# 5. Type D structures

# ① TENSOR PRODUCT

Setting: $A$ an $A_\infty$-algebra over $k$

$X$ a right $A_\infty$-module over $A$

$Y$ a left $A_\infty$-module over $A$

The usual definition of the tensor product $X \otimes_A Y$ is not good:

1) due to the non-associativity of the action, the relation $(x \cdot a, y) \sim (x, a \cdot y)$ is not transitive;

2) even for diff. algebra and diff. modules, if $X \underset{q.i.}{\sim} X'$ it does not follow necessarily that $X \otimes_A Y \underset{q.i.}{\sim} X' \otimes_A Y$

(this problem is usually circumvented by taking derived $\otimes$).

**Def:** The $A_\infty$ TENSOR PRODUCT is the chain complex

$$X \tilde{\otimes}_A Y := X \otimes_k TA \otimes_k Y$$

with

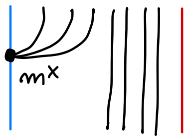
$$\partial = M_X \otimes id_Y + id_X \otimes M_A \otimes id_Y + id_X \otimes M_Y$$

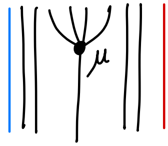
$$\stackrel{!}{=} \tilde{M}_X \otimes id_Y + id_X \otimes M_A \otimes id_Y + id_X \otimes \tilde{M}_Y$$

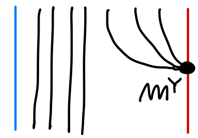
**RK:** This definition recovers the derived tensor product if

$\mu_i \equiv 0 \quad \forall i > 2$ and $m_j \equiv 0 \quad \forall j > 1$

Using the tree representation we have

$\partial = \sum$ 

$+ \sum$ 

$+ \sum$ 


<u>Ex</u>: $\partial^2 = 0$.

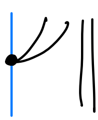
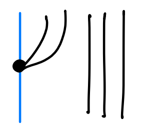
<u>Ex</u>: Unpack the definition of $\partial(x \otimes a_1 \otimes \cdots \otimes a_n \otimes y)$.

<u>Prop</u>: A homom. of $A_\infty$-modules $F : X \longrightarrow X'$ induces a chain map $F \tilde{\otimes} \mathrm{Id}_Y : X \tilde{\otimes} Y \longrightarrow X' \tilde{\otimes} Y$.

$A_\infty$-homotopic morphisms induce homotopic chain maps.

<u>Cor</u>: If $X \underset{A_\infty\text{-c.h.e.}}{\sim} X'$, then $X \tilde{\otimes} Y \underset{\text{c.h.e}}{\sim} X' \tilde{\otimes} Y$.

<u>Proof of proposition</u>

$F = \sum$  $\rightsquigarrow F \tilde{\otimes} \mathrm{Id}_Y = \sum$ 

$$(F \otimes id_Y) \circ \partial = F \otimes id_Y \circ \left( \tilde{M}_X \otimes id_Y + id_X \otimes M_Y \right)$$

$$= \underbrace{(F \circ \tilde{M}_X)}_{} \otimes id_Y + \underbrace{(F \otimes id_Y) \circ (id_X \otimes M_Y)}_{\text{These commute! Use tree repr.}}$$

these commute by definition of $A_\infty$-homom.

$$= \partial \circ (F \otimes id_Y)$$

$\square$

## ② TYPE D STRUCTURES

There's a smaller model for $A_\infty$ tensor product if the left $A_\infty$-module comes from a "type D" structure.

**Def:** Let $C$ be a counital coalgebra. A counital <u>COMODULE</u> over $C$ is a left $k$-module $N$ and a map

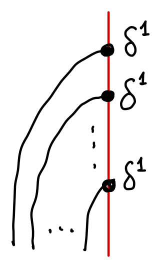
$$\delta : N \longrightarrow C \otimes N$$

such that

1) $(\Delta \otimes \mathrm{id}_N) \circ \delta = (\mathrm{id}_C \otimes \delta) \circ \delta$ $\circledast$
2) $(\varepsilon \otimes \mathrm{id}_N) \circ \delta = \mathrm{id}_N$

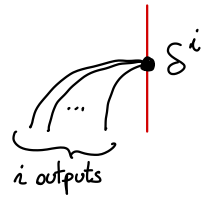
**Prop** (universal property of comodules over $TA$)

$$\begin{tikzcd}
A \otimes N & \\
\mathrm{Com}(TA) \otimes N \arrow[u] & N \arrow[ul, "\forall\, k\text{-linear } \delta^1"'] \arrow[l, "\exists!\, \delta \text{ counital comodule structure}"]
\end{tikzcd}$$

& $\delta = \sum$ 


completion of $TA$, i.e. $\prod_{i=0}^{\infty} A^{\otimes i}$

**Notation:** Denote by $\delta^i : N \longrightarrow A^{\otimes i} \otimes N$ the composition of $\delta$ with the projection onto the $A^{\otimes i} \otimes N$ summand.

