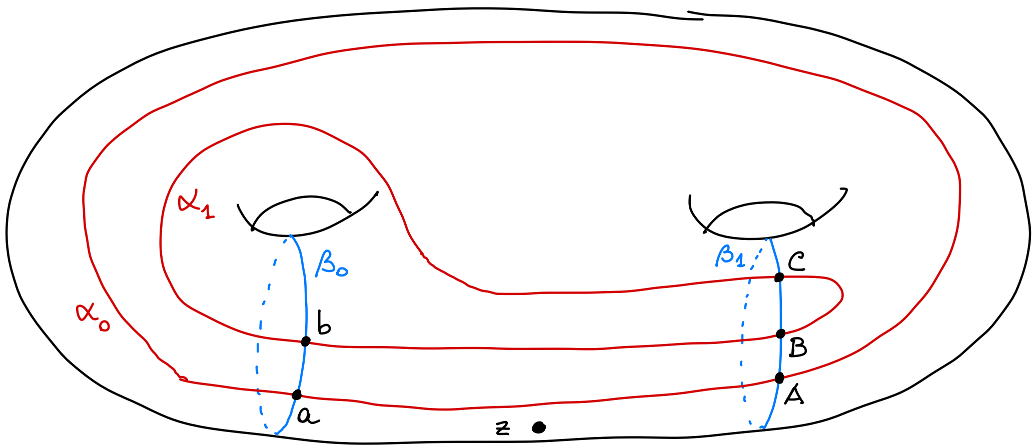
Lecture 3: curved DGAs, A_∞-modules, and type-D structures

1 Motivation from bordered Floer homology

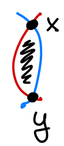
Consider the Heegaard diagram (for S^3)



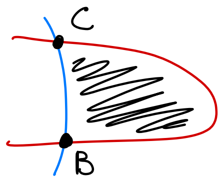
Heegaard Floer complex

- Generators: $\underline{x} = (x_0, x_1)$ w/ $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some $\sigma \in S_2$

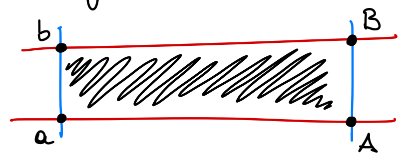
  3 generators: aB, aC, bA

- Differential: a holom. disc  gives an arrow $\underline{x} \longrightarrow y$

In our case we have two kinds of hol. discs:



$aB \longrightarrow aC$



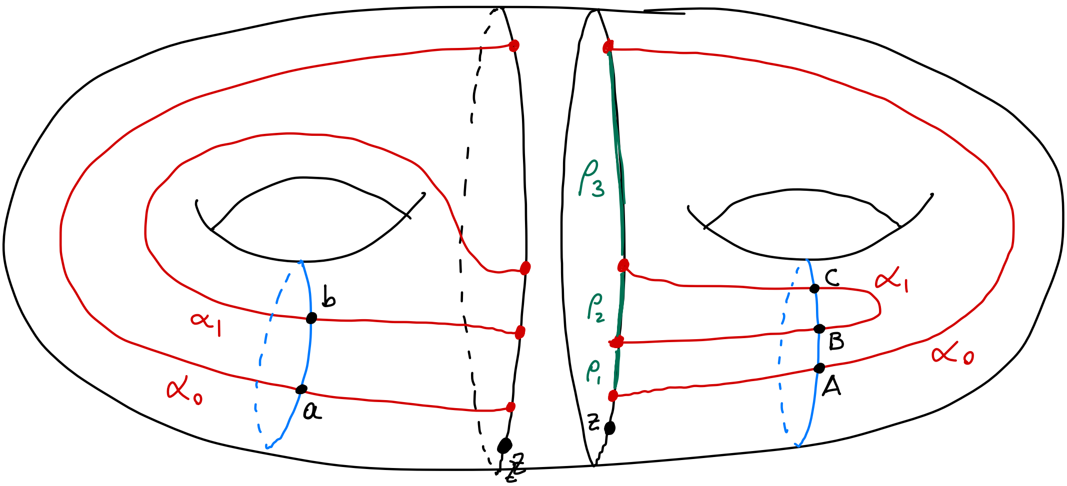
$aB \longrightarrow bA$

⇝ differential:

