Z}, k)$$

1)



added an extra bpt to exemplify bigon (cannot be a chord diagram b/c |B| is odd)

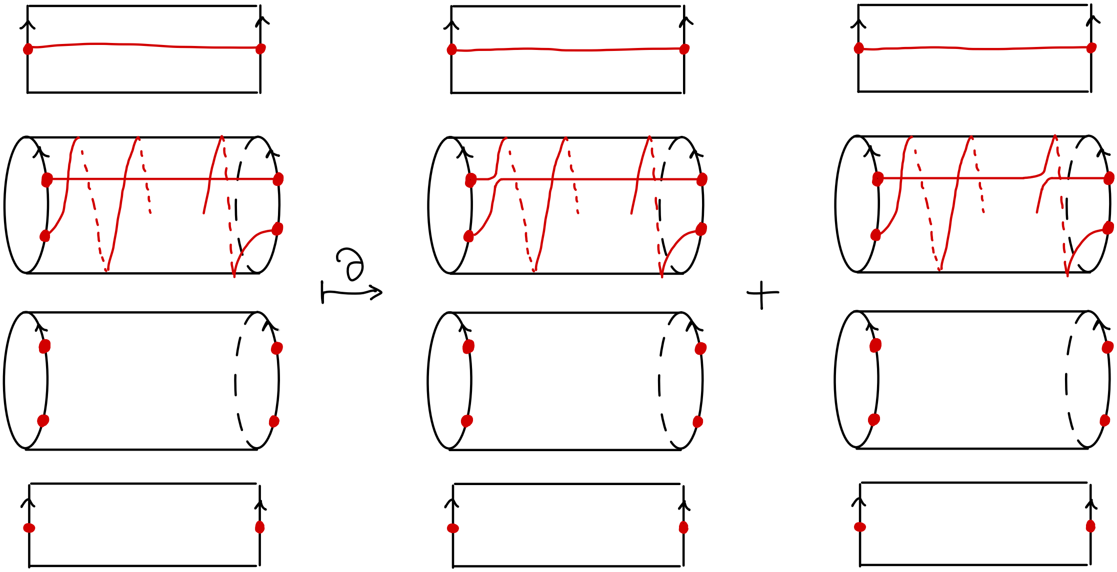
$$\downarrow \partial$$



0

(contains bigon)

2)



(resolving all the other crossings in the middle creates bigons)

Lemma: $\partial^2 = 0$

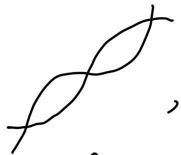
Pf: If we ignore the bigons, we get

$$\partial^2 s = \sum_{\substack{(c_1, c_2) \text{ distinct} \\ \text{crossings}}} \text{resolution of } s \text{ at } c_1 \text{ and } c_2$$

which is obviously $0 \pmod 2$.

We just need to check that $\partial t = 0$ in $\tilde{A}(Z, K)$ if t contains bigons.

If t contains exactly 1 bigon, then the two resolutions at the vertices of the bigon cancel out, and the others still contain a bigon.

If t contains ≥ 2 bigons, e.g. , then any resolution still contains bigons, so it is 0 in the algebra. $\square$

Thm: $\widetilde{\mathcal{A}}(\mathcal{Z},K)$ is a DG algebra, called the <u>PRE-STRANDS ALGEBRA</u>.

<u>Pf</u>: Exercise. [Find unit and check Leibniz rule.]

Rk: So far we have not used the matching on $\mathcal{Z}$.

③ STRANDS ALGEBRA

Def: Let $s=\{s_1,\dots,s_K\}$ be a K-strand on $\mathbb{Z}$, and let I denote the set of indices for which s_i is constant.

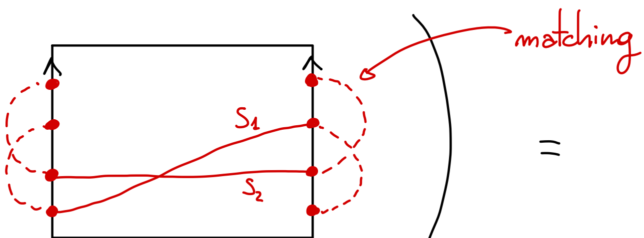
For $\iota \subseteq I$, define $s^\iota=\{s_1^\iota,\dots,s_K^\iota\}$ as the K-strand s.t.:

- for $i\notin\iota$, $s_i^\iota = s_i$;
- for $i\in\iota$, s_i^ι is the constant strand at $\varphi(s_i(0))$.