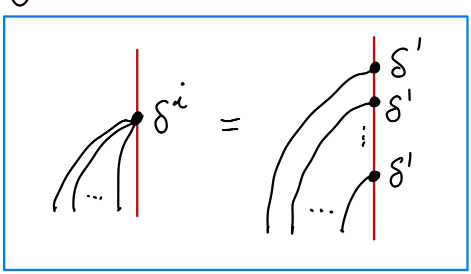



The (counital) comodule structure can be phrased in terms of the maps $\delta^i$ as follows:

- $\delta^{i+j} = (id_{A^{\otimes i}} \otimes \delta^j) \circ \delta^i$ — Take structure relation $\circledast$ and consider the projection onto $A^{\otimes i} \otimes A^{\otimes j} \otimes N$
- $\delta^0 = id_N$ (counitality)

**Proof of proposition:** Uniqueness

- counitality and commutativity of the diagram determine the projections $\delta^0$ and $\delta^1$ uniquely.
- the comodule condition says that each higher $\delta^i$ is determined by $\delta^1$:



Existence: Check that the given formula satisfies $\circledast$. $\square$

Def: A comodule $N$ over $\mathrm{Com}(TA)$ is BOUNDED if $\delta^i \equiv 0$ for all $i$ sufficiently large.

Rk: $N$ is bounded if it is a comodule over $TA$.

Rk: Since $\delta$ is determined by $\delta^1$, often people say that "$\delta^1$ is bounded".

Def: Let $A$ be an $A_\infty$-algebra and $(N, \delta)$ a counital comodule over $\mathrm{Com}(TA)$.

$(N, \delta)$ is a TYPE D STRUCTURE on $A$ if

- $A$ is operationally bounded or $(N, \delta)$ is bounded; and
- $(M_A \otimes id_N) \circ \delta = 0$.

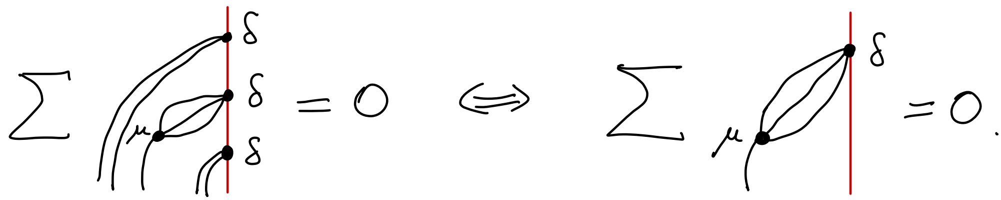
need this condition so the map $(M_A \otimes id_N) \circ \delta$ is well-defined

## Unpacking the definition

1. Using the universal property, $\delta$ is determined by a collection of maps

$$\delta^i : N \longrightarrow A^{\otimes i} \otimes N$$

2. Both $(M_A \boxtimes \mathrm{id}_N) \circ \delta$ and $0$ satisfy the univ. property for morphisms (see Section 6 below). Thus, they agree iff their projections onto $A \otimes N$ agree, i.e.




3. The last relation can be written as

$$\boxed{\sum_{n=1}^{\infty} (\mu_n \otimes \mathrm{id}_N) \circ \delta^n = 0.}$$

## Revisited definition

A type-D structure over an $A_\infty$-algebra $A$ is a left $k$-module $N$ with a $k$-linear map $\delta^1 : N \longrightarrow A \otimes N$ s.t.:

- •) $A$ is operationally bounded or $\delta^1$ is bounded, and
- •) $\sum_{n=1}^{\infty} (\mu_n \otimes \mathrm{id}_N) \circ \delta^n = 0$

where $\delta^n$ denotes the $n$-th iteration of $\delta^1$ defined recursively by

$$\delta^{i+1} = (\mathrm{id}_{A^{\otimes i}} \otimes \delta^1) \circ \delta^i.$$

## 3 EXAMPLES

1. $A$ = torus algebra $\mathcal{A}_{\mathbb{T}^2}$, over $k$ = idempotent ring.