$$\partial(aB) = aC + bA$$
$$\partial(aC) = 0$$
$$\partial(bA) = 0$$

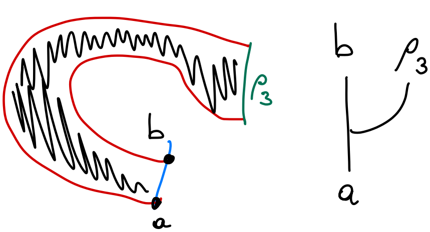
$$\left[\begin{tikzcd} & aB \arrow[dl] \arrow[dr] & \\ aC & & bA \end{tikzcd}\right]$$

Let's compute the complex in a different way:



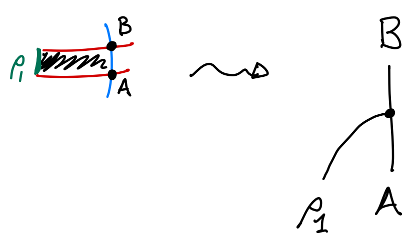
Generators: a_{ι_0}, b_{ι_1}

"Differential"

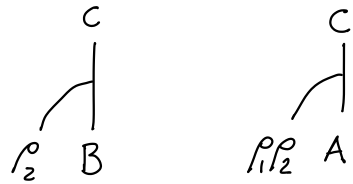


Generators: $\iota_1 A$, $\iota_0 B$, $\iota_0 C$

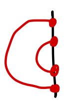
"Differential"



There are two more rectangles:



"A_∞-module" M | "Type-D structure" N

M and N are over a strands algebra $\mathcal{A}$. Note that $\mathcal{A}$ isn't exactly the torus algebra we have already seen, because here we use the matching  instead of .

<u>Follow to re-build the full complex:</u> $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}N$

<u>Generators:</u> $a_{\iota_0} \otimes_{\iota_1} A$, $a_{\iota_0} \otimes_{\iota_0} B$, $a_{\iota_0} \otimes_{\iota_0} C$

0

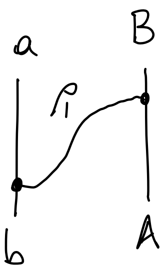
$b_{\iota_1} \otimes_{\iota_1} A$, $b_{\iota_1} \otimes_{\iota_0} B$, $b_{\iota_1} \otimes_{\iota_0} C$

0 0

bA, aB, aC

Differential:

 $\rightsquigarrow$ $\begin{array}{c} aB \\ \downarrow \\ aC \end{array}$

 $\rightsquigarrow$ $\begin{array}{c} aB \\ \downarrow \\ bA \end{array}$

No other pictures match exactly, so we don't get any other arrow in the complex.

② Curved DGAs

gp, central element

Def: A $\underline{\text{CURVED DIFFERENTIAL } (G,\lambda)\text{-GRADED } K\text{-ALGEBRA}}$ is

curvature, multiplication

(A, μ_0, μ_1, μ_2), where $\mu_i : A^{\otimes i} \longrightarrow A$,

K-algebra, differential

subject to the following structure relations:

$R_0)$ $\mu_1 \circ \mu_0 = 0$ (μ_0 is a cycle)

$R_1)$ $\mu_2(\mu_0, a) + \mu_2(a, \mu_0) + \mu_1 \circ \mu_1(a) = 0$

(i.e., $\partial^2 \neq 0$, but $\partial^2 a = [a, \mu_0]$)

$R_2)$ $\mu_1(\mu_2(a,b)) + \mu_2(\mu_1(a), b) + \mu_2(a, \mu_1(b)) = 0$

(Leibniz's rule)

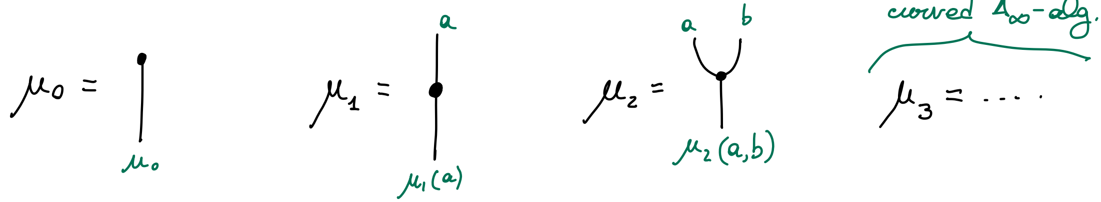
$R_3)$ $\mu_2(\mu_2(a,b), c) + \mu_2(a, \mu_2(b,c)) = 0$ (associativity)

