BORDERED KNOT FLOER HOMOLOGY

Lecture 1 - Motivation & first algebraic background

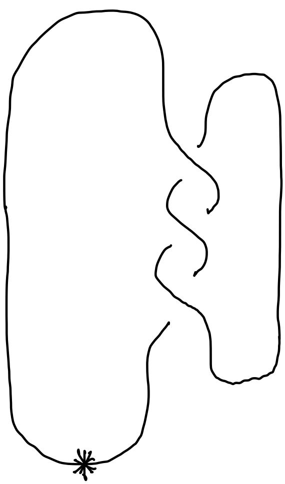
1 Motivation

$K \subseteq S^3$ Knot $\rightsquigarrow$ $\widehat{HFK}(K)$ bigraded vector space / $\mathbb{F}_2$

- detects Seifert genus, fibredness, Alex poly
- algorithmically computable (grid homology) [but computational complexity is factorial w/ grid index]

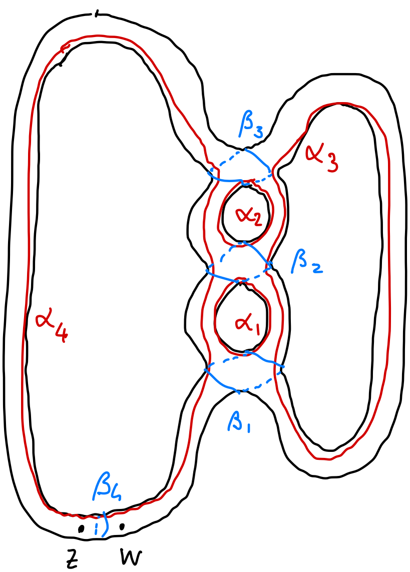
<u>Goal</u>: find a better algorithmic description of $\widehat{HFK}(K)$.

<u>Review</u>:



Knot diagram (w/ basepoint)

$\rightsquigarrow$



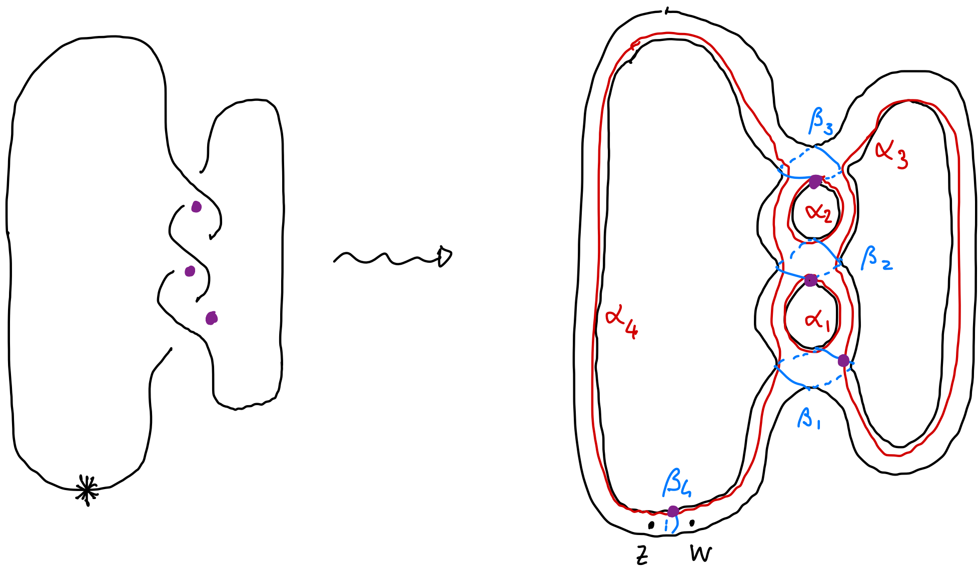
$\rightsquigarrow$ chain complex $\widehat{CFK}$

generators $/\mathbb{F}_2$: $\underline{x} = (x_1, x_2, x_3, x_4)$, where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some $\sigma \in S_4$

