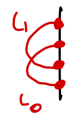
Lecture 2: Ozsváth-Szabó's algebra

① Idempotents & linear categories

Recall: $\mathcal{A}_1 = \mathrm{Path}_{\mathbb{F}_2}\left(\begin{tikzcd} \iota_0 \arrow[r, bend left, "\rho_1"] \arrow[r, bend right, "\rho_3"'] & \iota_1 \arrow[l, "\rho_2"'] \end{tikzcd} \right) \Big/ \begin{pmatrix} \rho_2\rho_1 = 0, \\ \rho_3\rho_2 = 0 \end{pmatrix}$

The constant paths I_{ι_0} and I_{ι_1} are <u>orthogonal idempotents</u>

$(\text{i.e., } I_{\iota_0} \cdot I_{\iota_0} = I_{\iota_0},\ I_{\iota_1} \cdot I_{\iota_1} = I_{\iota_1},\ I_{\iota_0} \cdot I_{\iota_1} = 0 = I_{\iota_1} \cdot I_{\iota_0})$

$\{\perp \text{ idempotents}\} \longleftrightarrow \{\text{vertices}\} \longleftrightarrow \left\{ \begin{array}{l} \text{sets of } \ell \text{ pairs} \\ \text{of matched bpts} \end{array} \right.$  $\}$

$\{I_{\iota_0}, I_{\iota_1}\}$ give a splitting of $\mathcal{A}_1$:

$$\mathcal{A}_1 \underset{\mathbb{F}_2\text{-mod}}{\overset{\text{as}}{=}} I_{\iota_0} \cdot \mathcal{A}_1 \cdot I_{\iota_0} \oplus I_{\iota_0} \cdot \mathcal{A}_1 \cdot I_{\iota_1} \oplus I_{\iota_1} \cdot \mathcal{A}_1 \cdot I_{\iota_0} \oplus I_{\iota_1} \cdot \mathcal{A}_1 \cdot I_{\iota_1}$$

In fact, the $\perp$ idempotents allow to reinterpret $\mathcal{A}_1$ as a <u>LINEAR CATEGORY</u> (i.e., category w/ $\mathrm{Mor}(x,y)$ are $\mathbb{F}_2$-v.s.):

$\mathrm{Obj}(\mathcal{A}_1) = \{\iota_0, \iota_1\}$

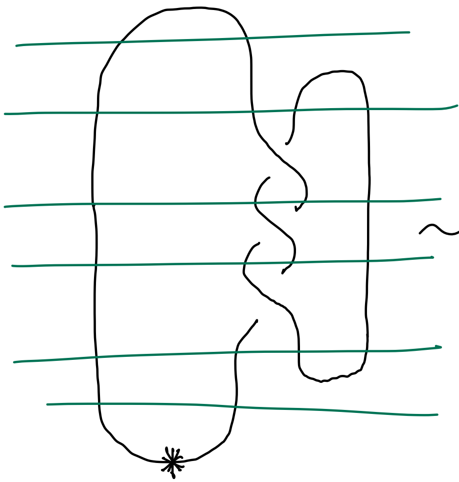
$\mathbb{F}_2$ - vector spaces

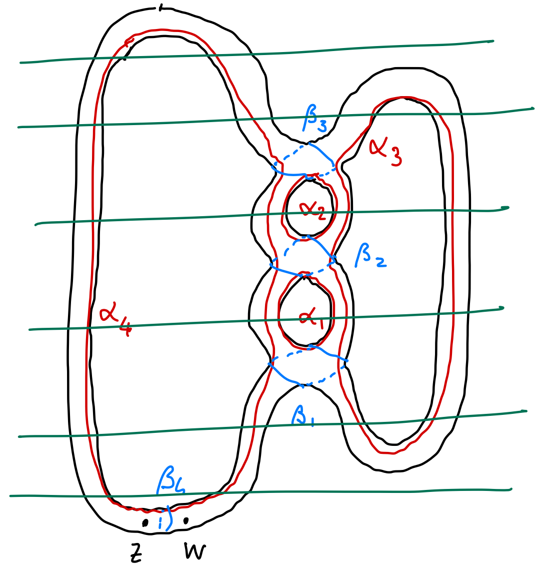
$\mathrm{Mor}(\mathcal{A}_1) = \mathrm{Mor}(\iota_0,\iota_0) \amalg \mathrm{Mor}(\iota_0,\iota_1) \amalg \mathrm{Mor}(\iota_1,\iota_0) \amalg \mathrm{Mor}(\iota_1,\iota_1)$