Then, the EQUALISER of s is

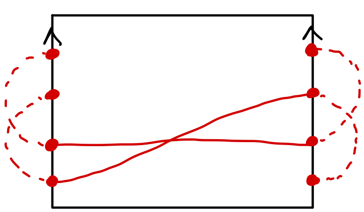
$$E(s) := \begin{cases} \sum_{\iota\subseteq I} s^\iota & \text{if } s(0)\cap\varphi(s(0))=\emptyset = s(1)\cap\varphi(s(1)) \\ 0 & \text{otherwise}\end{cases}$$

1) $E\Big($



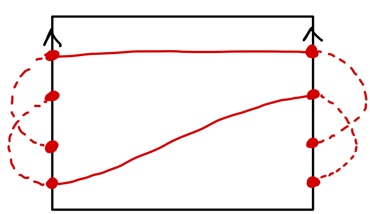
$\Big) =$

Here $I=\{2\}$

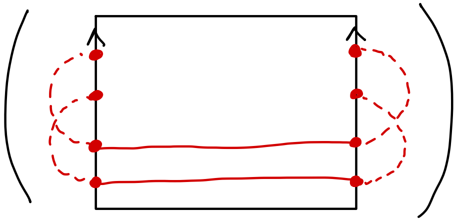


$\iota=\emptyset$

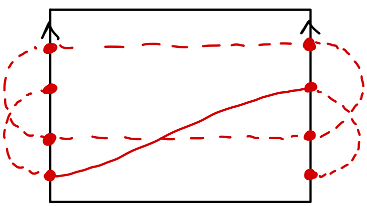
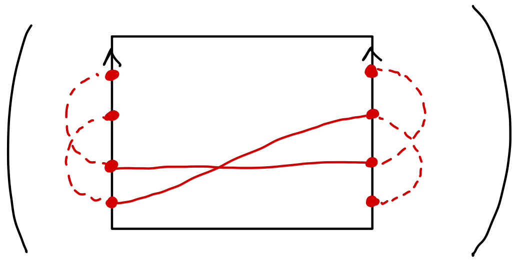
$+$

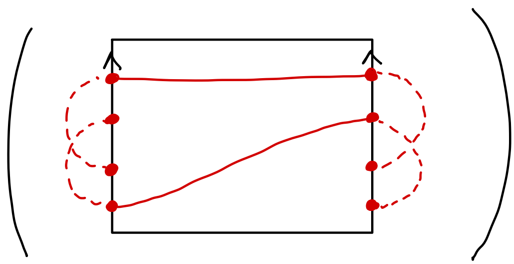


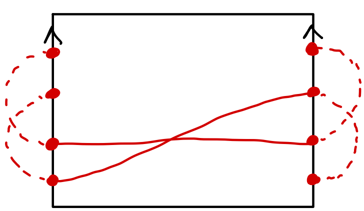
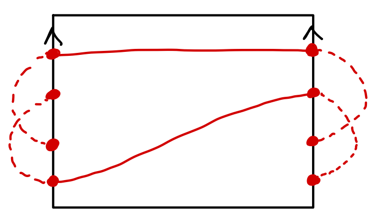
$\iota=\{2\}$

E  $=$ sum of 4 terms

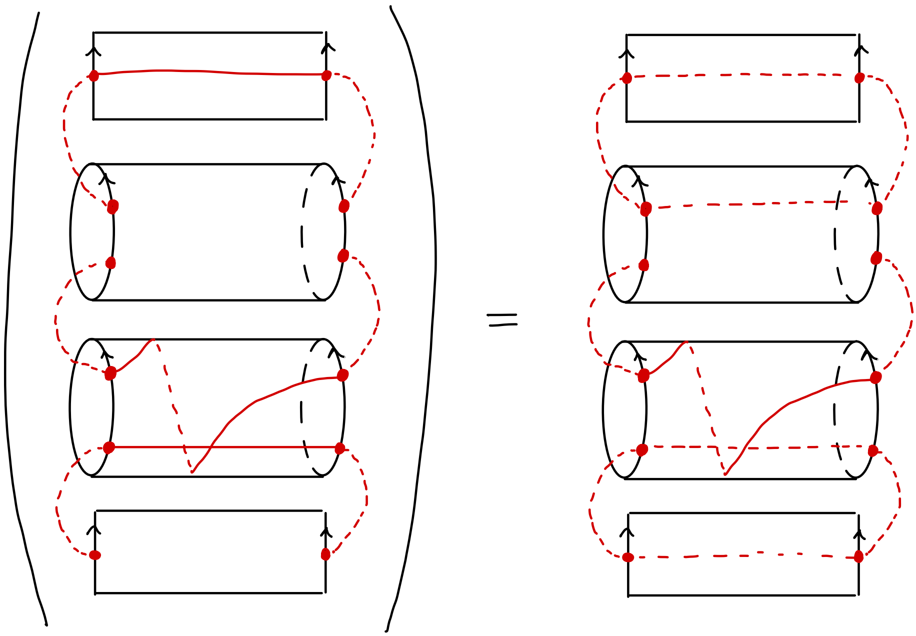
<u>Notation:</u> Replace all constant strands with dashed lines & also add dashed lines at the matched basepoints.

