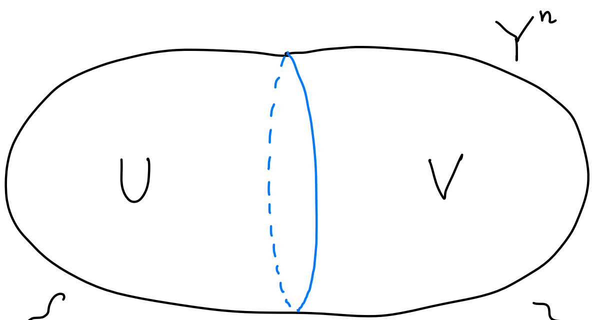
6. Manifolds with $\mathbb{T}^2$ boundary

① TQFT MOTIVATION

Typical TQFT framework



Σ^{n-1}

$Z(U)$	Σ^{n-1} ⇝	$Z(V)$
vector space / module	field / algebra	vector space / module
elem't of v.s.	vector space	elem't of v.s.
object in $\mathcal{C}$	category $\mathcal{C}$	object in $\mathcal{C}$

$$Z(Y) = \langle Z(U), Z(V) \rangle_{Z(\Sigma)} \quad \text{pairing on } Z(\Sigma).$$

Heegaard Floer theory as a (2+1)-dim'l TQFT

Y^3 = closed oriented 3-mfd, Σ^2 = closed oriented surface

$HF(\Sigma)$ = Lagrangians in $\mathrm{Sym}^g \Sigma$ + local systems

$HF(U)$ and $HF(V)$ are Lagrangians with local systems.

(cf. Auroux, Lekili-Perutz)

Special cases

1) U and V are the simplest 3-mfds possible, i.e. handlebodies.
Σ has high genus, so $HF(\Sigma)$ is complicated.
Then $HF(U)$ and $HF(V)$ are Lagrangians in $\mathrm{Sym}^g \Sigma$, usually denoted $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ (products of attaching curves).

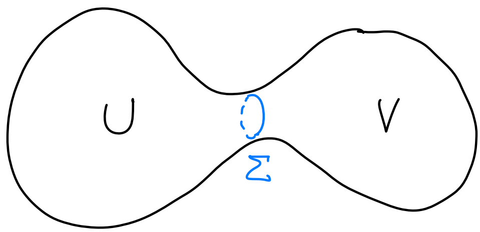
$$HF(Y) = \langle HF(U), HF(V) \rangle$$

pairing is Lagrangian intersection Floer homology

2) $\underbrace{\Sigma = S^2}_{\text{simplest one}}$, $Y = \underbrace{\overline{U}}\ \#\ \underbrace{\overline{V}}$

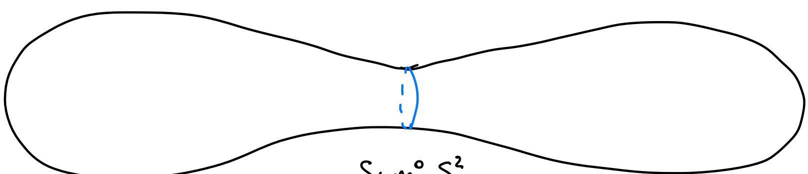
complicated (any 3-mfd)

$$\mathrm{Sym}^0(S^2) = pt$$



The Lagrangians for U and V are both $\{pt\}$, but there are interesting <u>local systems</u>.

a local system over a point is just a vector space / module



$$\mathrm{Sym}^0 S^2$$

$$\widehat{HF}(\overline{U}) \bullet \longrightarrow \bullet \longleftarrow \bullet\ \widehat{HF}(\overline{V})$$

Pairing is tensor product $\rightsquigarrow$ recover Künneth formula for $\widehat{HF}$.

Q: Is there something in between?

$\Sigma = \mathbb{T}^2$ next simplest surface after S^2.

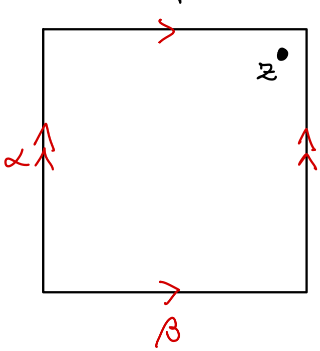
Lagrangians can already be interesting.