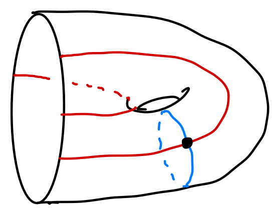
$N = \mathbb{F}_2\langle y \rangle$, with $\iota_0 \cdot y = y$, $\iota_1 \cdot y = 0$.

$$\delta^1(y) := \rho_{12} \otimes y$$

$\neq 0$ because the idempotents $\iota_0$ in the middle match.
(Recall that $k$ = idemp. ring)

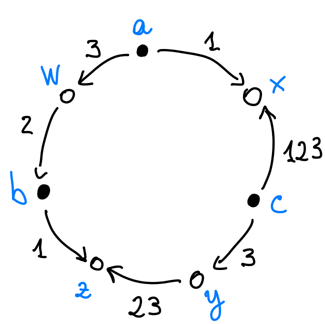
Check that it is a type D structure. For $\mathcal{A}_{\mathbb{T}^2}$ there is only $\mu_2$, so the structure relation is simply $(\mu_2 \boxtimes \mathrm{id}) \circ \delta^2 = 0$, which holds true because $\mu_2(\rho_{12}, \rho_{12}) = 0$.

RK: This type-D structure is associated to



It is algebraically much simpler than the $A_\infty$-module.

2. $N = \mathbb{F}_2\langle a, b, c, w, x, y, z \rangle$, with same $A$ and $k$.




$a, b, c \in \iota_0 \cdot N$

$w, x, y, z \in \iota_1 \cdot N$

$\delta^1$ is given by the arrows.

Note that $(\mu_2 \otimes \mathrm{id}) \circ \delta^2 = 0$, so the structure relation is satisfied.

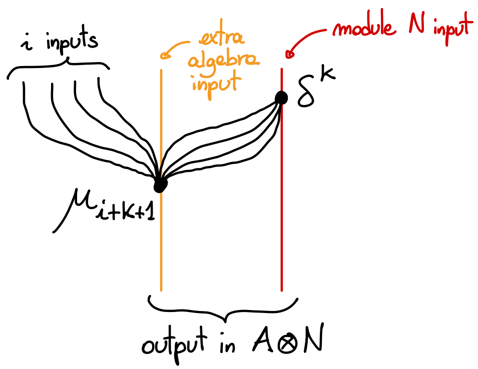
RK: This is the type D structure associated to the complement of the right handed trefoil.

Advantage: Type D structures are algebraically simpler than $A_\infty$-modules; you need only $\delta^1$.

## ④ BOX TENSOR PRODUCT

Prop: Let $N$ be a type D structure over $A$. Then $A \otimes_k N$ is a left $A_\infty$ module over $A$, with

$$m_i = \sum_{K=0}^{\infty}$$




Proof: Exercise.

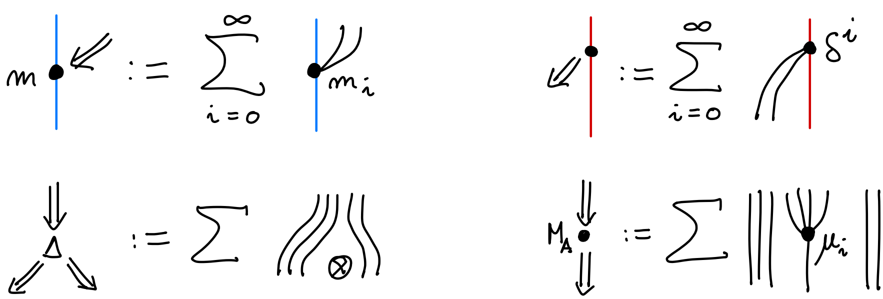
If a left $A_\infty$ module comes from a type D structure, then the $A_\infty$ tensor product can be computed more easily.