 $= E$ 

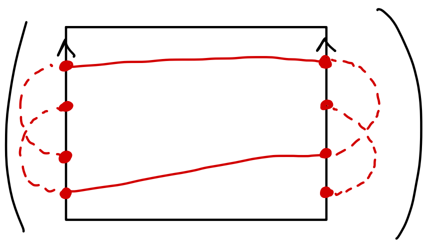
$= E$ 

$=$  $+$ 

2)

E 

This element is the sum of 4 3-strands.

RK: E  $= 0$

because $s(1) \cap \varphi(s(1)) \neq \phi$.

From the dashed notation it is clearer.

Def: The <u>STRANDS ALGEBRA</u> $\mathcal{A}(\mathcal{Z}, k)$ is the $\mathbb{F}_2$-vector subspace of $\widetilde{\mathcal{A}}(\mathcal{Z}, k)$ spanned by the equalisers.

Lemma: $\mathcal{A}(Z,K)$ is closed under multiplication and differential.

Proof: Multiplication

$$E(s)\cdot E(t) = \left(\sum_{\iota\subseteq I} s^{\iota}\right)\cdot\left(\sum_{\iota'\subseteq I'} t^{\iota'}\right)$$

Suppose $\exists\, \iota, \iota'$ such that $s^{\iota}(1) = t^{\iota'}(0)$, and choose

If $E(s)\cdot E(t) \neq 0$, $\exists\, \iota, \iota'$ such that $s^{\iota}(1) = t^{\iota'}(0)$, and replace s and t with s^{ι} and $t^{\iota'}$ respectively (equalisers do not change).

Then,

$$E(s)\cdot E(t) = \left(\sum_{\iota\subseteq I} s^{\iota}\right)\cdot\left(\sum_{\iota'\subseteq I'} t^{\iota'}\right)$$

concatenable iff you change the endpts in the same way

$$= \sum_{\iota\subseteq I\cap I'} s^{\iota}\cdot t^{\iota}$$

contains a bigon iff s.t does

$$= \sum_{\iota\subseteq I\cap I'} (s\cdot t)^{\iota} = E(s\cdot t)$$

Differential: consider $\partial(E(s))$.

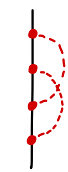
*) If you resolve crossings away from constant strands you get equalisers.

*) If you resolve a crossing involving a constant strand and a (necessarily) non-constant one, remove the matched constant lines; note the new lines created cannot be constant.

If bigons appear in one resolution, they are there for all the summands of the corresponding equaliser, because bigons cannot involve constant strands. $\square$

RK: $\mathcal{A}(\mathcal{Z},k)$ is not a subalgebra of $\tilde{\mathcal{A}}(\mathcal{Z},k)$ because $1_{\tilde{\mathcal{A}}(\mathcal{Z},k)} \notin \mathcal{A}(\mathcal{Z},k)$. However, $\mathcal{A}(\mathcal{Z},k)$ has its own unit.