differential : skip

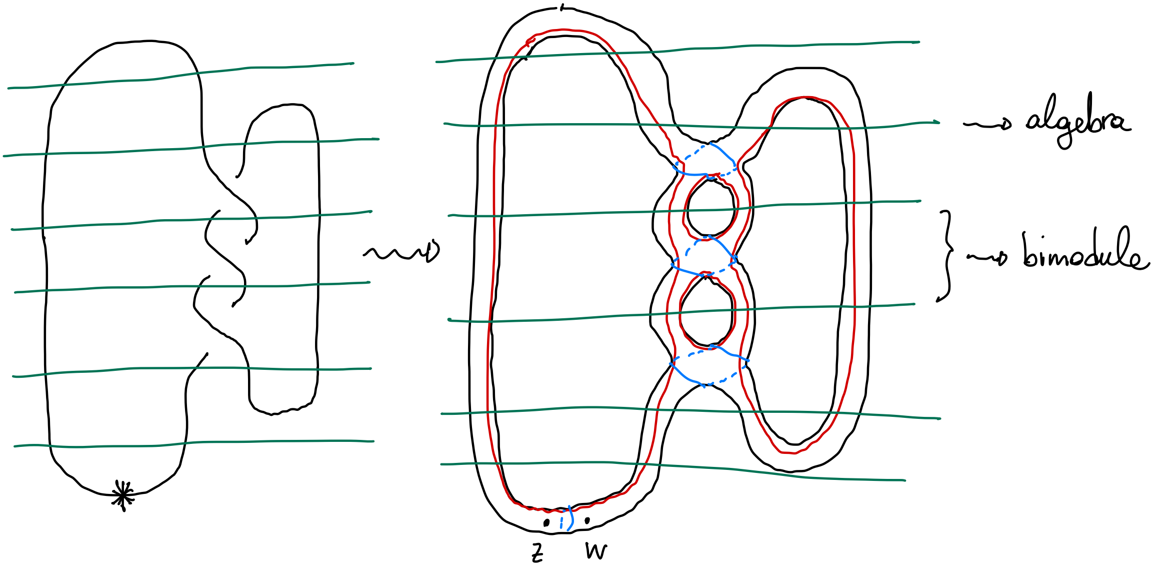
$\rightsquigarrow$ Take homology, $\widehat{HFK}(K)$, a knot invariant

Def: A <u>KAUFFMAN STATE</u> is a marking near each crossing, so that all regions of the diagram contain exactly 1 marking, except those adjacent to the basepoint.



RK: {Kauffman states} $\overset{1-1}{\longleftrightarrow}$ {generators of $\widehat{CFK}$}

Idea: build $\widehat{HFK}(K)$ step by step:



Hope: recover $\widehat{HFK}(K)$ as tensor product of bimodules

Too optimistic: gen'tors are local, but ∂ is global, so we need to keep track of it.

$\rightsquigarrow$ DGAs & DA-bimodules

$\rightsquigarrow$ after $\boxtimes$ recover $\widehat{CFK}(K)$.

2 DGAs

$\mathbb{K}$ = a commutative ring (usually $\mathbb{F}_2$)

Recall: a $\mathbb{K}$-algebra $\mathcal{A}$ is a ring w/ ring hom. $\mathbb{K} \longrightarrow \mathcal{A}$

(G, λ) = a group G with a central element $\lambda \in G$.

Def: $\mathcal{A}$ DIFFERENTIAL (G,λ)-GRADED $\mathbb{K}$-ALGEBRA (DGA) is:

- a $\mathbb{K}$-algebra $\mathcal{A}$
- a map $\partial : \mathcal{A} \longrightarrow \mathcal{A}$ such that
  - $\partial^2 = 0$
  - $\partial(ab) = (\partial a)b + a(\partial b)$
- a splitting $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ as $\mathbb{K}$-modules, such that
  - $\mathcal{A}_g \cdot \mathcal{A}_{g'} \subseteq \mathcal{A}_{gg'}$
  - $\partial(\mathcal{A}_g) \subseteq \mathcal{A}_{\lambda^{-1}\cdot g}$

The DGA is usually denoted $(\mathcal{A}, \partial, \cdot)$, or simply $\mathcal{A}$.