Hanselman-J. Rasmussen-Watson: construct a Lagrangian from a type D structure on $\mathcal{A}_{\mathbb{T}^2}$.

2 IMMERSED TRAIN TRACKS

Def: A 3-manifold with parameterised torus boundary is a cpt oriented M^3 together with $\partial M \cong \mathbb{T}^2$, $z \in \partial M$ basepoint, and $\alpha, \beta \subseteq \partial M$ two simple closed curves with $\alpha \cdot \beta = 1$.

Standard picture



$\mathbb{T}^2 =$ quotient of the unit square

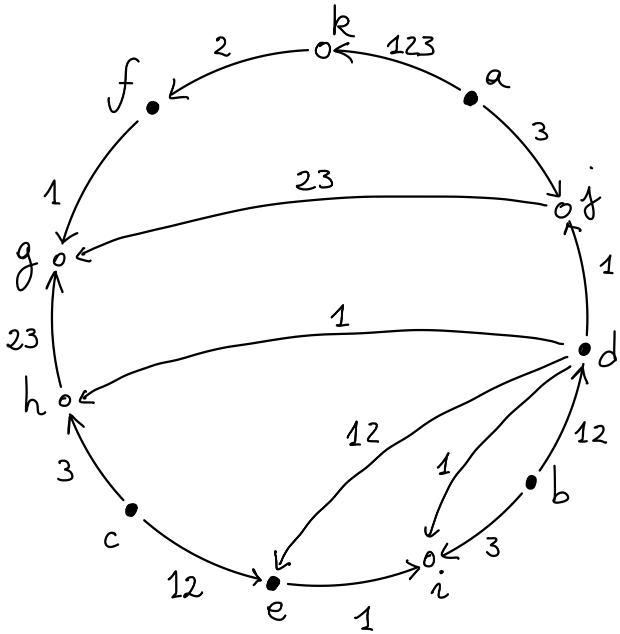
$z = (1-\varepsilon, 1-\varepsilon)$

$\beta =$ horiz. curve $\qquad \alpha =$ vertical curve

We look at ∂M from the outside, so for us $\beta \cdot \alpha = 1$.

From type D structures to train tracks

*) Represent a type D structure as a directed labelled graph



Dots are generators

$\bullet \in \iota_0 \cdot N$

$\circ \in \iota_1 \cdot N$

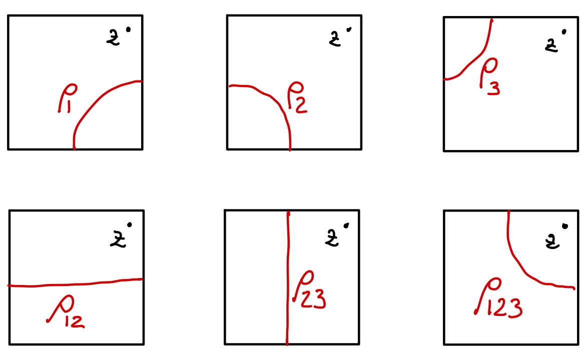
Arrows describe δ^1:

$\overset{x}{\bullet} \xrightarrow{\rho} \bullet y$

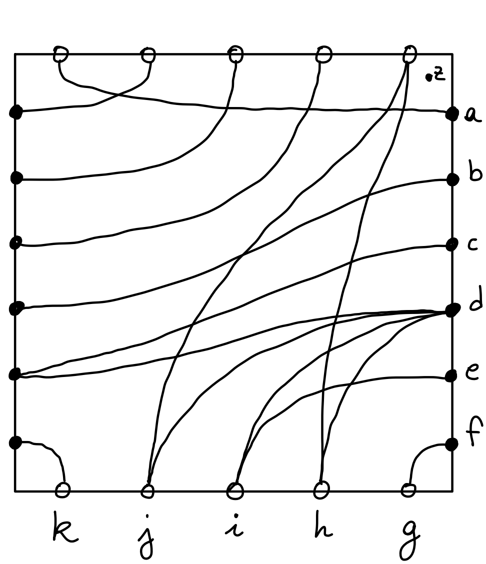
means $\delta^1(x) = \rho \otimes y$

*) Choose a point on α (resp. β) for each $\bullet$ (resp. $\circ$) gener.

