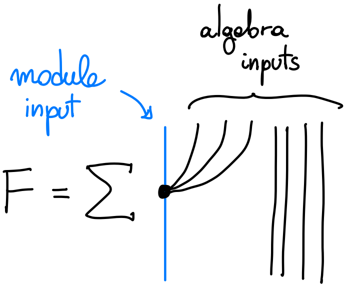
4. A_∞-modules

① A_∞-MODULES

Let A be an A_∞-algebra over k, with (counital, coassociative) coaugmented coderivation M_A satisfying $M_A \circ M_A \equiv 0$.
Let $\mu = \pi \circ M_A$ and μ_i denote the i-input maps.

Prop (universal property for modules)

$$\begin{tikzcd} Y & \\ Y \boxtimes TA \arrow[u, "\pi"] & X \boxtimes TA \arrow[ul, "\forall\, k\text{-linear } f"'] \arrow[l, dashed] \end{tikzcd}$$

& $F = \sum$ 

$\exists !$ k-linear F s.t.
$(id \boxtimes \Delta) \circ F = (F \boxtimes id) \circ (id \boxtimes \Delta)$ ← F "commutes" with the comultiplication

Pf: Uniqueness

- Commutativity forces the projection onto $Y \boxtimes k \simeq Y$.
- Let $F^{i \to j}$ denote the restriction of F to $X \boxtimes A^{\otimes i}$ composed with the projection onto $Y \boxtimes A^{\otimes j}$.

  We will show that the collection of maps $F^{i \to 0}$ determines all the $F^{i \to j}$.

Consider the relation $(\mathrm{id}_X \boxtimes \Delta)\circ F = (F \boxtimes \mathrm{id}_{TA})\circ(\mathrm{id}_X \boxtimes \Delta)$ and restrict it to $X \boxtimes A^{\otimes i}$. The right hand side is

$$(F \boxtimes \mathrm{id}_{TA}) \sum_{s=1}^{i} x \boxtimes (a_1 \otimes \cdots \otimes a_s) \boxtimes (a_{s+1} \otimes \cdots \otimes a_i) =$$

$$= \sum_{s=1}^{i} F\left(x \boxtimes (a_1 \otimes \cdots \otimes a_s)\right) \boxtimes (a_{s+1} \otimes \cdots \otimes a_s)$$

Consider the projection of this onto $Y \boxtimes k \boxtimes A^{\otimes j}$, which is

$$F^{i-j\to 0}\left(x \boxtimes (a_1 \otimes \cdots \otimes a_{i-j})\right) \boxtimes (a_{i-j+1} \otimes \cdots \otimes a_i)$$

Now let's turn to the left hand side, and let's do the same restriction to $X \boxtimes A^{\otimes i}$ and corestriction to $Y \boxtimes k \boxtimes A^{\otimes j}$:

$$\underbrace{pr_{0,j} \circ (\mathrm{id} \boxtimes \Delta)}\circ F^{i\to j}\left(x \boxtimes (a_1 \otimes \cdots \otimes a_i)\right)$$

this map is injective on $Y \boxtimes A^{\otimes j}$; in fact it is the canonical isom.

$$Y \boxtimes A^{\otimes j} \longrightarrow Y \boxtimes k \boxtimes A^{\otimes j}$$

Thus, $F^{i\to j}$ is completely determined.

<u>Existence</u>: try the given formula and show that it works. $\square$

Def: A right A_∞-MODULE over A is a k-module X with a k-linear map $X \boxtimes TA \xrightarrow{M_X} X \boxtimes TA$ satisfying

1) $(id_X \boxtimes \Delta) \circ M_X = (M_X \boxtimes id_{TA}) \circ (id_X \boxtimes \Delta)$
2) $\widetilde{M}_X := M_X + id_X \boxtimes M_A$ is a differential on $X \boxtimes TA$, i.e. $\widetilde{M}_X \circ \widetilde{M}_X \equiv 0$.

Unpacking the definition

1. By the univ. property, M_X is determined by maps

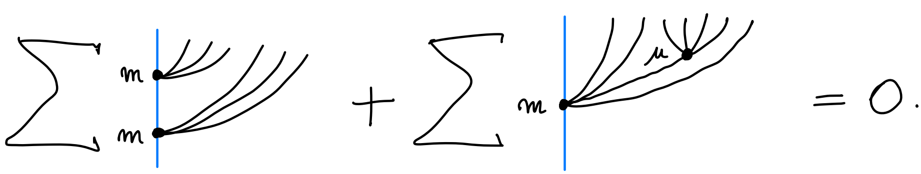
$$m_i : X \otimes A^{\otimes i} \longrightarrow X \qquad \text{for } i \geq 0.$$

[Notation is not universally agreed; many authors would call this map m_{i+1}, because there are $i+1$ inputs.]