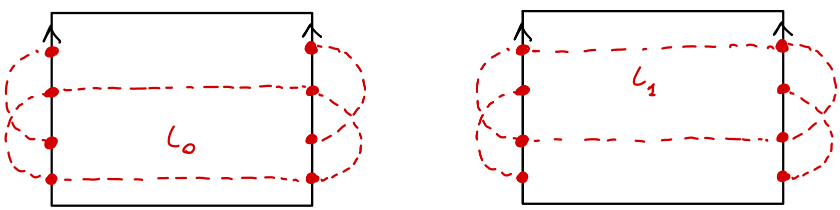
The torus algebra

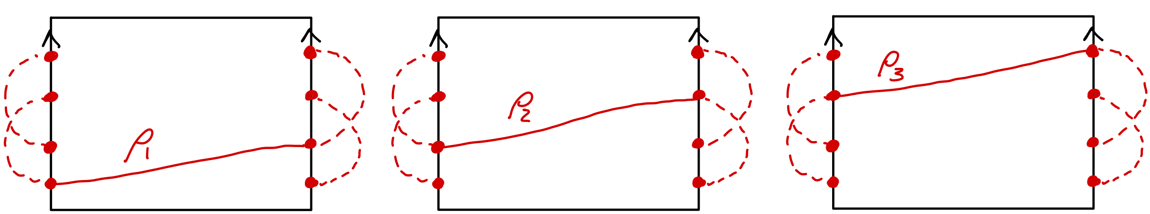
$\mathcal{Z} =$ 

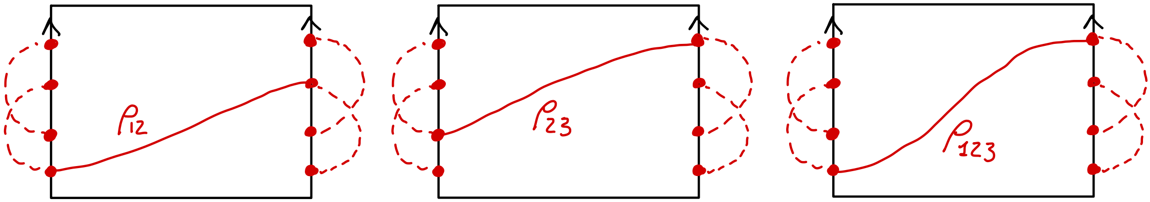
$$\mathcal{A}(\mathcal{Z},0) = \mathbb{F}_2$$

as a vector space

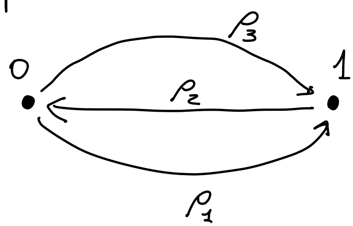
$$\mathcal{A}(\mathcal{Z},1) \cong \mathbb{F}_2\langle \iota_0, \iota_1, \rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123} \rangle$$







As an algebra, $\mathcal{A}(\mathcal{Z},1)$ is isomorphic to the <u>PATH ALGEBRA</u> over the directed graph

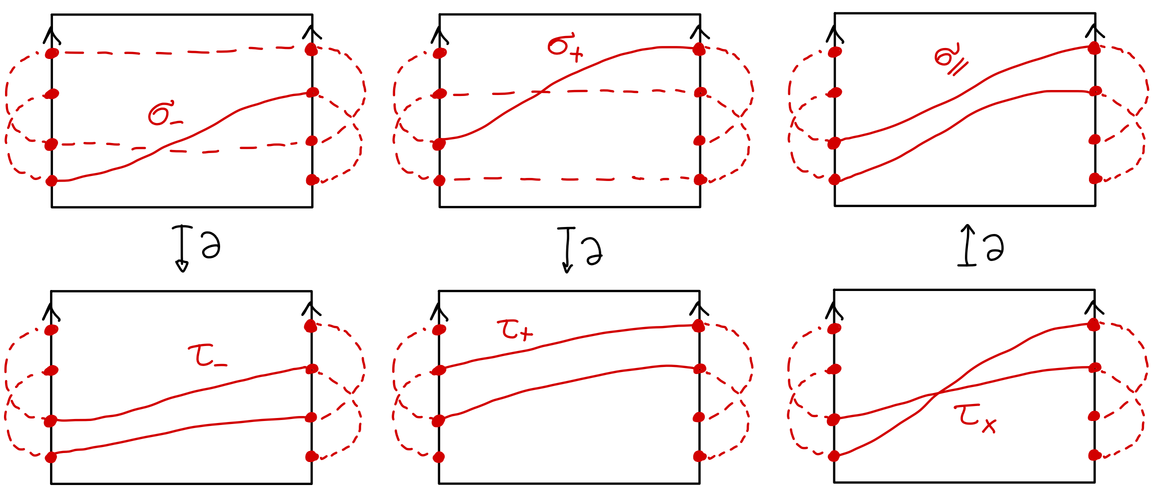


modulo the relations $\rho_2\rho_1 = \rho_3\rho_2 = 0$.