*) Draw an arrow between x and y for each edge of the directed graph:



Example:



Dual $\mathcal{A}_\infty$-module

Given a directed graph for a type D structure, one can construct also an $\mathcal{A}_\infty$-module over $\mathcal{A}_{\mathbb{T}^2}$:

*) swap $1 \longleftrightarrow 3$

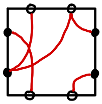
*) Get an idempotent decomposition $X = X\iota_0 \oplus X\iota_1$.

The other non-vanishing maps are given by the graph:

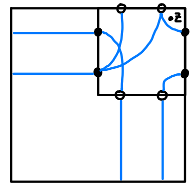
- for each directed path $\xi_{std} \to \xi_{end}$ you get a sequence of numbers;
- regroup them in maximal subsequences s_i of 123;
- get a map $m_j(\xi_{start}, s_1, s_2, \dots, s_j) = \xi_{end}$.

Pairing theorem

θ train track

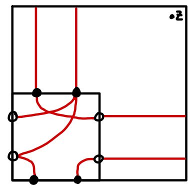


type A realisation



$A(\theta)$

type D realisation



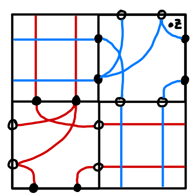
$D(\theta)$

note the reflection along $y=-x$

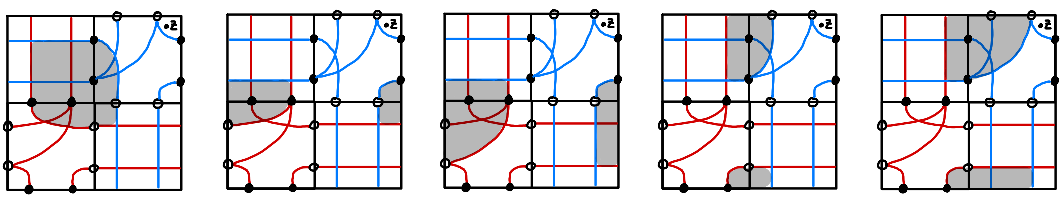
Immersed Lagrangian Floer homology:

- generators are intersection points of red & blue curves
- differential is given by counting bigons ($\leftrightarrow$ J-holom. discs)

e.g. pairing $A(\theta)$ and $D(\theta)$ one obtains a chain complex with 8 generators



there are 5 bigons:



This gives a chain complex $C(\theta_0, \theta_1)$.

The nice fact is that this is a geometric model for $\boxtimes$.

Def: A train track is called <u>BOUNDED</u> if its underlying decorated graph does not contain directed cycles.
(i.e. the associated type D structure is bounded)

Thm Let ϑ_0 and ϑ_1 be train tracks associated to dec. graphs, and suppose that ϑ_1 is bounded. Let

-) N_0 the $\mathcal{A}_\infty$-module associated to ϑ_0
-) N_1 the type-D structure associated to ϑ_1

Then there is an isomorphism of chain complexes

$$C(\vartheta_0, \vartheta_1) \cong N_0 \boxtimes_{\mathcal{A}} N_1$$

RK: There is a more general version where arrows of the form $\bullet \xrightarrow{\phi} \bullet$ are allowed.

3 IMMERSED CURVES

A change of basis in the type D structure results in a change in the train track associated to it. Luckily, there is a canonical form for many type D structures.

$\exists$ DG algebra $\tilde{\mathcal{A}}$ that extends $\mathcal{A}$ (with ρ_0 as well).

Def: A type D structure N over $\mathcal{A}$ is called EXTENDABLE if $\exists$ $\tilde{N}$ type D structure over $\tilde{\mathcal{A}}$ s.t. $N = \mathcal{A} \otimes_{\tilde{\mathcal{A}}} \tilde{N}$.

Thm: Bordered Floer type D structures are extendable.