2. Both $\widetilde{M}_X \circ \widetilde{M}_X$ and 0 satisfy condition 1.

Thus, they agree iff their projections on $X \boxtimes k \cong X$ agree.

In the usual tree notation, this is

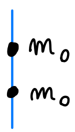


3. The resulting $(1+n)$-input relation (Rn) is

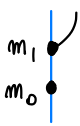
$$\sum_{i=0}^{n} m_{n-i}\left(m_i\left(x \otimes a_1 \otimes \cdots \otimes a_i\right) \otimes a_{i+1} \otimes \cdots \otimes a_n\right) +$$

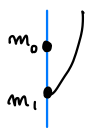
$$\sum_{i=1}^{n} \sum_{s=0}^{n-i} m_{n-i+1}\left(x \otimes a_1 \otimes \cdots \otimes a_s \otimes \mu_i\left(a_{s+1} \otimes \cdots \otimes a_{s+i}\right) \otimes a_{s+i+1} \otimes \cdots \otimes a_n\right) = 0$$

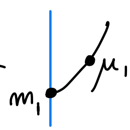
First relations

(R0) 

$= 0$, i.e., m_0 is a differential on X

(R1) 

$+$ 

$+$ 

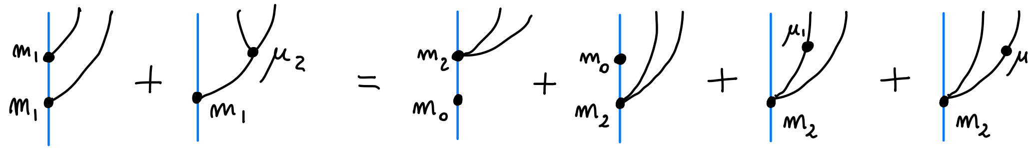
$= 0$

i.e. m_1 satisfies a Leibnitz rule:

$$m_0\left(m_1(x,a)\right) = m_1\left(m_0(x),a\right) + m_1\left(x,\mu_1(a)\right)$$

differentials are m_0 and μ_1

Said otherwise, m_1 is a chain map $X \otimes A \to X$, thus m_1 descends to a map in homology.

(R2) 

The action $m_1 : X \boxtimes A \longrightarrow X$ is not associative, but it is associative up to a homotopy $\Rightarrow$ the induced map in homology is associative.

$$m_1(m_1(x,a),b) - m_1(x,\mu_2(a,b)) = d(m_2)$$

differential in $\mathrm{Hom}(X \boxtimes A^{\otimes 2}, X)$

② EXAMPLES

1) $A :=$ torus algebra $\mathcal{A}_{\mathbb{T}^2} \overset{\text{as a vector space}}{=} \mathbb{F}_2\langle \iota_0, \iota_1, \rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123} \rangle$, which can be seen as a quotient of a path algebra.

$k := \mathbb{F}_2\langle \iota_0, \iota_1 \rangle$ subring of idempotents

$X := \mathbb{F}_2\langle x \rangle$, with k-action given by $x \cdot \iota_0 = x$, $x \cdot \iota_1 = 0$

The nonzero module maps are:

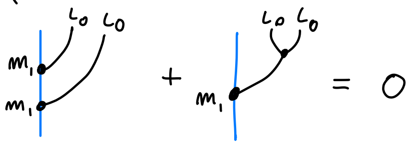
$$m_2(x, \iota_0) = x$$

$$m_{n+2}(x, \rho_3, \underbrace{\rho_{23}, \dots, \rho_{23}}_{n \text{ times}}, \rho_2) = x \qquad \forall n \geq 1$$

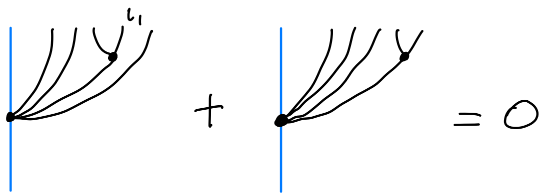
Let's check that this is an A_∞-module.