Moreover, $\mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and

- $\mu_0 \in \mathcal{A}_{\lambda^{-2}}$
- $\mu_1 : \mathcal{A}_g \to \mathcal{A}_{\lambda^{-1} g}$
- $\mu_2 : \mathcal{A}_g \otimes \mathcal{A}_{g'} \to \mathcal{A}_{g \cdot g'}$

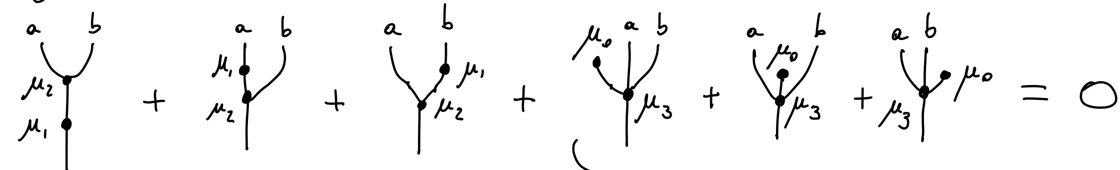
How structure relations are produced:

these exist for curved A_∞-alg.



The relation R_n is obtained by summing over all possible compositions of two such operations that take n inputs in total.

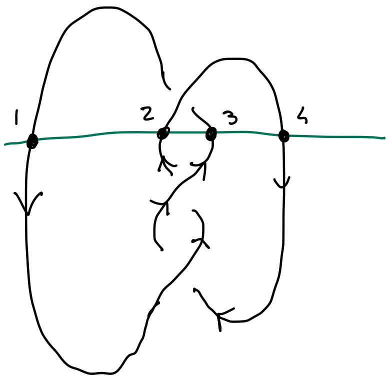
e.g. R_2



all 0 because μ_3 does not exist

$\rightsquigarrow \mu_1(\mu_2(a,b)) + \mu_2(\mu_1(a),b) + \mu_2(a,\mu_1(b)) = 0$

Example: curved DGA associated to a slice of knot projection



Orientation gives a function

$M : \{1,4\} \longrightarrow \{2,3\}$

$1 \longmapsto 2$

$4 \longmapsto 3$

Def: A MATCHING is a bijection $M: A \longrightarrow B$, where $A \cup B = \{1, \dots, 2n\}$, $|A| = |B| = n$.

Def: The curved DGA $\mathcal{B}_*(n, M)$ is defined as:

- the algebra $\mathcal{B}(2n, n)$
- $\mu_0 = \sum_{i \in A} U_i \cdot U_{M(i)}$
- $\mu_1 \equiv 0$
- the gradings are induced by the set $S = \mathrm{Im}(M)$.

RK: μ_0 central, so $\partial^2 = 0$ and $\partial^2 = [\mu_0, -]$ are equivalent.

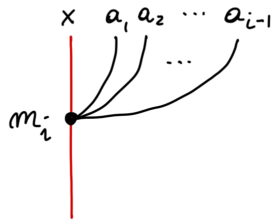
RK: $\lambda = -1$ for Maslov grading, 0 otherwise