Recover $\mathcal{A}_1$ as $\bigoplus_{i,j} \mathrm{Mor}(i,j)$

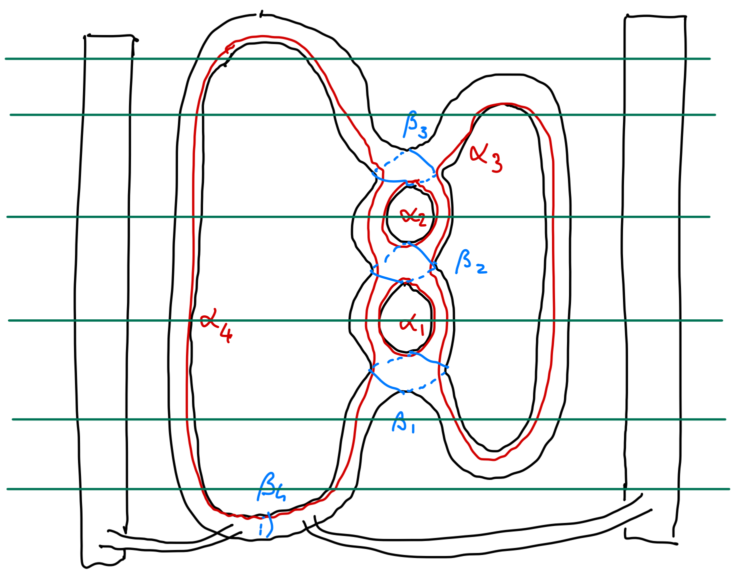
② Motivation for Ozsváth-Szabó's algebra

Recall:



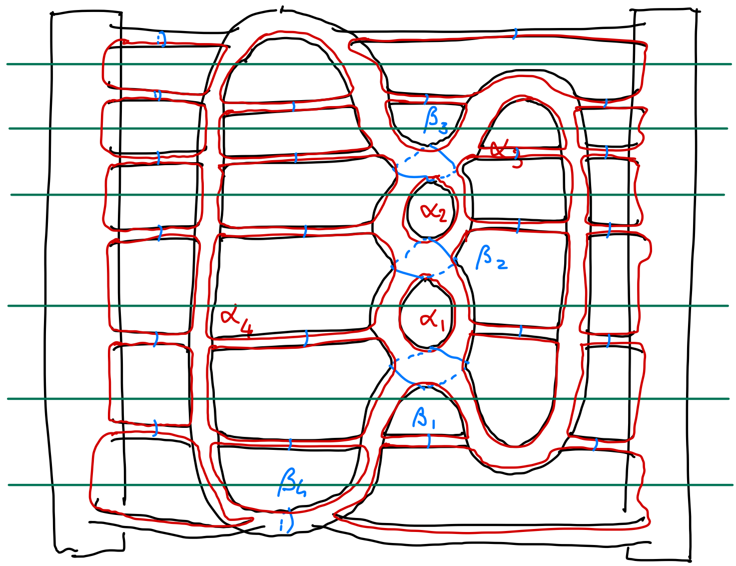


Problems: ① no basepoints/sutures in the avg layer

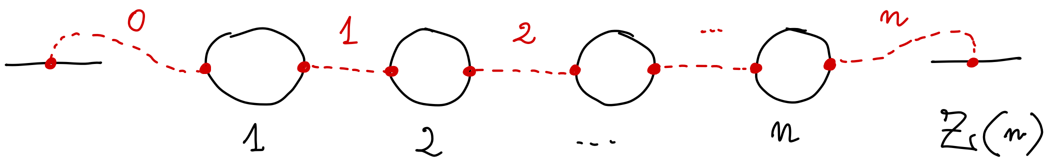


The rectangles on the sides have boundary, so new $\partial\Sigma$ consists of two closed curves.

(2) Most layers are disconnected, & red curves go from bottom to top.



