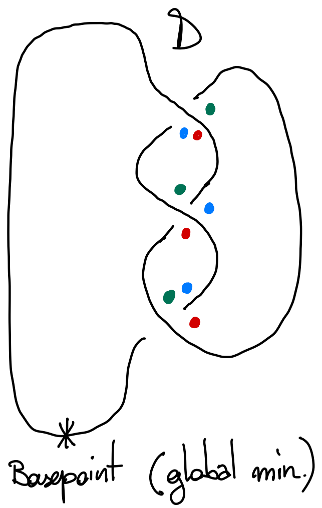
Lecture 4: partial Kauffman states and elementary bimodules

① Upper and lower Kauffman states

Motivation



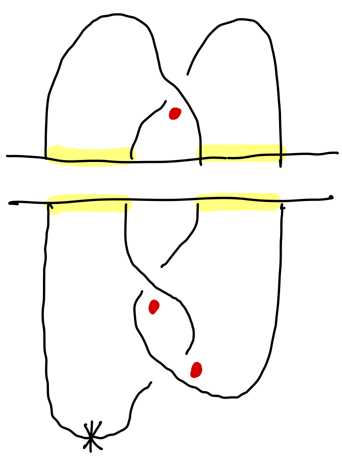
Def: KAUFFMAN STATE is

$: \{\text{crossings}\} \longrightarrow \left\{\begin{matrix}\text{connected cpts} \\ \text{of } \mathbb{R}^2 \setminus D\end{matrix}\right\}$

- injective
- $\forall c$, $K(c)$ is a region adjac. to c
- $\operatorname{im} K$ does not contain the regions adjacent to basepoint

EX: All regions not adjacent to basepoint must be in $\operatorname{im} K$.

Picture above has 3 Kauffman states (•, •, •). Each of them can be recovered by gluing an upper and a lower Kauffman state.



In order for the gluing to be meaningful, we need to specify which regions on the top still need to receive a dot

⇝ idempotents.

Def: An <u>UPPER</u> (resp. <u>LOWER</u>) <u>KNOT DIAGRAM</u> is the intersection of a knot diagram sitting in the xy plane with the half-plane $\{y \geq y_0\}$ (resp. $\{y \leq y_0\}$).

Here we assume that $\{y = y_0\}$ is a generic section, consisting of $2n$ points (n oriented upwards and n oriented downwards).

We label: the $2n$ points $\quad 1, 2, \dots, 2n \quad \longleftrightarrow$ lines

the intervals b/w points $\quad 0, 1, 2, \dots, 2n \quad \longleftrightarrow$ positions

$$\begin{array}{ccccccccccc} 0 & \bullet & 1 & \bullet & 2 & \bullet & \cdots & \bullet & 2n-1 & \bullet & 2n \\ & 1 & & 2 & & 3 & & & & 2n & \end{array}$$

Def: An <u>UPPER KAUFFMAN STATE</u> for an upper knot diagram is a pair (K, I), where:

- $K : \{\text{crossings}\} \longrightarrow \{\text{bounded regions}\}$
- I is an n-idempotent in $\mathcal{B}(2n, n)$

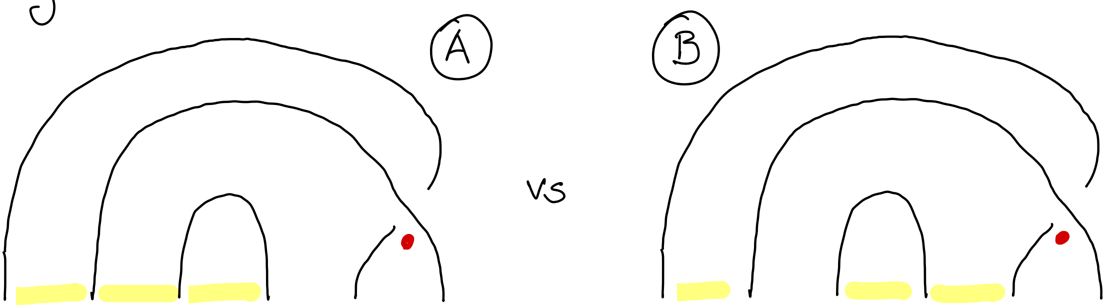
such that:

- $\{0, 2n\} \cap I = \emptyset$;
- K injective;
- $\forall c$, $K(c)$ is a region adjacent to c;

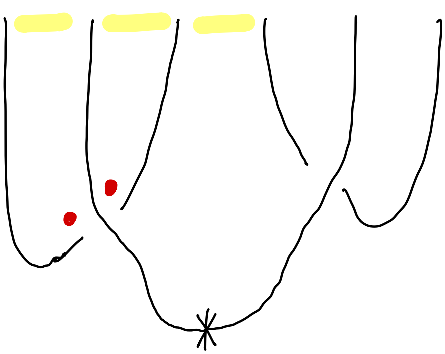
- $\forall$ <u>occupied region</u> R (i.e. $R \in \operatorname{im} K$), no interval of ∂R is in I;
- $\forall$ <u>unoccupied region</u> R (i.e. $R \notin \operatorname{im} K$), exactly 1 interval of ∂R is in I.