RK: Alternative notation: $\mu_1(a) = \partial(a)$
$\mu_2(a,b) = a \cdot b$

RK: $(\mathcal{A}, \partial, \cdot)$ DGA $\Rightarrow$ $H(\mathcal{A})$ is a G-graded $\mathbb{K}$-alg.

Examples

- $\mathcal{A}_0 = \mathbb{F}_2$ with $\partial \equiv 0$ and $G = \{0\}$ is a DGA.
- $$\mathcal{A}_1 = \mathrm{Path}_{\mathbb{F}_2}\left(\begin{tikzcd} \iota_0 \arrow[r, bend left, "\rho_1"] \arrow[r, bend right, "\rho_3"'] & \iota_1 \arrow[l, "\rho_2"'] \end{tikzcd}\right) \Big/ \left(\rho_2\rho_1 = 0 = \rho_3\rho_2\right)$$

path algebra notation
$\rho_2\rho_1$ is the concatenation of ρ_2 and then ρ_1

Notation:

$$\rho_{12} = \rho_1\rho_2 \qquad \rho_{23} = \rho_2\rho_3 \qquad \rho_{123} = \rho_1\rho_2\rho_3$$

constant paths

As a vector space / $\mathbb{F}_2$, $\mathcal{A}_1 \cong \mathbb{F}_2\langle \rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}, c_{\iota_0}, c_{\iota_1}\rangle$

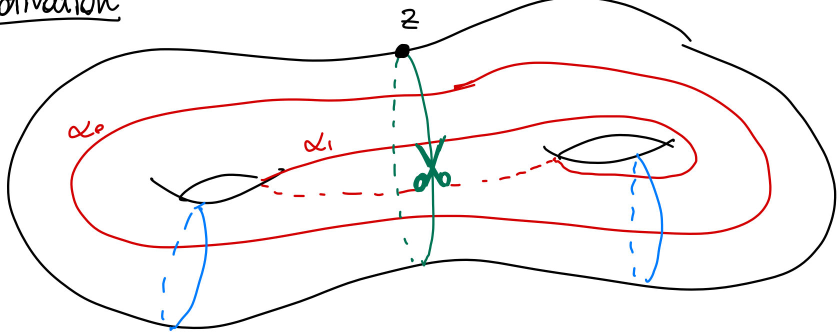
Set $\partial \equiv 0$.

Gradings: skip for now.

In fact, $\mathcal{A}_0$ & $\mathcal{A}_1$ are DGAs arising from bordered HF.
They are instances of "strands algebras", which we review below
(we only focus on the torus algebra).

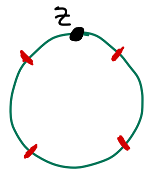
(3) THE TORUS ALGEBRA in BORDERED HF

Motivation

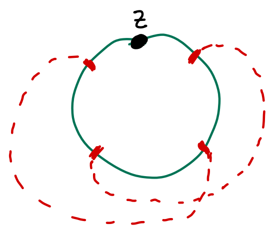


In BHF you want to cut a <u>3-mfd</u> into <u>2 pieces w/ boundary</u>
(Heegaard diag.) (2 Heeg. diag. w/ curves & arcs)

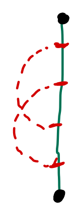
In the example above we cut along



The red points are matched in pairs, because 2 of them lie on α_0 and 2 on α_1

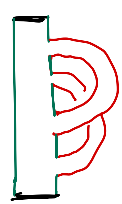


For convenience, the circle is cut in corresp. of z :



Call this diagram $\mathcal{Z}_{\mathbb{T}}$

<u>RK</u>: The name "torus algebra" is justified by the fact that this is a surgery diagram for the torus.