New typical section: (w/ matching of basepoints):

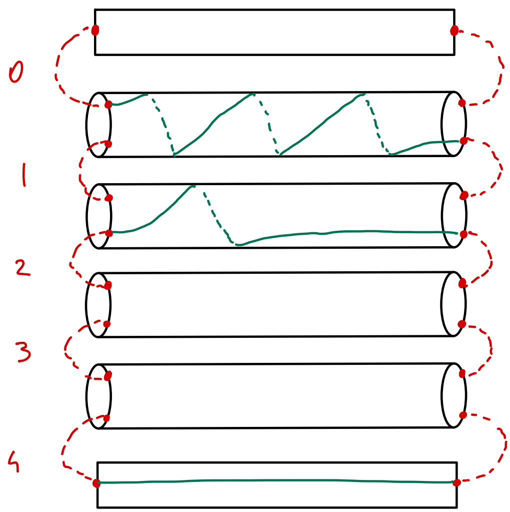


<u>Today's goal</u>: motivated by ℓ-strands on $Z(n)$, define an algebra $\mathcal{B}(n,\ell)$, with a splitting

$$\mathcal{B}(n,\ell) = \bigoplus_{I,J \text{ idemp.}} I \cdot \mathcal{B}(n,\ell) \cdot J$$

③ Idempotents on $\mathcal{Z}(n)$

Typical ℓ-strand on $\mathcal{Z}(n)$



$s(0)$ occupies the following pairs of matched bpts: $0, 2, 4$.

$s(1)$ " " " " " " " : $1, 2, 4$.

Def: An ℓ-IDEMPOTENT for $\mathcal{Z}(n)$ is an increasing function

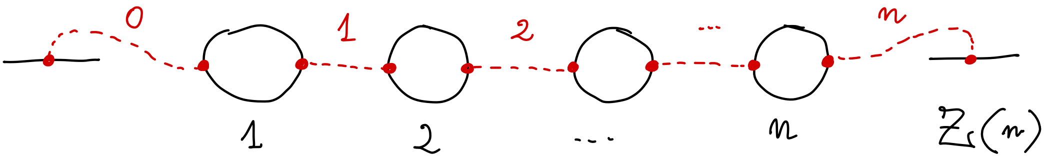
$$I : \{1, \dots, \ell\} \longrightarrow \{0, 1, \dots, n\}$$

We often represent I by $\operatorname{im} I$, and write $|I| = \ell$.

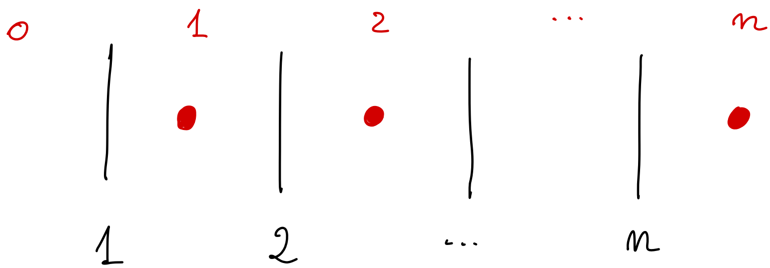
e.g. $I = \{1, 2, 4\}$ is a 3-idempotent, i.e. $|I| = 3$.

Rk: $\{\ell\text{-idempotents}\} \longleftrightarrow \left\{\begin{array}{l}\text{sets of } \ell \text{ pairs of} \\ \text{matched bpts in } \mathcal{Z}(n)\end{array}\right\}$

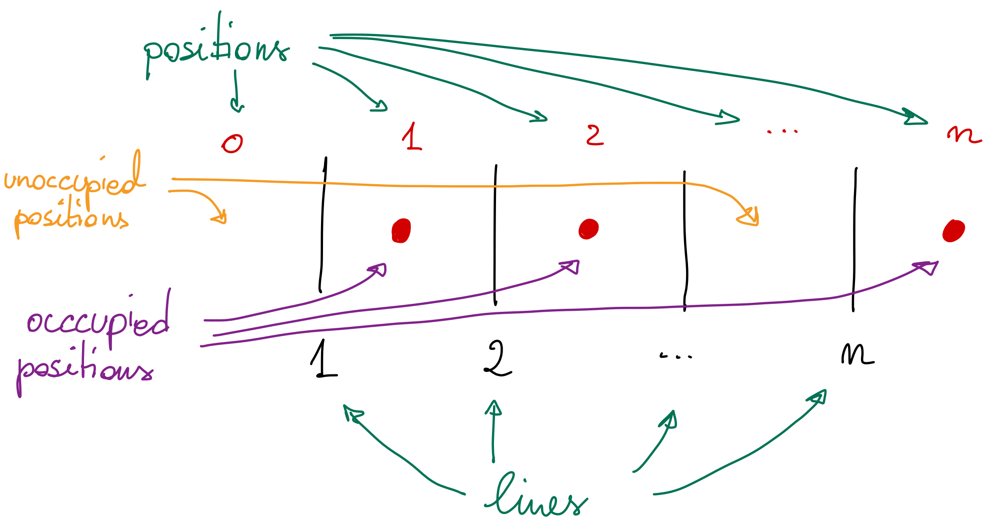
Representation:



minimalism



Terminology:



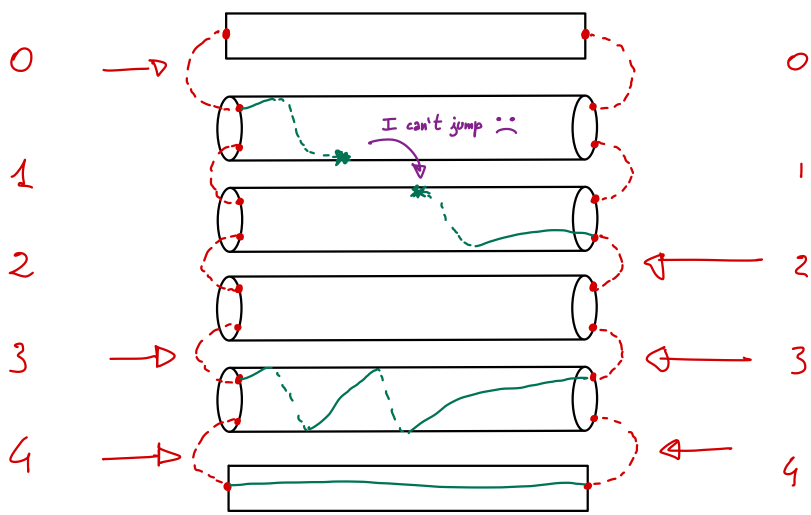
Def: Suppose $|I| = |J|$. We say that I and J are

<u>CLOSE</u> if $|I(i) - J(i)| \leq 1 \quad \forall i$

<u>FAR</u> otherwise

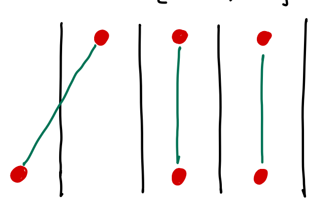
Geometric motivation: if two idempotents are far, there are no ℓ-strands between them.

e.g. $I = \{0, 3, 4\}$ $\qquad\qquad$ $J = \{2, 3, 4\}$



Def: If I and J are close, $I(i) = r - 1$, and $J(i) = r$, we say <u>line r is crossed</u>

$J = \{1, 2, 3\}$



e.g. line 1 is crossed here

$I = \{0, 2, 3\}$

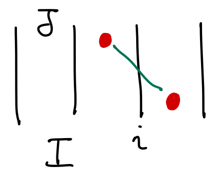
④ Ozsváth–Szabó's algebra

Def: Define the graph $Q(n,\ell)$ by setting

vertices = ℓ-idempotents on $\mathcal{Z}(n)$

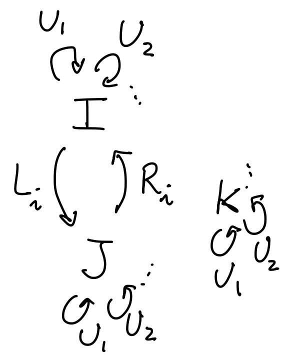
edges = $\forall I$, add edges $U_1, \dots, U_n : I \circlearrowleft$

if I and J are close and differ by 1 crossed line (line i) as below



add edges $L_i : I \to J$
$R_i : J \to I$

Bad side picture



Def: $\mathcal{B}(n,\ell) = \mathrm{Path}_{\mathbb{F}_2}\left(Q(n,\ell)\right) / \text{relations}$