<u>RK</u>: For unoccupied regions, the distinguished interval keeps track of where in the lower diagram the marking of the Kauffman state will appear.

e.g.:



[these are different U.K.S.'s because they have different I]



(A) pairs with the picture on the left to give a whole Kauffman state. However, (B) does not pair with the picture on the left (they have non-matching idempotents).

<u>RK</u>: The upcoming definition of lower Kauffman state is not exactly the mirror of upper Kauffman state, because we suppose that the basepoint $*$ is the global minimum.

<u>Def</u>: A <u>LOWER KAUFFMAN STATE</u> for a lower knot diagram is a pair (K, I), where:

- $K : \{\text{crossings}\} \longrightarrow \{\text{bounded regions } \underline{\text{not adj. to } *}\}$
- I is an n-idempotent in $\mathcal{B}(2n, n)$

such that:

- $\{0, 2n\} \cap I = \emptyset$;
- K injective;
- $\forall c$, $K(c)$ is a region adjacent to c;
- $\forall$ <u>occupied region</u> R (i.e. $R \in \operatorname{im} K$), all intervals of ∂R are in I;
- $\forall$ <u>unoccupied region</u> R (i.e. $R \notin \operatorname{im} K$, not adj. to $*$), all intervals of ∂R except one are in I
- For the region R adjacent to $*$, all intervals of ∂R are in I.

Significance of these definitions

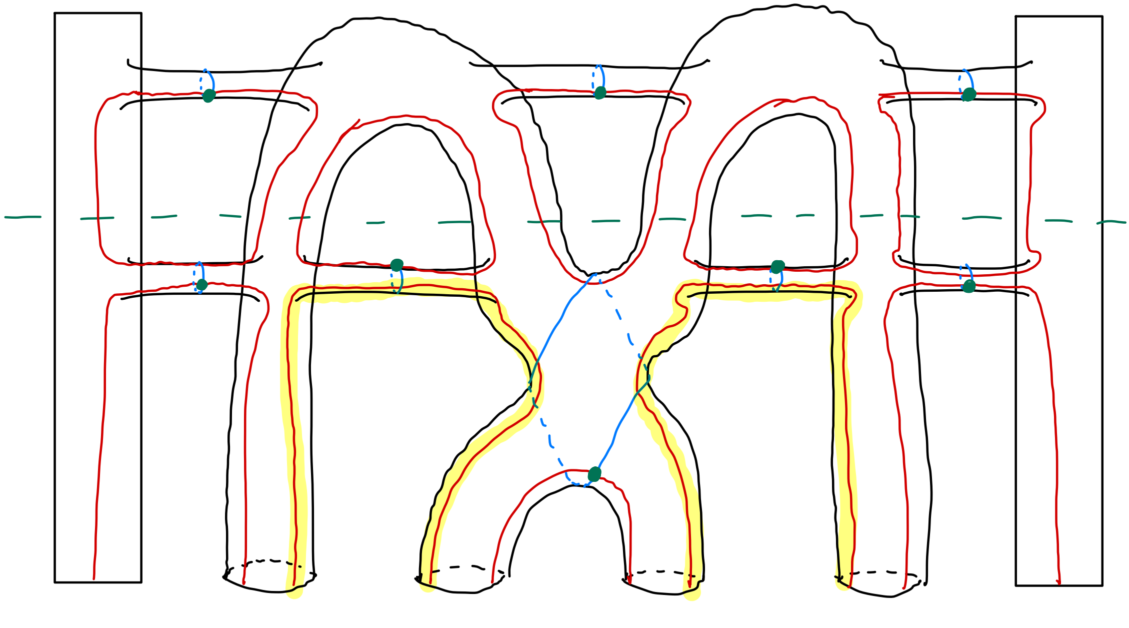
Kauffman states are the generators of CFK.

Upper (resp. lower) Kauffman states generate a type-D structure X (resp. A_∞-module M) associated to the upper (resp. lower) knot diagram.

The box tensor product $M_{\mathcal{B}} \boxtimes^{\mathcal{B}} X$ recovers CFK.

"Upper generators" of knot Floer complex

Recall that we defined a Heegaard diagram associated to a knot projection. Let's focus on the upper portion thereof.