+ 2-handle

Goal: define a DGA associated to $Z_{\mathbb{T}}$

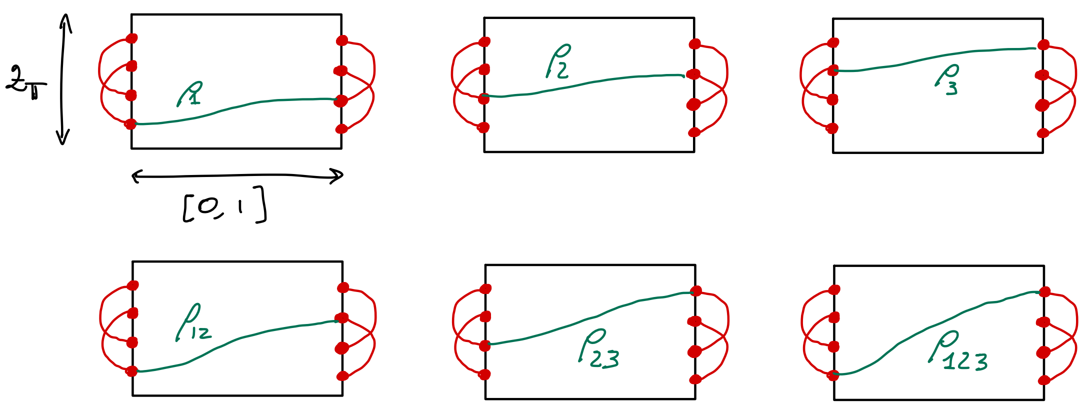
Def: An <u>ℓ-STRAND</u> on $Z_{\mathbb{T}}$ is a collection $s=\{s_1,\dots,s_\ell\}$ of continuous, pw smooth functions $s_i:[0,1]\to Z_{\mathbb{T}}$ s.t.:

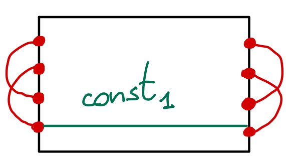
- $s(0)$ (resp. $s(1)$) is a subset of the marked basepoints which does not contain matched basepoints
- $\partial/\partial t\,(s_i(t)) \geq 0$
- $s_i \pitchfork s_j$,

up to pw smooth isotopy.

e.g. <u>0-strands</u>: only the empty strand

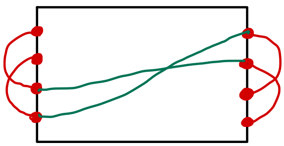
e.g. <u>1-strands</u>:



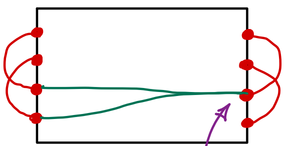


There are 4 more 1-strands, namely $const_2$, $const_3$, $const_4$.

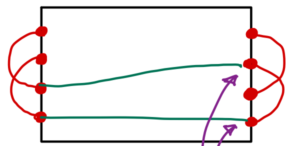
e.g. <u>2-strands</u>



is a 2-strand



and



are <u>NOT</u> 2-strands

problem here

matched basepoints

e.g. <u>3-strands</u>: they don't exist, because of the matched basepoint condition.

<u>Def</u>: An ℓ-strand is <u>non-degenerate</u> if $\forall i, j$

$$\mathrm{geom}(s_i, s_j) = |\mathrm{alg}(s_i, s_j)|$$

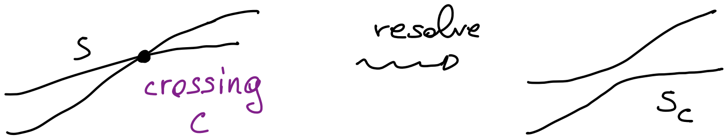
<u>Def</u>: <u>CONCATENATION</u> of s, t non-deg. ℓ-strands

$$s \cdot t := \begin{cases} \boxed{\; s \;|\; t \;} & \text{if } s(1) = t(0) \text{ and } s \cdot t \text{ non-deg.} \\ 0 & \text{otherwise} \end{cases}$$

e.g. $\rho_1 \cdot \rho_2 = \rho_{12}$, ...