Relations:

- $R_i R_{i+1} = 0 = L_{i+1} \cdot L_i$
- $L_i R_i = U_i = R_i L_i$
- $U_i = 0$ if $I \cap \{i-1, i\} = \emptyset$ — the path based at I
- U_i are "central" (i.e., $U_i L_j = L_j U_i, \dots$)
- if $|i-j| > 1$, then
$$L_i L_j = L_j L_i$$
$$R_i R_j = R_j R_i$$
$$L_i R_j = R_j L_i$$

Rk: $\{\ell\text{-idempotents}\} \longleftrightarrow \left\{\begin{array}{c}\text{constant paths} \\ \text{in } Q(n,\ell)\end{array}\right\}$

Rk: ℓ-idempotents form a set of orthogonal idemp. in $\mathcal{B}(n,\ell)$.

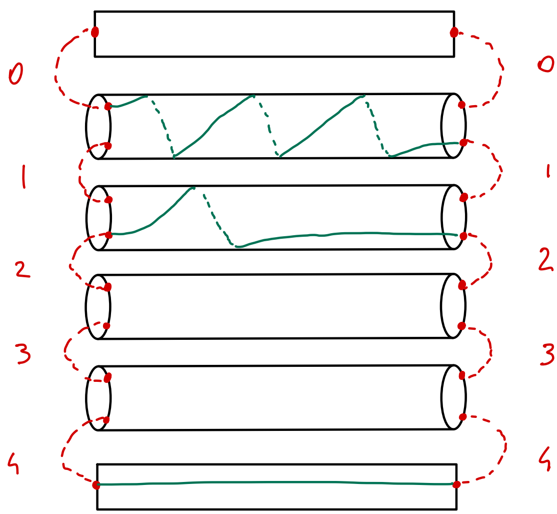
Rk: $\mathcal{B}(n,\ell)$ isn't a strands algebra on $\mathcal{B}(n,\ell)$, but a close relative of it.

⑤ Strands algebra interpretation

Def: An ℓ-STRAND on $Z(n)$ is a set $s = \{s_1, \dots, s_\ell\}$ of continuous, pw smooth functions $s_i : [0,1] \longrightarrow Z(n)$ s.t.:

- $s(0)$ (resp. $s(1)$) does not contain matched basepts.
- $\dfrac{\partial s_i}{\partial t} \geq 0 \quad \forall i$
- $s_i \pitchfork s_j \quad \forall i, j$

e.g.



$s(0) \rightsquigarrow I_0 = \{0, 2, 4\}$

$s(1) \rightsquigarrow I_1 = \{1, 2, 4\}$

s goes from I_0 to I_1

Def: s is <u>non-degenerate</u> if $\mathrm{geom}(s_i, s_j) = |\mathrm{alg}(s_i, s_j)|$.