Local Knot Floer generator $\xleftrightarrow{1-1}$ Upper Kauffman state

Intersection pts near crossings $\longleftrightarrow$ Function K

Unoccupied red arcs $\longleftrightarrow$ n-idempotent I

EX: Check that "lower generators" of the Knot Floer complex are in 1-1 correspondence with lower Kauffman states.

EX: An upper generator and a lower generator glue to give a whole generator of CFK iff the idempotents match.

Goal: Define type-D structure spanned by upper K.S.'s, and A_∞-module spanned by lower K.S.'s. Then recover CFK as a box tensor product.

② The strategy

We can try to define the type-D structure / A_∞-module in two ways:

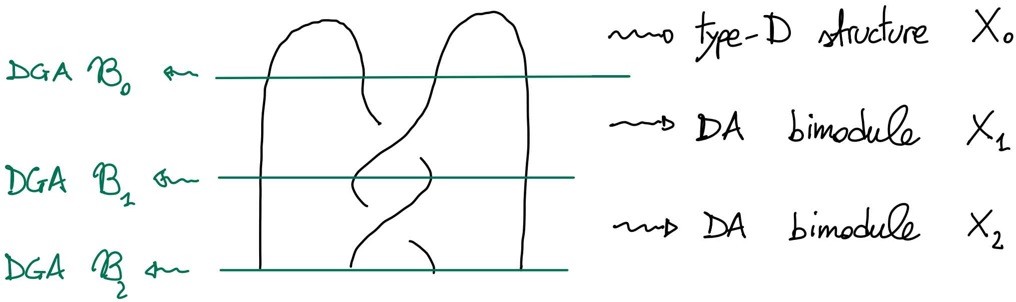
<u>Way #1</u> (closer to the original definition of CFK):

generators: upper/lower Kauffman states

"differential": count holomorphic discs with punctures

<u>Way #2</u>:

•) break the upper/lower diagram further



•) define a type-D structure / A_∞-module / DA bimodule for each elementary configuration, so that:

* generators = partial Kauffman states

* "differential" is defined combinatorially

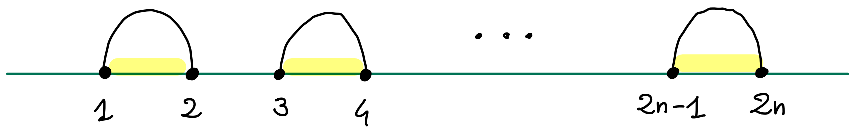
•) recover ${}^{\mathcal{B}_2}X$ by ${}^{\mathcal{B}_2}(X_2)_{\mathcal{B}_1} \boxtimes {}^{\mathcal{B}_1}(X_1)$.

We follow way #2.

<u>RK</u>: Ozsváth-Szabó proved that the two ways are equivalent!

③ Type-D structure for maxima

Up to isotopy, we can assume that all maxima of the knot projection happen at the top of the diagram.



There is a unique upper Kauffman state (ϕ, I_{odd}), where

- $\phi : \{\text{crossings}\} \longrightarrow \{\text{bounded regions}\}$ is the empty function
- $I_{odd} := \{1, 3, \dots, 2n-1\}$ (illustrated above)

matching

We define a curved type-D structure Ω_n over the algebra $\mathcal{B}(n, M)$.

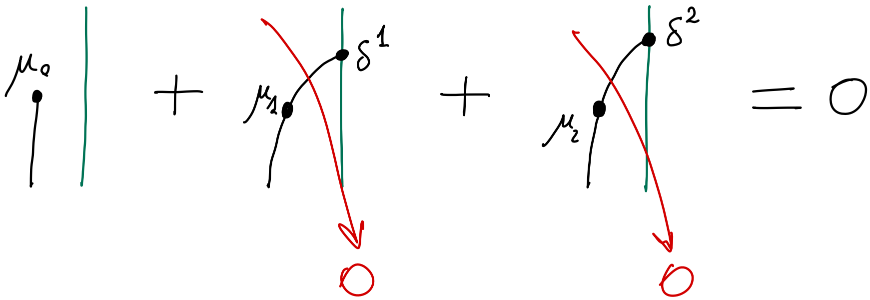
This consists of:
- a module over $\mathcal{I}(n)$, the subring of $\mathcal{B}(n, M)$ generated by n-idempotents; and
- a map $\delta^1 : \Omega_n \longrightarrow \mathcal{B}(n, M) \otimes \Omega_n$.

The module Ω_n is generated over $\mathbb{F}_2$ by a single element Z $(Z = (\phi, I_{odd}))$. For an n-idempotent I, define

$$I \cdot Z = \begin{cases} Z & \text{if } I = I_{odd} \\ 0 & \text{otherwise} \end{cases}$$

Define $\delta^1 \equiv 0$.