③ Curved A_∞-modules & type-D structures

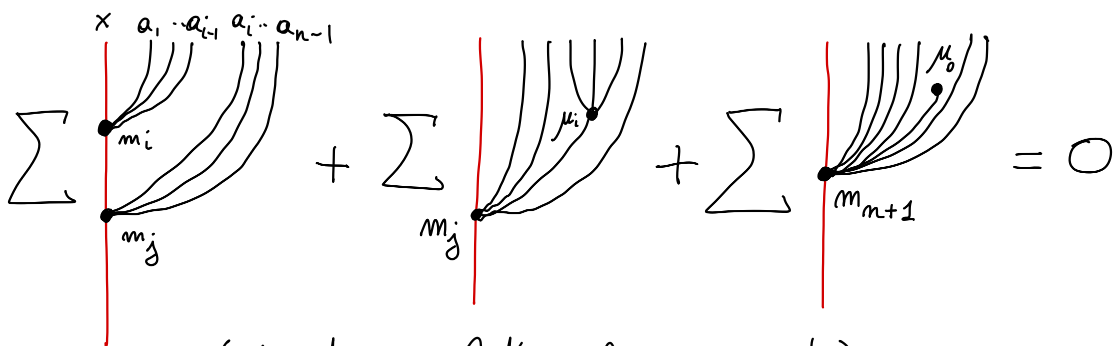
Def: A <u>CURVED A_∞-MODULE</u> M over a curved DGA $\mathcal{A}$ is a (right) R-module M with R-module maps

ring of idempotents

$$m_i : M \otimes_R \underbrace{\mathcal{A} \otimes_R \cdots \otimes_R \mathcal{A}}_{i-1 \text{ times}} \longrightarrow M \qquad \forall i \geq 1$$



subject to <u>structure relations</u>:



(structure relation for n inputs)

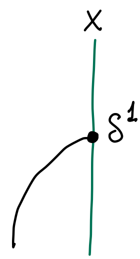
$$\sum_{i+j=n+1} m_i\left(m_j(x \otimes a_1 \otimes \cdots \otimes a_{j-1}) \otimes a_j \otimes \cdots \otimes a_{n-1}\right) +$$

$$\sum_{i+j=n+1} \sum_{\ell=1}^{n-j} m_i\left(x \otimes a_1 \otimes \cdots \otimes a_{\ell-1} \otimes \mu_j(a_\ell \otimes \cdots \otimes a_{\ell+j-1}) \otimes a_{\ell+j} \otimes \cdots \otimes a_{n-1}\right) +$$

$$\sum_{j=1}^{n} m_{n+1}\left(x \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes \mu_0 \otimes a_j \otimes \cdots \otimes a_{n-1}\right) = 0$$

Def: A **CURVED TYPE-D STRUCTURE** X over $\mathcal{A}$ is a (left) R-module w/ an R-module hom.

$$\delta^1 : X \longrightarrow \mathcal{A} \otimes_R X$$

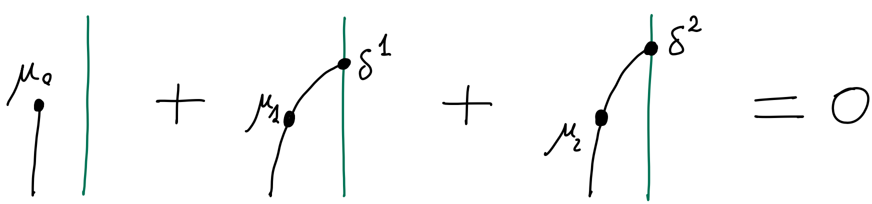


subject to a structure relation.

Def: $$\delta^i : X \longrightarrow \overbrace{\mathcal{A} \otimes_R \mathcal{A} \otimes_R \cdots \otimes_R \mathcal{A}}^{i \text{ times}} \otimes X$$

$$\delta^i := \mathrm{id}_{\underbrace{\mathcal{A} \otimes_R \cdots \otimes_R \mathcal{A}}_{i-1 \text{ times}}} \otimes \delta^1$$

Type-D structure relation:



$$\mu_0 \otimes \mathrm{id}_X + (\mu_1 \otimes \mathrm{id}_X) \circ \delta^1 + (\mu_2 \otimes \mathrm{id}_X) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta^1) \circ \delta^1 = 0$$

④ Box tensor product

Some notation:

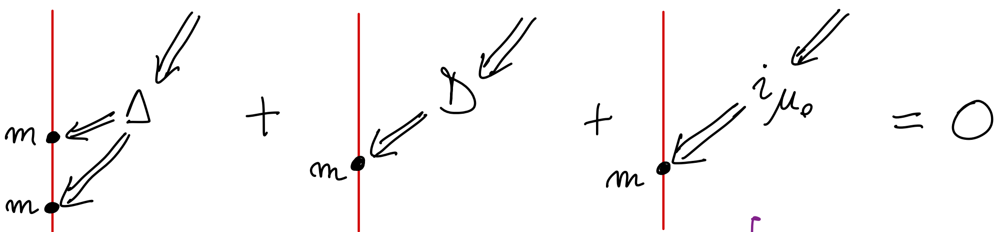
$$\Delta(a_1\otimes\cdots\otimes a_n) = \sum_{i=1}^{n-1} (a_1\otimes\cdots\otimes a_i)\otimes(a_{i+1}\otimes\cdots\otimes a_n)$$

$$D(a_1\otimes\cdots\otimes a_n) = \sum_{i=1}^{n}\sum_{\ell=1}^{n-i+1} a_1\otimes\cdots\otimes a_{\ell-1}\otimes\mu_i(a_\ell\otimes\cdots\otimes a_{\ell+i-1})\otimes a_{\ell+i}\otimes\cdots\otimes a_n$$

$$i_\omega(a_1\otimes\cdots\otimes a_n) = \sum_{i=0}^{n} a_1\otimes\cdots\otimes a_i\otimes\omega\otimes a_{i+1}\otimes\cdots\otimes a_n$$

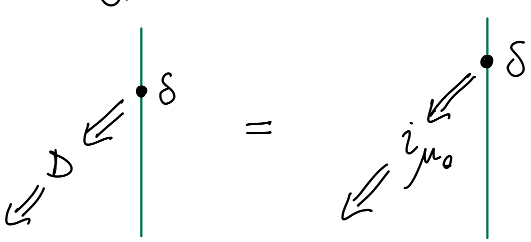
$$m := m_1 + m_2 + m_3 + \ldots, \qquad \delta := 1 + \delta^1 + \delta^2 + \ldots$$

Curved A_∞-module structure relation:



[Simplify by defining $cD := D + i_{\mu_0}$]

Curved type-D structure relation:

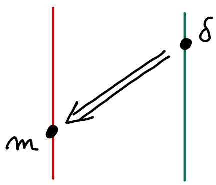


[EX: Prove it.

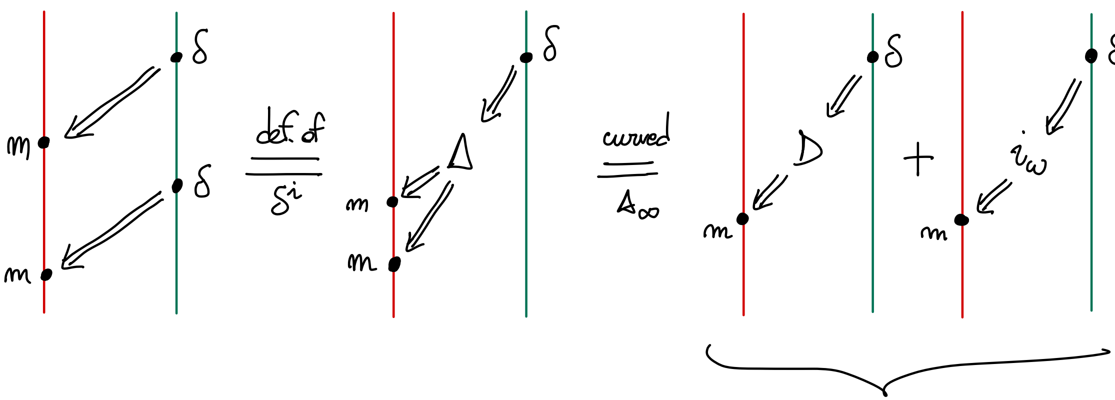
Hint: Fix # of outputs, select 1 output, and apply str. relat.]

Def: Given an A_∞-module M & a type-D structure N such that $m_i \equiv 0$ for $i \gg 0$ or $\delta^i \equiv 0$ for $i \gg 0$, we define the BOX TENSOR PRODUCT $M \boxtimes_{\mathcal{A}} N$ as $M \otimes_R N$, with differential

$$\partial = \sum_{j=0}^{\infty} (m_{j+1} \otimes id_N) \circ (id_M \otimes \delta^j)$$

i.e.: $\partial =$ 

RK: $\partial^2 = 0$.