The only μ_i we care about is μ_2, so we need to check sequences of inputs that give an allowable string after we do a μ_2 or a Δ:

- (ι_0, ι_0)



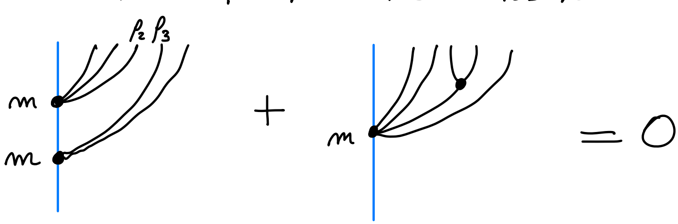
- $(\rho_3, \rho_{23}, \dots, \rho_{23}, \iota_1, \rho_{23}, \dots, \rho_2)$



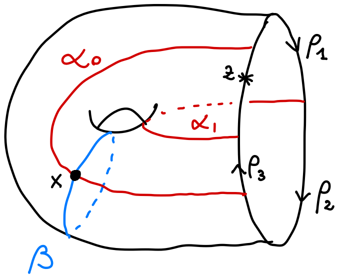
- $(\iota_0, \rho_3, \dots, \rho_2)$ and $(\rho_3, \dots, \rho_2, \iota_0)$

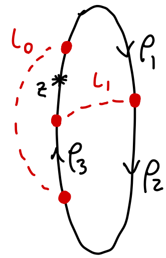
  Same as before, but use $m_1(x, \iota_0) = x$.

- $(\rho_3, \rho_{23}, \dots, \rho_{23}, \rho_2, \rho_3, \rho_{23}, \dots, \rho_{23}, \rho_2)$



Motivation: this example is the $\mathcal{A}_\infty$-module over $\mathcal{A}_{\mathbb{T}^2}$ associated to the bordered Heegaard diagram "for the unknot"

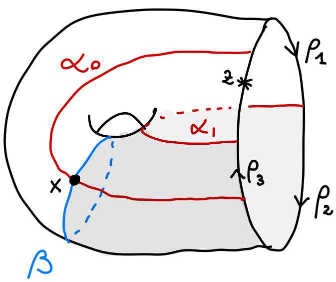




$$\left.\begin{array}{l} m_2(x, \iota_0) = x \\ m_2(x, \iota_1) = 0 \end{array}\right\}$$

get a generator of CF iff x is paired with a generator on the "right" part of the Heegaard diagram (which we do not see here) occupying the curve α_1.

$m_2(x, \rho_3, \rho_2)$ corresponds to discs with domain as in the picture

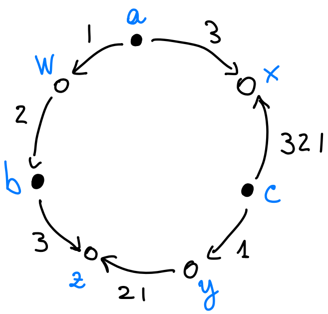


[there is a cut along the red curve]

Higher maps $m_{n+2}(x, \rho_3, \rho_{23}, \rho_{23}, \dots, \rho_{23}, \rho_2)$ correspond to multiples of the previous domain (but there are no cuts on the α-curve after the first one).

2) Same A and k as before, but now

$$X = \mathbb{F}_2\langle a, b, c, w, x, y, z\rangle$$



$$m_1(\xi, \iota_0) = \begin{cases} \xi & \text{if } \xi = a, b, c \\ 0 & \text{if } \xi = w, x, y, z \end{cases}$$

$$m_1(\xi, \iota_1) = \begin{cases} 0 & \text{if } \xi = a, b, c \\ \xi & \text{if } \xi = w, x, y, z \end{cases}$$

$\rightsquigarrow$ We get an idempotent decomposition $X = X\iota_0 \oplus X\iota_1$.

The other non-vanishing maps are given by the graph:

- •) for each directed path $\xi_{std} \to \xi_{end}$ you get a sequence of number;
- •) regroup them in maximal subsequences s_i of 123;
- •) get a map $m_j(\xi_{start}, s_1, s_2, \dots, s_j) = \xi_{end}$.