Let's check that the curved type-D relation is satisfied:



(because $\delta^1 \equiv 0$, so $\delta^2 \equiv 0$)

The curvature μ_0 is given by the matching M. This in turn is induced by the upper diagram, so

$$\mu_0 = \sum_{i=1}^{n} U_{2i-1} \cdot U_{2i}$$

Consider $\mu_0 \otimes Z$, where the tensor product is over the ring of idempotents $\mathcal{I}(n)$. Then $U_{2i-1} \cdot U_{2i} \cdot I_{odd} = 0$, because, using the relations of the algebra $\mathcal{B}(n, M)$, we can factor

$$U_{2i-1} \cdot U_{2i} \cdot I_{odd} = L_{2i-1} \cdot \underbrace{R_{2i-1} \cdot R_{2i}}_{=0} \cdot L_{2i} \cdot I_{odd}$$

Thus, $\mu_0 \otimes Z = 0 \Rightarrow$ curved type-D relations satisfied.

4 Partial Kauffman states

Def: A PARTIAL KNOT DIAGRAM is the intersection of a knot diagram with the slice $\{y_2 \leq y \leq y_1\}$.

We assume that the sections $\{y = y_1\}$ and $\{y = y_2\}$ are generic, consisting of $2n_1$ and $2n_2$ points respectively.

We denote $\mathcal{B}_1 = \mathcal{B}(n_1, M_1)$ and $\mathcal{B}_2 = \mathcal{B}(n_2, M_2)$.

We denote n_1-idempotents by I_1 and n_2-idempotents by I_2.

A bounded region R has boundary $\partial R = \underset{\{y=y_1\}}{\underset{\cap\!\shortmid}{\partial_1 R}} \amalg \underset{\{y=y_2\}}{\underset{\cap\!\shortmid}{\partial_2 R}}$

Def: A PARTIAL KAUFFMAN STATE for a partial knot diagram is a triple (K, I_1, I_2), where

- $K : \{\text{crossings}\} \longrightarrow \{\text{bounded regions}\}$
- I_1 is an n_1-idempotent;
- I_2 is an n_2-idempotent;

such that:

- $\{0, 2n_1\} \cap I_1 = \phi$ and $\{0, 2n_2\} \cap I_2 = \phi$;
- K injective;

- $\forall c$, $K(c)$ is a region adjacent to c;
- $\forall$ <u>occupied region</u> R (i.e. $R \in \operatorname{im} K$),
  all intervals of $\partial_1 R$ are in I_1 and no interval of $\partial_2 R$ is in I_2;
- $\forall$ <u>unoccupied region</u> R (i.e. $R \notin \operatorname{im} K$), either
  * all intervals of $\partial_1 R$ except one are in I_1 and no interval of $\partial_2 R$ is in I_2; or
  * all intervals of $\partial_1 R$ are in I_1 and exactly one interval of $\partial_2 R$ is in I_2.

<u>Facts:</u>

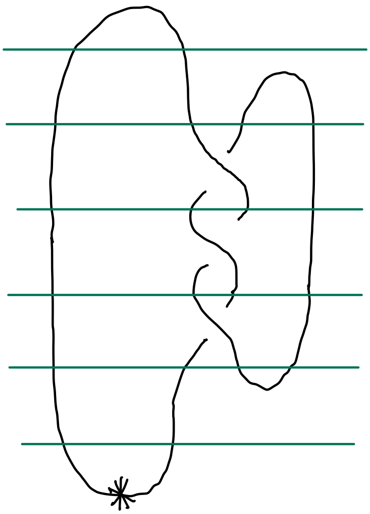
- "partial generators" of CFK $\xleftrightarrow{1\text{-}1}$ partial Kauffman states
  occupied α arcs at top $\longleftrightarrow$ incoming idempt. I_1
  unoccupied α arcs at bottom $\longleftrightarrow$ outgoing idempt. I_2
- Given (K, I_1, I_2) for a partial diagram $\{y_2 \leq y \leq y_1\}$ and (K', I_2', I_3') for a partial diagram $\{y_3 \leq y \leq y_2\}$, they glue to a partial Kauffman state $(K \cup K', I_1, I_3')$ iff $I_2 = I_2'$.
- Variations: U.K.S. + P.K.S. $\rightsquigarrow$ U.K.S.

  P.K.S. + L.K.S. $\rightsquigarrow$ L.K.S.

5 Elementary configurations

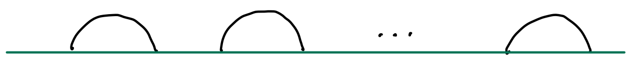
GOAL: define DA bimodules for elementary configurations.