Thm: Every extendable type D structure can be represented by a collection of immersed curves in $\mathbb{T}^2 - \{z\}$, decorated with local systems. ← these come from two parallel curves with switches between them

Thm: The immersed curve construction gives a bijection

$$\left\{ \begin{matrix} \text{extendable type D} \\ \text{structures over } \mathcal{A} \end{matrix} \right\} \Big/ \begin{matrix}\text{hty} \\ \text{equivalence}\end{matrix} \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{immersed multi-curves} \\ \text{with local systems} \end{matrix} \right\}$$

Rk: There are no known examples of non-trivial local systems.

Naturality

$$\begin{tikzcd}[column sep=large, row sep=large]
(M,\alpha,\beta) \arrow[r] \arrow[drr, dashed, "f_{\alpha,\beta}(\widehat{CFD}(M,\alpha,\beta))"] & \text{type D structure} \arrow[r] & \text{multi-curve in } \mathbb{T}^2 \arrow[d, "\substack{\text{identify } \mathbb{T}^2 \text{ with } \partial M \\ \text{by sending horiz} \to \beta \\ \text{vert} \to \alpha}"] \\
 & & \text{multi-curve on } \partial M \\
(M,\alpha',\beta') \arrow[r] \arrow[urr, dashed, "f_{\alpha',\beta'}(\widehat{CFD}(M,\alpha',\beta'))"'] & \text{type D structure} \arrow[r] & \text{multi-curve in } \mathbb{T}^2 \arrow[u, "\substack{\text{horiz.} \to \beta' \\ \text{vert.} \to \alpha'}"']
\end{tikzcd}$$

Thm: $f_{\alpha,\beta}(\widehat{CFD}(M,\alpha,\beta)) \simeq f_{\alpha',\beta'}(\widehat{CFD}(M,\alpha',\beta'))$

regular homotopy in $\partial M - \{z\}$

Cor: Given M manifold with torus boundary and $z \in \partial M$, the multi-curve $HF(M) \subseteq \partial M - \{z\}$ is an invariant.

Pairing thm revisited

mfds with torus boundary

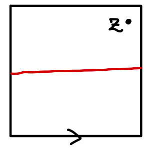
$M_i \rightsquigarrow \gamma_i \subseteq \partial M_i - \{z_i\}$ for $i \in \{0,1\}$

$h : (\partial M_1, z_1) \longrightarrow (\partial M_0, z_0)$ orient. rev. homeom. Then

$$\widehat{HF}(M_0 \cup_h M_1) \cong HF(\gamma_0, h(\gamma_1))$$

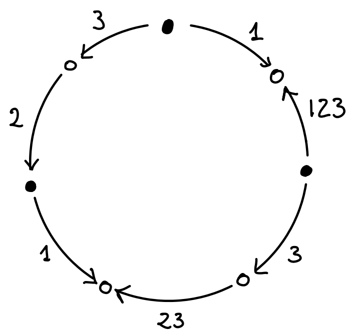
4 EXAMPLES

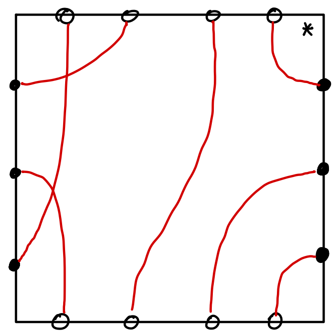
Solid torus



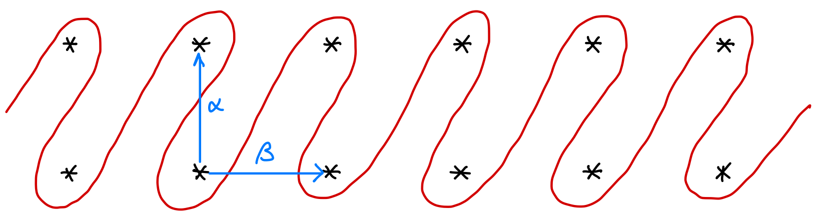
β = meridian

Right-handed trefoil complement





Easier to see in $\mathbb{R}^2 - \mathbb{Z}^2$, the maximal abelian cover of $\mathbb{T}^2 - \{z\}$.