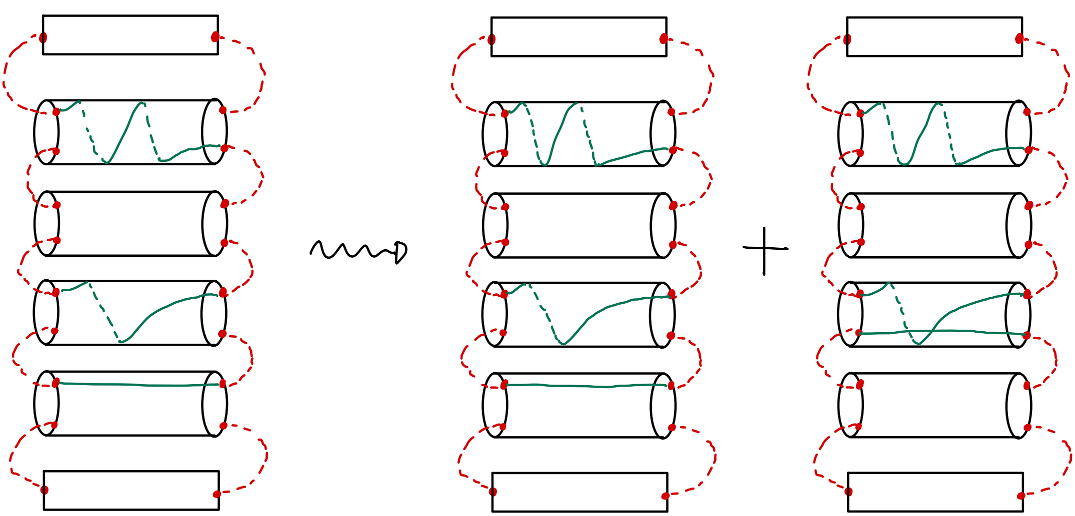
Def: <u>CONCATENATION</u>

$$s \cdot t = \begin{cases} \boxed{s}\boxed{t} & \text{if } s(1) = t(0) \text{ and } s \cdot t \text{ non-deg.} \\ 0 & \text{otherwise} \end{cases}$$

Def: $\partial s = \sum_{\substack{c \text{ crossings} \\ s_c \text{ non-deg.}}} s_c$ resolution of s at the crossing c

Def: $\tilde{A}(n, \ell)$ = algebra gen.'d by non-deg. ℓ-strands

Def: Equalising a strand



$s \rightsquigarrow E(s)$

Def: $\mathcal{A}(n,\ell) \subseteq \tilde{\mathcal{A}}(n,\ell)$ is the subalgebra spanned by $E(s)$.

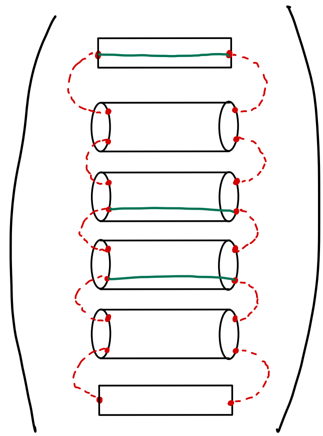
Rk: $\mathcal{A}(n,\ell)$ and $\tilde{\mathcal{A}}(n,\ell)$ are DGAs.

Idempotents are idempotents

I

rotate + turn into ℓ-strand $\rightsquigarrow$

E



$=: I \in \mathcal{A}(n,\ell)$

The ℓ-idempotents give a set of orthogonal idempotents in $\mathcal{A}(n,\ell)$.

$$\rightsquigarrow \quad \mathcal{A}(n,\ell) \xrightarrow[\text{cplxes}/\mathbb{F}_2]{\text{as chain}} \bigoplus_{I,J} I \cdot \mathcal{A}(n,\ell) \cdot J$$

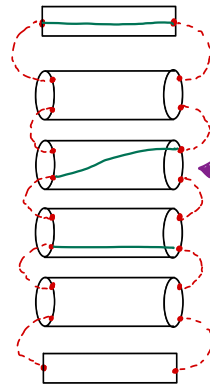
Thm (Lekili–Polishchuk, Manion–M.–Willis) $\partial \equiv 0$ here

$\exists$ quasi-isomorphism $\psi : \mathcal{B}(n,\ell) \longrightarrow \mathcal{A}(n,\ell)$

$$I \cdot \mathcal{B}(n,\ell) \cdot J \longrightarrow I \cdot \mathcal{A}(n,\ell) \cdot J$$

$$\Longrightarrow \mathcal{B}(n,\ell) = H(\mathcal{A}(n,\ell)).$$

Definition of the map

$(L_i : I \longrightarrow J) \longmapsto E\Big($  $\Big)$

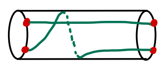
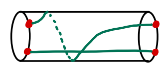
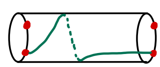
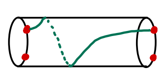
add constant strands as necessary so that the left idempotent is I (and the right is J)

i-th cylinder

$(R_i : I \longrightarrow J) \longmapsto E\Big($  $\Big)$

only showing i-th cylinder for simplicity

$(U_i : I \longrightarrow I) \longmapsto$

- $E\Big($  $\Big) + E\Big($  $\Big)$ if $\{i, i+1\} \subseteq I$
- $E\Big($  $\Big)$ if $i \notin I$ and $i+1 \in I$
- $E\Big($  $\Big)$ if $i \in I$ and $i+1 \notin I$
- 0 if $I \cap \{i, i+1\} = \emptyset$