Counterclockwise

$$m_1(a, \rho_1) = w$$

$$m_1(a, \rho_{12}) = b$$

$$m_1(a, \rho_{123}) = z$$

Clockwise

$$m_1(a, \rho_3) = x$$

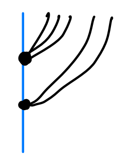
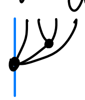
$$m_1(w, \rho_2) = b$$
$$m_1(w, \rho_{23}) = z$$

$$m_1(b, \rho_3) = z$$

$$m_2(y, \rho_2, \rho_1) = z$$

$$m_3(c, \rho_3, \rho_2, \rho_1) = x \qquad m_1(c, \rho_1) = y$$
$$m_2(c, \rho_{12}, \rho_1) = z$$

Ex: This is an $\mathcal{A}_\infty$-module over $\mathcal{A}_{\mathbb{T}^2}$.

<u>Idea</u>: if no idempotents are involved, then non-trivial terms  come from broken paths on the loop-type graph. Each such term always cancels with a term , because there is always a non-trivial μ_2 at the breaking point, e.g. $\underset{w}{\circ} \xrightarrow{2} \underset{b}{\bullet} \xrightarrow{3} \underset{z}{\circ}$.

Def: An $\mathcal{A}_\infty$-module is <u>STRICTLY UNITAL</u> if

$$m_1(x, 1_k) = x \quad \forall x \in X$$
$$m_n(x, \bullet, \dots, \bullet, 1_k, \bullet, \dots, \bullet) = 0 \quad \forall x \in X, \ n > 1$$

Both examples are strictly unital $\mathcal{A}_\infty$-modules.

Def: An $\mathcal{A}_\infty$-algebra is <u>OPERATIONALLY BOUNDED</u> if $\mu_i \equiv 0$ for i sufficiently large.

All strands algebras are operationally bounded (they are DGAs).

Def: An $\mathcal{A}_\infty$-module is <u>BOUNDED</u> if $m_i \equiv 0 \quad \forall i \gg 1$.

Example 2 is bounded, but example 1 is <u>NOT</u>.

③ MORPHISMS and HOMOTOPIES

Def: A <u>HOMOMORPHISM of $\mathcal{A}_\infty$-MODULES</u> over A is a map $F : X \boxtimes TA \longrightarrow Y \boxtimes TA$ s.t.

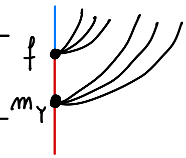
1) $(id_Y \boxtimes \Delta) \circ F = (F \boxtimes id_{TA}) \circ (id_X \boxtimes \Delta)$;
2) $\widetilde{M}_Y \circ F = F \circ \widetilde{M}_X$ (chain map)

<u>Unpacking the definition</u>

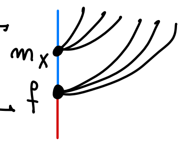
1. By univ. property, F is determined by a collection of maps

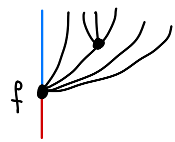
$$f_i : X \boxtimes A^{\otimes i} \longrightarrow Y \quad \text{for } i \geq 0$$

2. $\widetilde{M}_Y \circ F - F \circ \widetilde{M}_X$ and 0 satisfy condition ①, so by the universal property they are equal iff the projections on Y agree (i.e., 1 module output + 0 algebra outputs):

$\pi \circ \widetilde{M}_Y \circ F = \sum$ 

(no μ_A because it produces algebra outputs)

$\pi \circ F \circ \widetilde{M}_X = \sum$ 

$+ \sum$ 

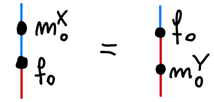
3. Thus, the A_∞ relation (Rn) with n algebra inputs is

$$\sum_{i=1}^{n} m^Y_{n-i}\left(f_i(x \otimes a_1 \otimes \cdots \otimes a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n\right) +$$

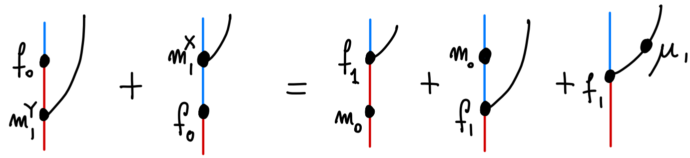
$$\sum_{i=1}^{n} f_{n-i}\left(m^X_i(x \otimes a_1 \otimes \cdots \otimes a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n\right) +$$

$$\sum_{i=1}^{n} \sum_{s=0}^{n-i} f_{n-i+1}\big(x \otimes a_1 \otimes \cdots \otimes a_s \otimes \mu_i(a_{s+1} \otimes \cdots \otimes a_{s+i}) \otimes a_{s+i+1} \otimes \cdots \otimes a_n\big) = 0$$

First relations

(R0) 

, i.e. f_0 is a chain map $X \longrightarrow Y$

(R1) 

i.e., f_0 commutes with the action m_1 up to a homotopy f_1.