α = meridian

β = Seifert longitude

Left-handed trefoil

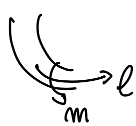
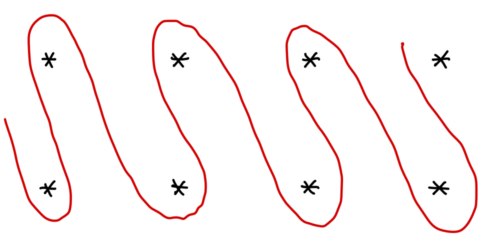
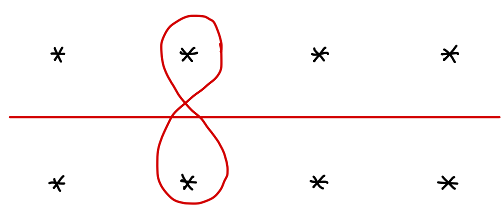


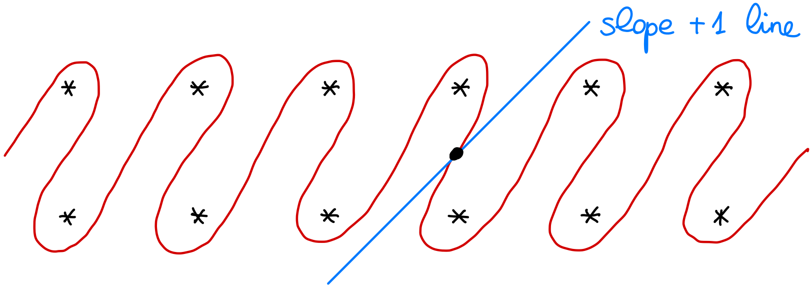
Figure-eight Knot





Surgeries: pair w/ line of corresponding slope.

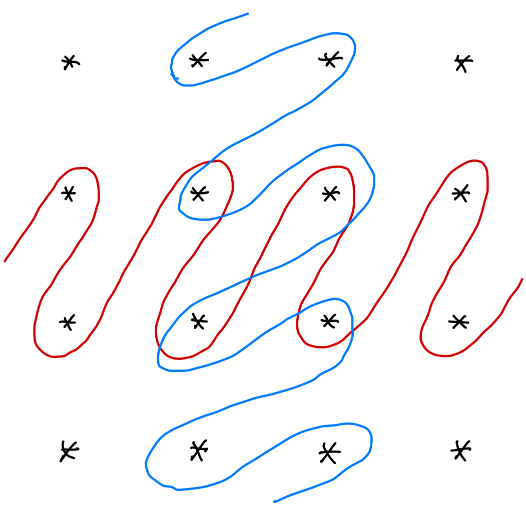
e.g. 1-surgery on RHT (Poincaré sphere w/ opposite orientation)



$$\widehat{HF}(-\Sigma(2,3,5)) = \mathbb{F}_2$$

Splice

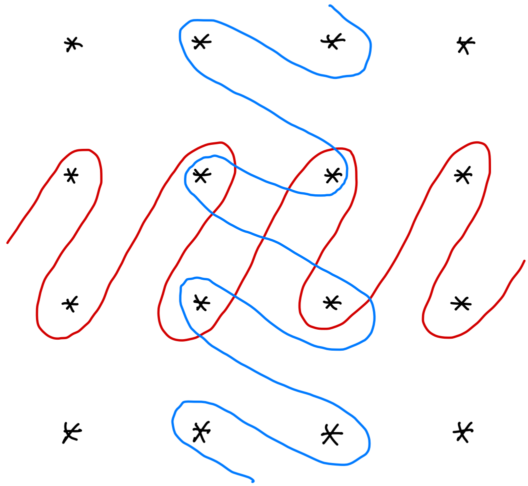
Def: The SPLICE of two knots K_0 and K_1 in S^3 is the 3-manifold obtained by gluing their knot exteriors using the identification $\mu_0 \longleftrightarrow \lambda_1$ and $\lambda_0 \longleftrightarrow \mu_1$



Splice of two RHT

$$\widehat{HF}(Y) \cong \mathbb{F}_7$$

[7 generators, no bigons]



Splice of RHT and LHT

$\widehat{HF}(Y) \cong \mathbb{F}_9$

$\left[\begin{array}{l}\text{9 generators,} \\ \text{no bigons}\end{array}\right]$