<u>Def</u>: $\widetilde{\mathcal{A}}_{\mathbb{T}}(\ell) := \mathbb{F}_2$-algebra generated by non-deg. ℓ-strands

Def. $\partial : \tilde{A}_{\mathbb{T}}(\ell) \longrightarrow \tilde{A}_{\mathbb{T}}(\ell)$ differential

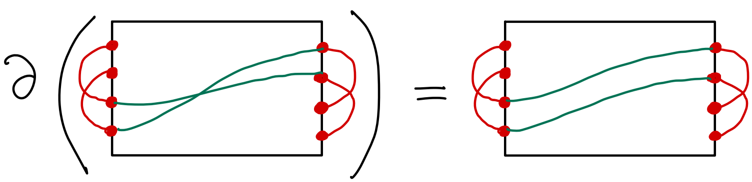


$$\partial s := \sum_{\substack{c \text{ crossings} \\ s_c \text{ non-deg.}}} s_c$$

Fact/EX: $\partial^2 = 0$

e.g.: for 0-strands & 1-strands $\partial \equiv 0$.

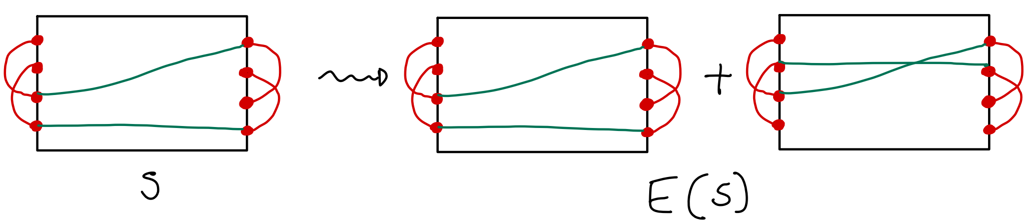
e.g. for 2-strands the differential is non-trivial:



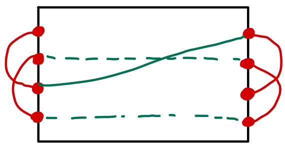
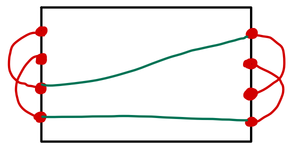
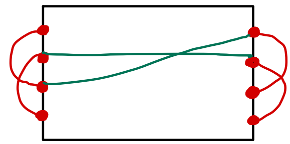
Given an ℓ-strand s, one defines the "equalised" ℓ-strand $E(s)$: for each subset A of the constant strands of s, define s_A to be the ℓ-strand obtained by erasing those constant strands & replacing them with their matched constant strands. Note that $s_\phi = s$.

Define $E(s) := \sum_A s_A$.

e.g.



Notation (dashed lines notation):

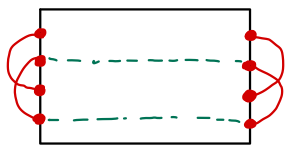
 $:=$  $+$ 

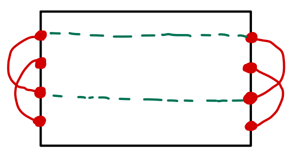
Def: $\mathcal{A}_{\mathbb{T}}(\ell) \subseteq \widetilde{\mathcal{A}}_{\mathbb{T}}(\ell)$ is the subalgebra gen'd by equalised ℓ-strands.

Rk: $\mathcal{A}_{\mathbb{T}}(0) \cong \mathcal{A}_0 \cong \mathbb{F}_2$

Rk: $\mathcal{A}_{\mathbb{T}}(1) \cong \mathcal{A}_1$, and the isomorphism maps ρ_1 to ρ_1, ...

Moreover:

 $\longleftrightarrow$ c_{L_0}

 $\longleftrightarrow$ c_{L_1}

Rk: $\mathcal{A}_{\mathbb{T}}(2)$ has rank 7 over $\mathbb{F}_2$, and non-trivial ∂.

$H(\mathcal{A}_{\mathbb{T}}(2)) \cong \mathbb{F}_2$.

Gradings: next week.