What are the elementary configurations?



Step 1: break into pieces each one containing 1 local max, min, or crossing.

Step 2: move all maxima to the top (you may create more crossings while doing so):

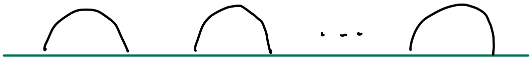


Step 3: make sure that the basepoint is in correspondence of the global minimum.

Step 4: move local minima to the far left (you may introduce new crossings while doing so.

After doing all these moves, we can build any knot type by assembling the following elementary pieces:

- Maxima

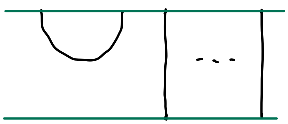


$\rightsquigarrow$ type-D structure (already discussed)

- <u>Positive and negative crossings</u>

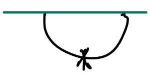
$\rightsquigarrow$ DA bimodules $\mathcal{P}$ and $\mathcal{N}$

- <u>Local minimum</u>



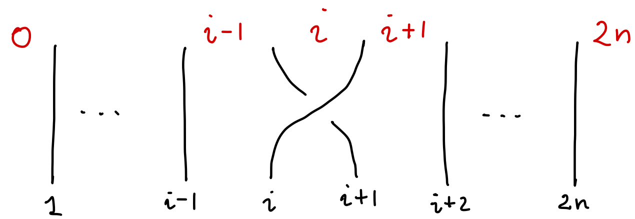
$\rightsquigarrow$ DA bimodule $\mho$

- <u>Global minimum</u> $\rightsquigarrow$ A_∞-module $\widehat{t\mho}$



⑥ Crossing bimodules

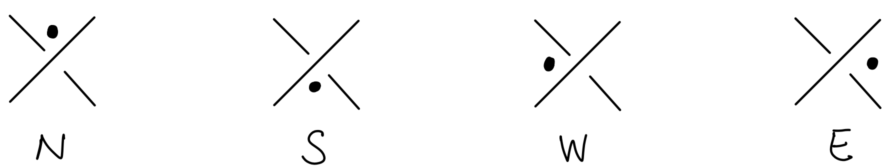
Positive crossing



We denote the intervals adjacent to the crossing by A, B, C.

$A \leftrightarrow i-1 \quad , \quad B \leftrightarrow i \quad , \quad C \leftrightarrow i+1$

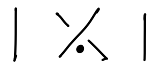
The positive crossing bimodule $\mathcal{P}$ is generated over $\mathbb{F}_2$ by partial Kauffman states. There are 4 macrofamilies:



Manion's notation: break them further into families depending on their incoming idempotent (top right) and outgoing one (bottom left).

The number of intervals among A, B, and C in I_{in} is the same as in I_{out}.

$|\{A, B, C\} \cap I| = 0$

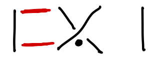
${}_{\phi}S^{\phi}$ 

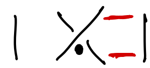
$|\{A, B, C\} \cap I| = 1$

${}_{B}N^{B}$ 

${}_{B}W^{A}$ 

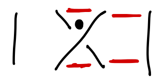
${}_{B}E^{C}$ 

${}_{A}S^{A}$ 

${}_{C}S^{C}$ 

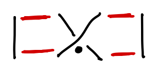
$\underbrace{|\{A, B, C\} \cap I|}_{\substack{\| \\ \text{"local weight" of } I \\ lw = 2}} = 2$

${}_{AB}N^{AB}$ 

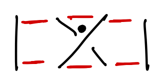
${}_{BC}N^{BC}$ 

${}_{BC}W^{AC}$ 

${}_{AB}E^{AC}$ 

${}_{AC}S^{AC}$ 

$|\{A, B, C\} \cap I\} = 3$

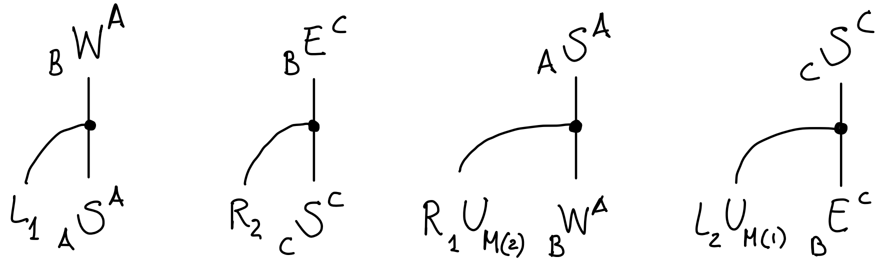
${}_{ABC}N^{ABC}$ 

We now define the various maps δ_n^1.