**Def:** Let $(X, m)$ be a right $A_\infty$ module over $A$, $m : X \otimes TA \to X$
and $(N, \delta)$ be a type D structure over $A$.
Suppose that at least one of them is bounded.
The <u>BOX TENSOR PRODUCT</u> $X \boxtimes N$ is the $k$-module $X \otimes_k N$ with differential

$$\partial^{\boxtimes} = (m \otimes id_N) \circ (id_X \otimes \delta)$$

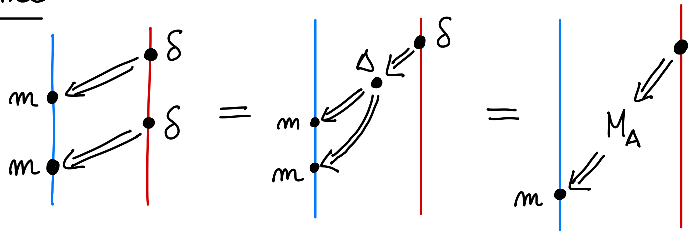
**Lemma:** $\partial^{\boxtimes} \circ \partial^{\boxtimes} = 0$.

We use graphical notation, with shortcut to avoid $\Sigma$ symbol.




**Ex:** Rewrite the structure relations for $A_\infty$-algebra, right $A_\infty$-module, and type D structure in terms of this notation.

## Proof of lemma

$\partial^{\boxtimes} \circ \partial^{\boxtimes} =$ 

$= 0.$ $\square$

**Thm** Let $A$ be an operationally bounded, strictly unital $A_\infty$-alg., $X$ a strictly unital right $A_\infty$-module over $A$, and $N$ be a type-D structure over $A$.
Assume <u>either</u> $N$ bounded <u>or</u> $X$ bounded & $N$ hty equiv. to a bounded type D structure. Then

$$X \boxtimes N \simeq X \,\tilde{\otimes}\, (\underbrace{A \otimes N})$$

$A_\infty$ tensor product

homotopy equivalent

left $A_\infty$-module

# 5 EXAMPLES

1) $A =$ torus algebra, $k =$ idempotent ring

operationally bounded

$X = \mathbb{F}_2\langle x\rangle$ in idempotent $\iota_0$, with maps

$m_1(x, \iota_0) = x$ and $m_{n+2}(x, \rho_3, \underbrace{\rho_{23}, \dots, \rho_{23}}_{n \text{ times}}, \rho_2) = x$.

$N = \mathbb{F}_2\langle y\rangle$ in idempotent $\iota_0$, $\delta^1(y) = \rho_{12} \otimes y$.

both unbounded, but in this special case the box tensor product still makes sense.

$X \boxtimes N$ is $X \otimes_k N$ as a vector space. Since the idempotents match, $X \otimes_k N = \mathbb{F}_2\langle x \otimes y\rangle$.

$$\partial^{\boxtimes}(x \otimes y) = \sum (m_i \otimes id) \circ (id \otimes \delta^i)(x \otimes y)$$
$$\overset{!}{=} (m_1 \otimes id) \circ (id \otimes \delta^1)(x \otimes y)$$
$$\overset{!}{=} m_1(x \otimes \rho_{12}) \otimes y = 0$$

We just computed that $\widehat{HF}(S^3) = \mathbb{F}$.

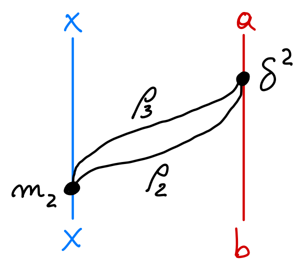
2) Same $A$, $k$, $X$ as before, but for $N$ we use the type D structure of the complement of $T_{2,3}$, which is bounded.

$$N = \mathbb{F}_2\langle \underbrace{a, b, c}_{\iota_0 \cdot N}, \underbrace{w, x, y, z}_{\iota_1 \cdot N} \rangle$$

In $X \otimes_k N$ only $x \otimes a$, $x \otimes b$, $x \otimes c$ survive, because the idempotents match.

The only non-trivial differential that we get is

$$\partial^{\boxtimes}(x \otimes a) = x \otimes b$$




Thus, again the homology ($\widehat{HF}(S^3)$) is $\mathbb{F}_2$. However, this time we computed it as the $\widehat{HF}$ of $\infty$ surgery on the trefoil.