$= 0$ by definition of type-D structure $\square$

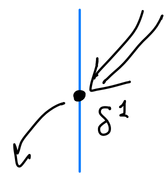
⑤ DA bimodules

Def: A <u>CURVED DA BIMODULE OVER $\mathcal{A}-\mathcal{B}$</u> ${}^{\mathcal{A}}X_{\mathcal{B}}$ is an $R_A - R_B$ bimodule, with maps for $i \geq 1$

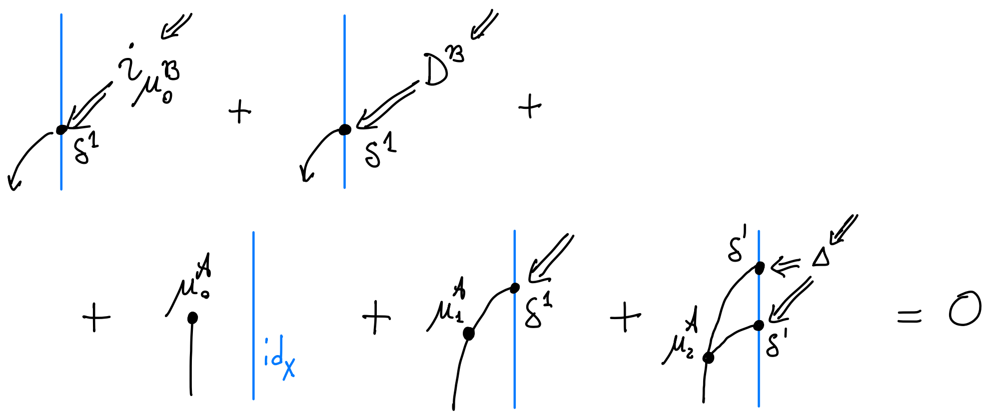
$$\delta^1_i : X \otimes_{R_B} \underbrace{\mathcal{B} \otimes_{R_B} \mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{i-1 \text{ times}} \longrightarrow \mathcal{A} \otimes_{R_A} X$$

subject to a structure relation.

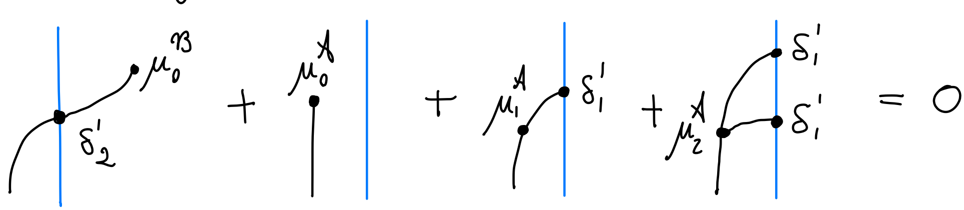
Notation: $\delta^1 = \delta^1_1 + \delta^1_2 + \cdots$



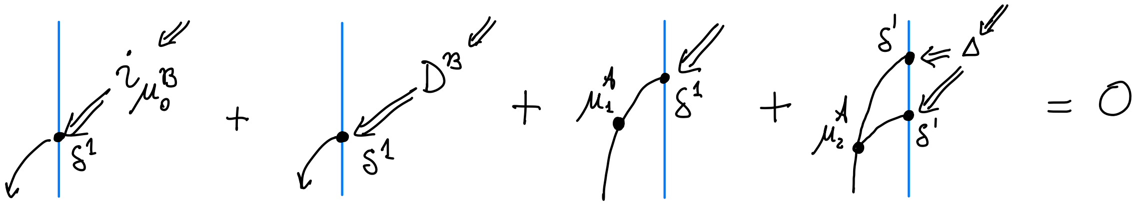
Curved DA relations



More explicitly, the 0 input relation is



while the n input relation, for $n > 0$, is



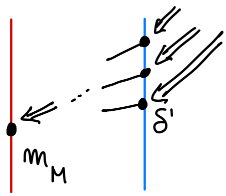
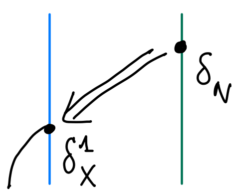
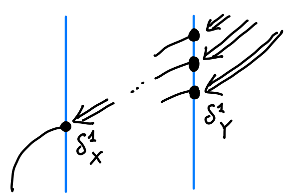
Def: <u>BOX TENSOR PRODUCTS</u> (variations)