6 Gradings on $\mathcal{B}(n,\ell)$

1) <u>Unrefined Alexander grading</u> over $\mathbb{Z}^{2n} = \mathbb{Z}\langle \tau_1, \beta_1, \dots, \tau_n, \beta_n \rangle$

Define it on the arrows of $Q(n,\ell)$ by setting:

$$w^{un}(\gamma) = \begin{cases} \tau_i & \text{if } \gamma = R_i \\ \beta_i & \text{if } \gamma = L_i \\ \tau_i + \beta_i & \text{if } \gamma = U_i \end{cases}$$

Relations are homogeneous $\rightsquigarrow$ grading on $\mathcal{B}(n,\ell)$

2) <u>Refined Alexander grading</u> over $\left(\frac{1}{2}\mathbb{Z}\right)^n = \frac{1}{2}\mathbb{Z}\langle e_1, \dots, e_n \rangle$

Let $\varphi : \mathbb{Z}^{2n} \longrightarrow \left(\frac{1}{2}\mathbb{Z}\right)^n$

$$\tau_i \longmapsto \frac{1}{2} e_i$$

$$\beta_i \longmapsto \frac{1}{2} \cdot e_i$$

Define $w := \varphi \circ w^{un}$.

For the next two gradings, we need to fix a subset $S \subseteq \{1, \dots, n\}$.

3) <u>Collapsed Alexander grading</u> over $\frac{1}{2}\mathbb{Z}$

$$\mathrm{Alex}_S := -\sum_{i\in S} w_i + \sum_{i\notin S} w_i$$

[geom. motivation: S are the arcs of the knot proj loc. pointing upwards]

4) <u>Maslov grading</u> over $\mathbb{Z}$

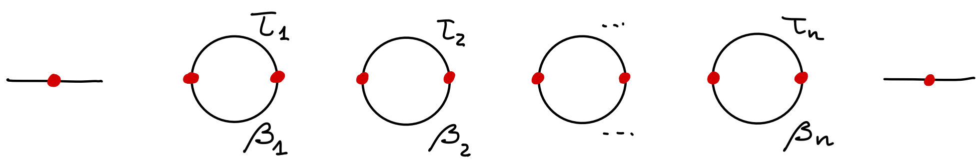
$$m_S := -2\cdot\sum_{i\in S} w_i$$

<u>Rk</u>: $\Delta := m_S - \mathrm{Alex}_S$ is independent of the choice of S

Gradings on $\mathcal{A}(n,\ell)$

1) <u>Unrefined Alexander grading</u>

$H_1(\mathcal{Z}(n), B) \cong \mathbb{Z}^{2n}$, spanned by $\tau_1, \beta_1, \dots, \tau_n, \beta_n$



Given s ℓ-strand on $\mathcal{Z}(n)$, define

$$w^{un}(s) := [s] \in \underbrace{H_1(\mathcal{Z}(n)\times[0,1], B\times[0,1])}_{\begin{array}{c}\cong \\ H_1(\mathcal{Z}(n), B) \\ \parallel \\ \mathbb{Z}\langle \tau_1, \beta_1, \dots, \tau_n, \beta_n\rangle\end{array}}$$

2) Refined Alexander grading over $\left(\frac{1}{2}\mathbb{Z}\right)^n$

As before, define $w(s) := \varphi \circ w^{un}(s)$

Geom. interpretation: the i-th component $w_i(s)$ of $w(s)$ indicates the total winding number of the ℓ-strand over the i-th cylinder.

For the next two gradings, we need to fix $S \subseteq \{1, \ldots, n\}$.

3) Single Alexander grading

$$\mathrm{Alex}_S := -\sum_{i \in S} w_i + \sum_{i \notin S} w_i \qquad \text{(same as before)}$$

4) Maslov grading over $\mathbb{Z}$

$$\mathrm{Cyl}_i(s) := \begin{cases} \#(\text{crossings on } i\text{-th cyl}) - w_i(s) & \text{if } s \text{ has 2 strands on the } i\text{-th cylinder} \\ 0 & \text{otherwise} \end{cases}$$

$$m_S(s) := -2 \cdot \sum_{i \in S} w_i(s) + \sum_{i=1}^{n} \mathrm{Cyl}_i(s)$$

Thm: The q.i. $\mathcal{B}(n,\ell) \longrightarrow \mathcal{A}(n,\ell)$ preserves all gradings