3) Same $A$, $k$, $N$, but now let's change the $A_\infty$-module.
Now $X = \mathbb{F}_2\langle \xi \rangle$ with structure maps

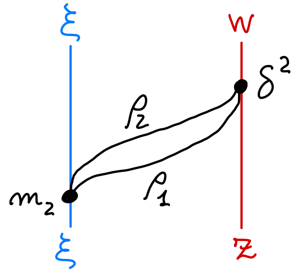
$$m_1(\xi, \iota_1) = \xi \quad \text{and} \quad m_{n+2}(\xi, \rho_2, \rho_{12}, \dots, \rho_{12}, \rho_1) = \xi$$

(this is the 0-surgery $A_\infty$-module).

$$X \otimes_k N = \mathbb{F}_2\langle \xi\otimes w, \xi\otimes x, \xi\otimes y, \xi\otimes z\rangle.$$

The only non-trivial differential is

$$\partial^{\boxtimes}(\xi\otimes w) = \xi\otimes z$$



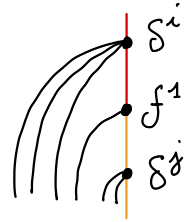

Thus, $\widehat{HF}(S^3_0(T_{2,3})) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$.

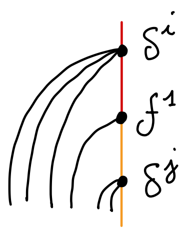
## ⑥ MORPHISMS and HOMOTOPIES

**Prop** (universal property)

$$\begin{tikzcd} A\otimes N' & \\ (\mathrm{Com}\, TA)\otimes N' \arrow[u, "\pi"] & N \arrow[ul, "\forall\, k\text{-linear } f^1"'] \arrow[l, dashed, "\exists!\ \text{counital } f \text{ s.t.}"] \end{tikzcd}$$

$$(\Delta\otimes \mathrm{id}_{N'})\circ f = (\mathrm{id}\otimes f)\circ\delta + (\mathrm{id}\otimes\delta)\circ f$$

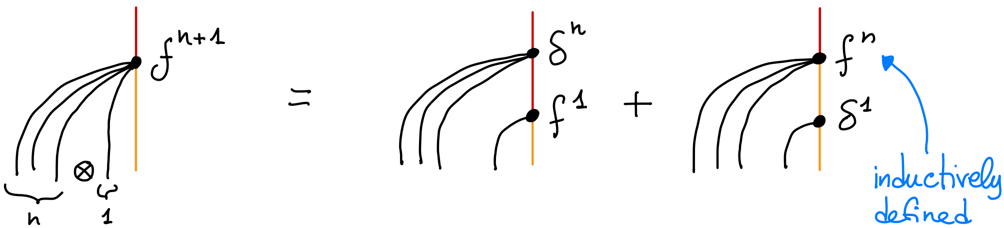
& $f = \sum$ 


**Rk:** $f = \sum_{n=0}^{\infty} f^n$, where $f^n = \sum_{i+j+1=n}$ 

Proof: Uniqueness

- Counitality and commutativity determine $f^0 \equiv 0$ and $f^1$
- Take the projection onto $A^{\otimes n} \otimes A^{\otimes 1} \otimes N'$:

$$(\Delta \otimes id_{N'}) \circ f = (id \otimes f) \circ \delta + (id \otimes \delta) \circ f$$




$\Rightarrow f^{n+1}$ is inductively determined.

Existence: Check that the formula works.

Def: A type D structure homomorphism is a map

$$f : N \longrightarrow Com(TA) \otimes N'$$

such that: 1) $(\Delta \otimes id_{N'}) \circ f = (id \otimes f) \circ \delta + (id \otimes \delta) \circ f$

2) $(M_A \otimes id_{N'}) \circ f = 0$

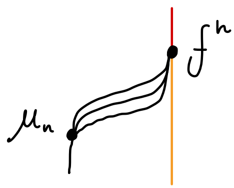
under the assumption that $f$ is BOUNDED (i.e. $f^n \equiv 0$ for $n \gg 1$) or $A$ is operationally bounded (this guarantees that 2) makes sense).

## Unpacking the definition

1. By the universal property, $f$ is defined by a map $f^1$.
2. $(M_A \otimes id_{N'}) \circ f$ and $0$ both satisfy condition 1).

   Thus, they agree if and only if their projections agree.

$$\sum_{n=0}^{\infty}$$




$$= 0$$

3. The relation is that $\sum_{n=0}^{\infty} (\mu_n \otimes id_{N'}) \circ f^n = 0$.

**Def:** $f, g : N \longrightarrow (\mathrm{Com}\, TA) \otimes N'$ are <u>HOMOTOPIC</u> if

$\exists h : N \longrightarrow (\mathrm{Com}\, TA) \otimes N'$ satisfying

1) $(\Delta \otimes id_{N'}) \circ h = (id \otimes h) \circ \delta + (id \otimes \delta) \circ h$

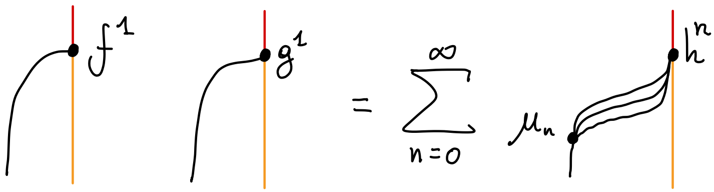
2) $f - g = (M_A \otimes id) \circ h$

assuming $h$ bounded or $A$ operationally bounded.