Def: The <u>IDENTITY MORPHISM</u> is $\mathrm{Id} : X \boxtimes TA \circlearrowleft$.

Equivalently, this is given by maps Id_i as follows:

- $\mathrm{Id}_0 : X \longrightarrow X$, $\quad x \longmapsto x$
- $\mathrm{Id}_i : X \boxtimes A^{\otimes i} \longrightarrow X$ is the zero map $\forall i > 0$

RK: The composition $G \circ F$ of two homom. of A_∞-modules has associated maps

$$(g \circ f)_n \left(x \boxtimes (a_1 \otimes \cdots \otimes a_n)\right) =$$

$$\sum_{i=0}^{n} g_{n-i}\Big(f_i\big(x \boxtimes (a_1 \otimes \cdots \otimes a_i)\big) \otimes a_{i+1} \otimes \cdots \otimes a_n \Big)$$

Def: Suppose A is a strictly unital A_∞-algebra.
$F : X \boxtimes TA \longrightarrow Y \boxtimes TA$ is <u>strictly unital</u> if
$$f_i\left(x \boxtimes (\bullet \otimes \cdots \otimes \bullet \otimes 1_A \otimes \bullet \otimes \cdots \otimes \bullet)\right) = 0.$$

The identity morphism is strictly unital.

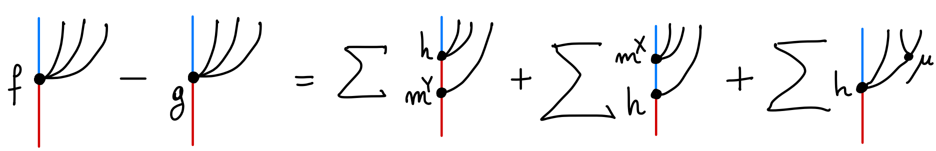
Def: $F, G : X \boxtimes TA \longrightarrow Y \boxtimes TA$ homom. of A_∞-modules are <u>A_∞-homotopic</u> if $\exists H : X \boxtimes TA \longrightarrow Y \boxtimes TA$ s.t.

1) $(\mathrm{id} \boxtimes \Delta) \circ H = (H \boxtimes \mathrm{id}) \circ (\mathrm{id} \boxtimes \Delta)$
2) $F - G = \widetilde{M}_Y \circ H - H \circ \widetilde{M}_X$

<u>Unpacking the definition</u>

1. By univ. property, H is determined by a collection of maps
$$h_i : X \boxtimes A^{\otimes i} \longrightarrow Y \qquad \text{for } i \geq 0$$

2. $F - G$ and $\widetilde{M}_Y \circ H + H \circ \widetilde{M}_X$ both satisfy condition 1.
Thus, they agree if and only if the projections onto Y agree.



3. We obtain A_∞ relations by fixing the number of inputs.

$$(Rn)\quad f_n(x \boxtimes (a_1 \otimes \cdots \otimes a_n)) - g_n(x \boxtimes (a_1 \otimes \cdots \otimes a_n)) =$$

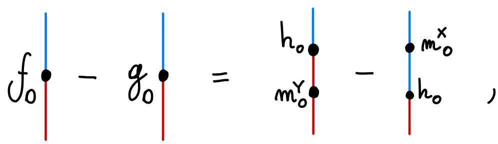
$$\sum_{i=0}^{n} m^Y_{n-i}\left(h_i(x \boxtimes (a_1 \otimes \cdots \otimes a_i)) \otimes a_{i+1} \otimes \cdots \otimes a_n\right) +$$

$$\sum_{i=0}^{n} h_{n-i}\left(m^X_i(x \boxtimes (a_1 \otimes \cdots \otimes a_i)) \otimes a_{i+1} \otimes \cdots \otimes a_n\right) +$$

$$\sum_{i=1}^{n} \sum_{s=0}^{n-i} h_{n-i+1}\left(x \boxtimes a_1 \otimes \cdots \otimes a_s \otimes \mu_i(a_{s+1} \otimes \cdots \otimes a_{s+i}) \otimes a_{s+i+1} \otimes \cdots \otimes a_n\right) = 0$$

First relation

(R0)



i.e. f_0 and g_0 are chain homotopic.