Let ${}^{\mathcal{A}}X_{\mathcal{B}}$, ${}^{\mathcal{B}}Y_{\mathcal{C}}$ be DA bimodules,

$M_{\mathcal{A}}$ an A_∞-module

${}^{\mathcal{B}}N$ a type-D structure

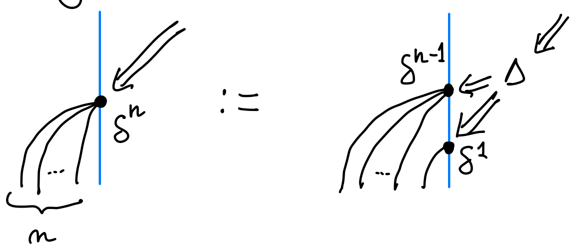
Then we can define 3 variations of the box tensor product:

$M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X_{\mathcal{B}}$	${}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}N$	${}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}Y_{\mathcal{C}}$
A_∞-module / $\mathcal{B}$	type-D / $\mathcal{A}$	DA $\mathcal{A}$-$\mathcal{C}$ bimodule
$M \otimes_{R_{\mathcal{A}}} X$	$X \otimes_{R_{\mathcal{B}}} N$	$X \otimes_{R_{\mathcal{B}}} Y$
$m :=$ 	$\delta^1 :=$ 	$\delta^1 :=$ 

How to show that we get the correct structure?

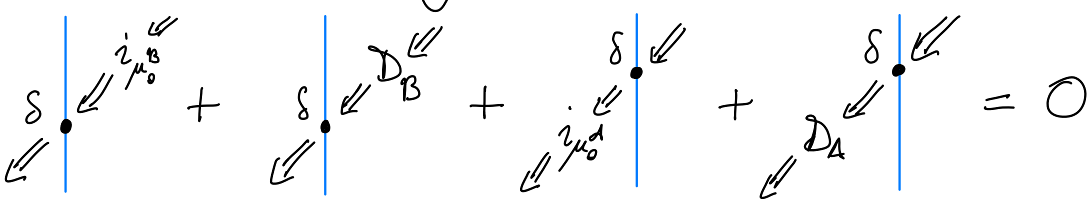
It is convenient to use the several-output curved DA relations.

Define iteratively



Also define $\delta := \mathrm{id}_X + \delta^1 + \delta^2 + \dots$

RK: A DA-bimodule satisfies the relation

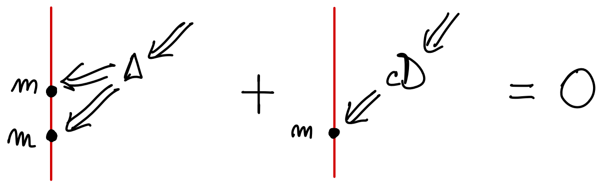


Idea of proof:

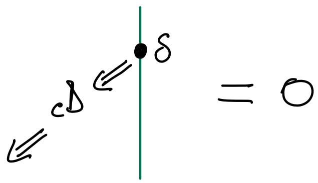
1) Fix the number of outputs, call it n.
2) Express each summand of the above relation as a sum of n elements, where the i-th element is associated to the i-th output.
3) Apply the curved DA relation to the i-th output. $\square$

Short versions of curved structure relations

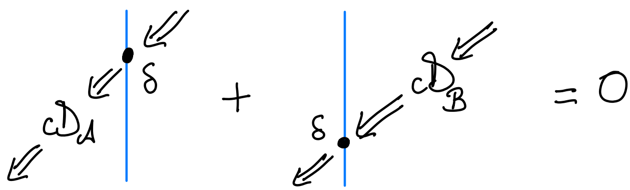
A_∞-module:



Type-D structure:



DA bimodules:



Cor: $M_{\mathcal{A}} \boxtimes {}^{\mathcal{A}}X_{\mathcal{B}}$ is a curved A_∞-module over $\mathcal{B}$

${}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}N$ is a curved type-D structure

${}^{\mathcal{A}}X_{\mathcal{B}} \boxtimes {}^{\mathcal{B}}Y_{\mathcal{C}}$ is a curved DA $(\mathcal{A}$-$\mathcal{C})$-bimodule