## Unpacking the definition

1. By the universal property, $h$ is defined by a map $h^1$.
2. LHS and RHS of 2) both satisfy condition 1).

   Thus, they agree if and only if their projections agree.

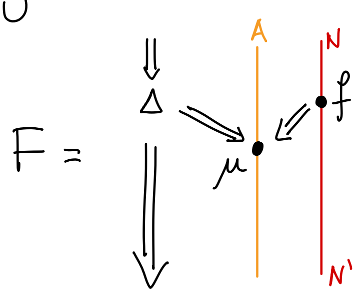



3. The relation is that $f^1 - g^1 = \sum_{n=0}^{\infty} (\mu_n \otimes \mathrm{id}_{N'}) \circ h^n$.

**Prop:** A morphism of type D structures $f : N \longrightarrow TA \otimes N'$ induces a morphism of $\mathcal{A}_\infty$-modules

$$F : TA \otimes (A \otimes N) \longrightarrow TA \otimes (A \otimes N')$$

given by the formula



Ex: A homotopy between type D structure homomorphisms induces an $\mathcal{A}_\infty$-homotopy between the corresponding left $\mathcal{A}_\infty$-module maps.

## (7) SKETCH of the PROOF of BOX TENSOR THEOREM

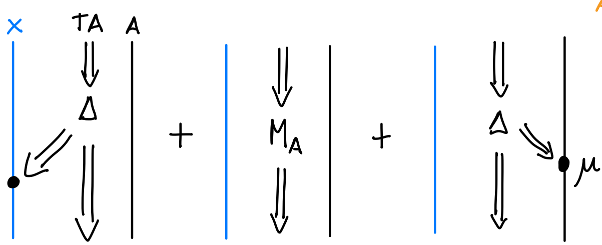
### Step 1: bar resolution

Given a right $\mathcal{A}_\infty$-module $X$ over $A$, consider its bar resolution

reduced bar algebra

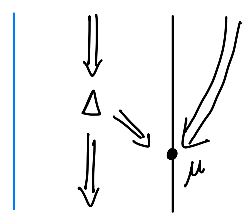
$$\overline{X} := X \otimes_k \overline{TA} = \underbrace{X \otimes_k TA \otimes_k A}_{\text{this is } X \tilde{\otimes}_A A}$$

with

$m_0 =$ 



this is $\partial_{X \tilde{\otimes}_A A}$

$m_i =$ 



for $i > 0$

Fact: $\overline{X}$ is $\mathcal{A}_\infty$-chain homotopy equivalent to $X$.

Alternative interpretation: $A$ is a left/right $\mathcal{A}_\infty$-bimodule over $A$-$A$, and $\overline{X}$ is the $\mathcal{A}_\infty$ tensor product of $X_A$ w/ ${}_AA_A$.

## Step 2: the quasi-isomorphism

We have a chain htpy equivalence $X \boxtimes_A N \underset{\text{c.h.e.}}{\simeq} \overline{X} \boxtimes_A N$.

As $k$-vector spaces, we have isomorphisms

$$\begin{aligned}
\overline{X} \boxtimes_A N &\cong (X \otimes_k TA \otimes_k A) \otimes_k N \\
&\cong X \otimes_k TA \otimes_k (A \otimes_k N) \\
&\cong X \tilde{\otimes}_A \underbrace{(A \otimes_k N)}_{\text{left } A_\infty \text{ module associated to } N}
\end{aligned}$$

Fact/Exercise: the differentials on $\overline{X} \boxtimes_A N$ and on $X \tilde{\otimes}_A (A \otimes_k N)$ are the same on the nose under these isomorphisms.

Thus, $X \boxtimes_A N$ is chain homotopy equivalent to $X \tilde{\otimes}_A (A \otimes_k N)$.

RK: the various boundedness hypotheses are required so that the tensor product operations are defined and chain maps/homotopies induce well-defined chain maps/homotopies on the tensor product.

$\square$