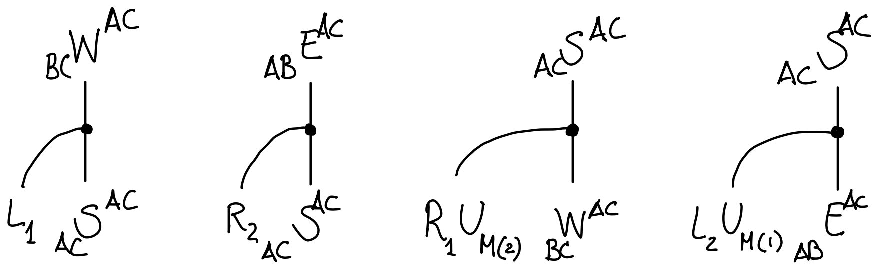
For simplicity, we assume $i = 1$ and $i+1 = 2$ when we write U_1, U_2, $L_1, \dots$

$\boxed{\delta^1_1}$

-) $\ell w = 0$ nothing here
-) $\ell w = 1$



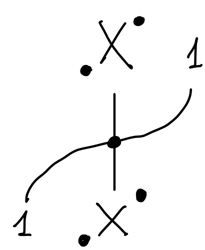
-) $\ell w = 2$: stabilised versions of the above



-) $\ell w = 3$: nothing here

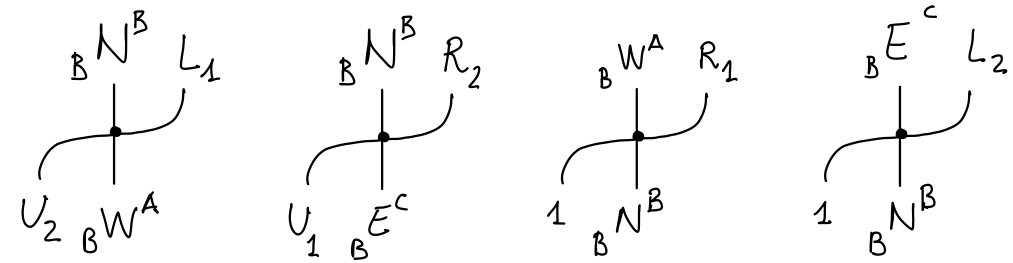
$\boxed{\delta^1_2}$

-) First, for every generator $_\bullet X^\bullet$, no matter what its local weight is, we have a δ^1_2 contribution as on the right.



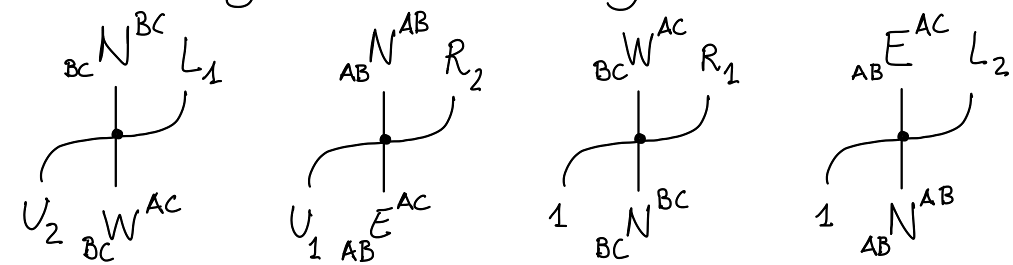
Moreover :