[ι_0 and ι_1 are the constant paths at the corresponding vertices.]

idempotent

$$\underline{\mathcal{A}(\mathcal{Z},2)} \cong \mathbb{F}_2\langle \iota, \sigma_-, \sigma_+, \sigma_\parallel, \tau_-, \tau_+, \tau_\times \rangle$$



Exercise: $H_*(\mathcal{A}(\mathcal{Z},2)) \cong \mathbb{F}_2$, so $\mathcal{A}(\mathcal{Z},2)$ is quasi-isom. to $\mathcal{A}(\mathcal{Z},0)$.

④ IDEMPOTENTS and UNIT

Let $\mathcal{Z} = (P, B, \varphi)$ be a chord diagram and let $x \subseteq B/_{(b\sim\varphi(b))}$ a subset of cardinality k.

Def: $I_x := E(\mathrm{Const}_S)$, where $S \subseteq B$ is any lift of x.

Ex: 1) $\{ I_x \mid x \subseteq B/\sim, |x| = k \}$ are orthogonal idempotents in $\mathcal{A}(\mathcal{Z},k)$.

2) All idempotents of $\mathcal{A}(\mathcal{Z},k)$ form an abelian subring $\mathcal{I}(\mathcal{Z},k) \subseteq \mathcal{A}(\mathcal{Z},k)$.

3) In fact, $\{I_x\}$ is a basis of the subring of idempotents $\mathcal{I}(\mathcal{Z},k) \subseteq \mathcal{A}(\mathcal{Z},k)$, seen as an $\mathbb{F}_2$-vector space.

Def: An idempotent $\iota \in \mathcal{A}$ is called MINIMAL if it cannot be decomposed as the sum of non-zero orthogonal idempotents.

Ex: Suppose that:

- •) the idempotents of $\mathcal{A}$ form an abelian subring $\mathcal{I}$; and
- •) $\{I_i\}$ is a basis of orthogonal idempotents.

Then $\{I_i\} = \{\text{minimal idempotents of } \mathcal{A}\}$.

This shows that the set $\{I_x\}$ is uniquely determined by the algebraic structure of $\mathcal{A}(\mathcal{Z}, k)$.

Def: $1 := \sum_x I_x$ is the unit of $\mathcal{A}(\mathcal{Z}, k)$.

Thm: $(\mathcal{A}(\mathcal{Z}, k), \partial, \cdot, 1)$ is a unital DGA.

RK: As a vector space, we have a splitting

$$\mathcal{A}(\mathcal{Z}, k) = \bigoplus_{x,y} I_x \cdot \mathcal{A}(\mathcal{Z}, k) \cdot I_y$$

Moreover, ∂ respects this splitting, and

$$\cdot : (I_x \cdot \mathcal{A} \cdot I_y) \times (I_y \cdot \mathcal{A} \cdot I_z) \longrightarrow I_x \cdot \mathcal{A} \cdot I_z$$

Note that in particular $\partial(I_x) = 0$.

Example: the torus algebra $\mathcal{A}(2,1)$

There are two idempotents, ι_0 and ι_1.

$$\iota_0 \cdot \mathcal{A} \cdot \iota_0 = \mathbb{F}_2 \langle \iota_0, \rho_{12} \rangle \qquad \iota_1 \cdot \mathcal{A} \cdot \iota_1 = \mathbb{F}_2 \langle \iota_1, \rho_{23} \rangle$$

$$\iota_0 \cdot \mathcal{A} \cdot \iota_1 = \mathbb{F}_2 \langle \rho_1, \rho_3, \rho_{123} \rangle \qquad \iota_1 \cdot \mathcal{A} \cdot \iota_0 = \mathbb{F} \langle \rho_2 \rangle$$

DG category representation

We can form a category $\mathcal{C}_{\mathcal{A}(2,k)}$ by letting:

*) objects = minimal idempotents

*) $Mor(I_x, I_y) = I_x \cdot \mathcal{A} \cdot I_y$

diff. k-module

[Mod_k^∂ is a monoidal category]

*) $1_{Mor(I_x, I_x)} = I_x \cdot 1_{\mathcal{A}} \cdot I_x$

*) $\circ : Mor(I_y, I_z) \otimes Mor(I_x, I_y) \to Mor(I_x, I_z)$

$\mathcal{C}_{\mathcal{A}(2,k)}$ is a DG category (def: same as above + associativity & unity axioms).

RK: There are also A_∞-categories, but they are not categories (do not satisfy associativity & unit axioms).