- $\ell_W = 1$

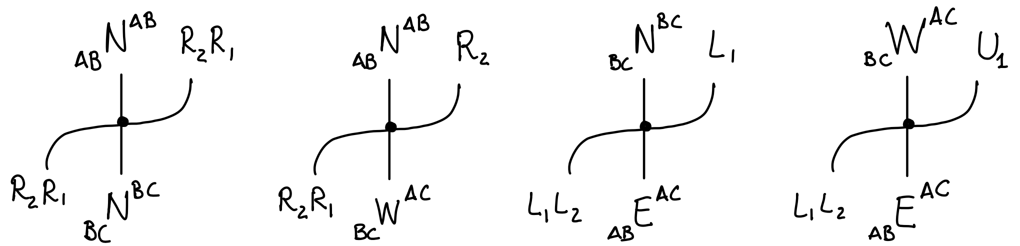


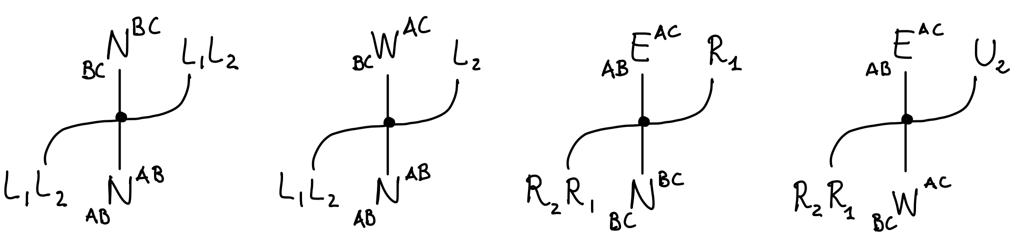
- $\ell_W = 2$

First, we get stabilised versions of the above

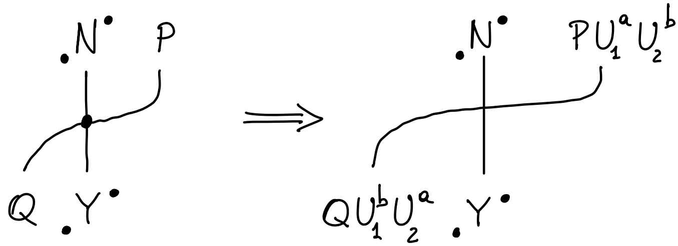


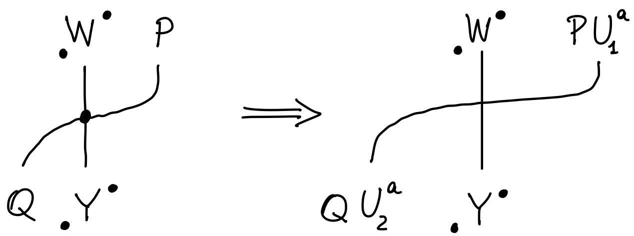
Then, we have additional terms:

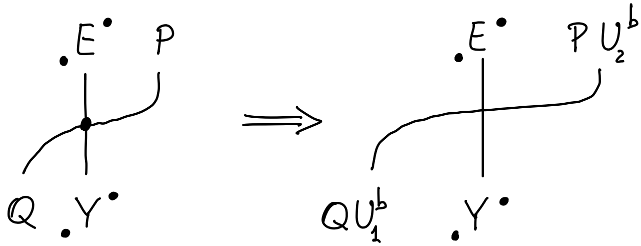


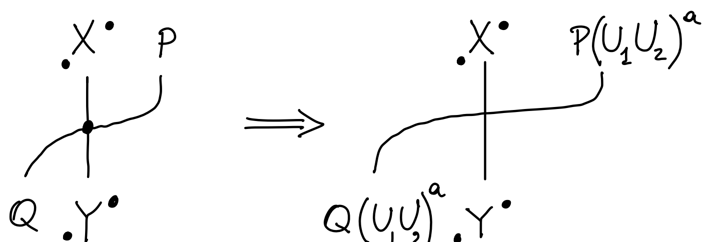


•) Finally, S_2^1 is "equivariant" as follows:

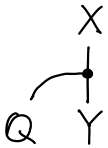


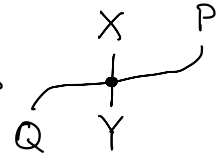


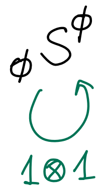


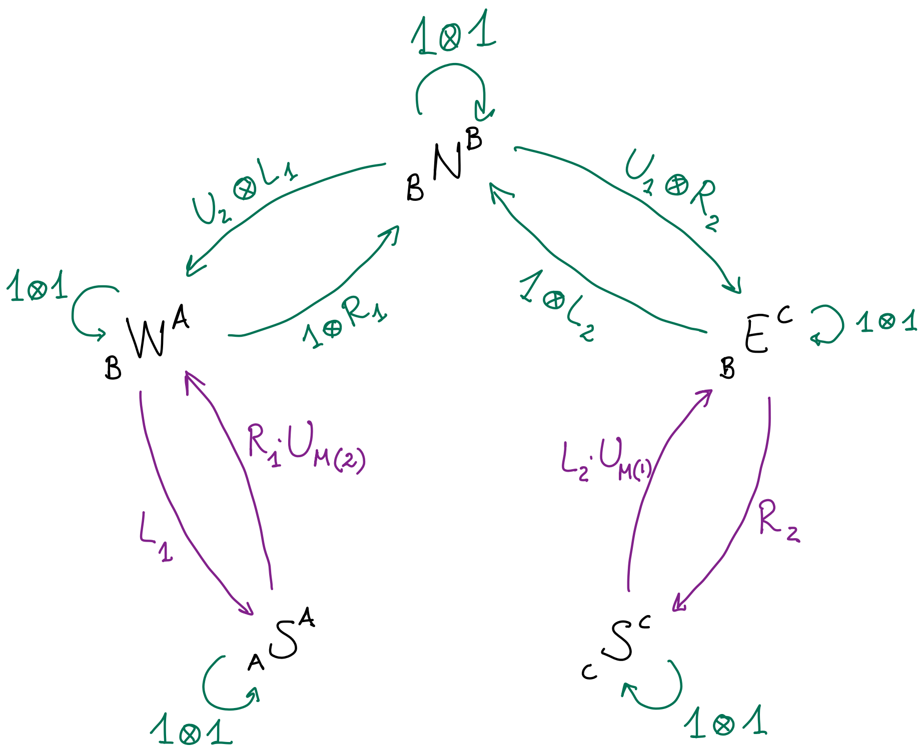


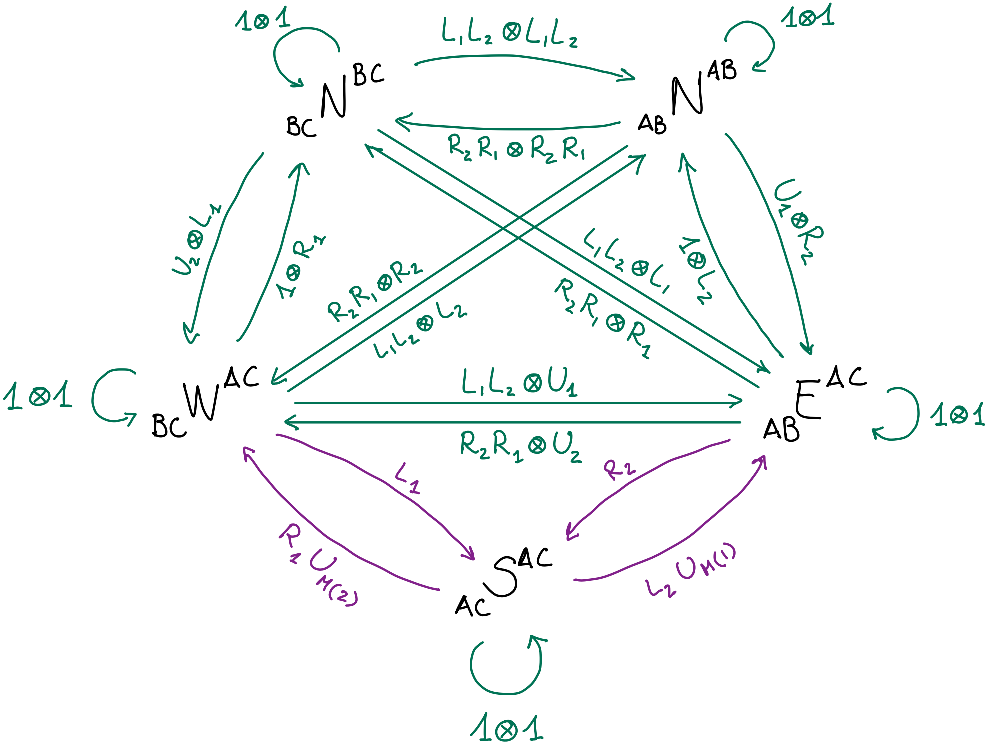
The information of S_1^1 and S_2^1 can be summarised in diagrams:

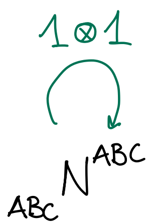
$X \xrightarrow{Q} Y$ is 

, $X \xrightarrow{Q \otimes P} Y$ is 





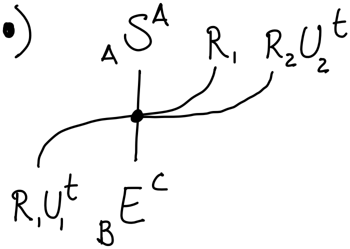


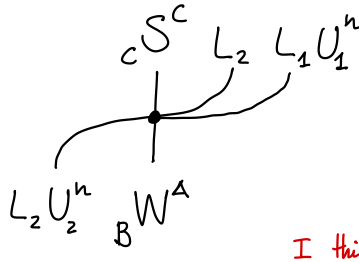


$\boxed{\delta_3^1}$ δ_3^1 is $(U_1 U_2)$-equivariant, meaning that

$$\delta_3^1(X,\, U_1U_2 a,\, b) = \delta_3^1(X,\, a,\, U_1U_2 b) = U_1U_2 \cdot \delta_3^1(X, a, b)$$

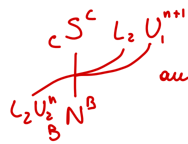
If U_1U_2 does not divide either a or b, we have the following δ_3' contributions:

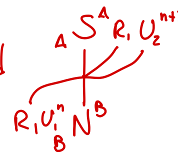
•) 

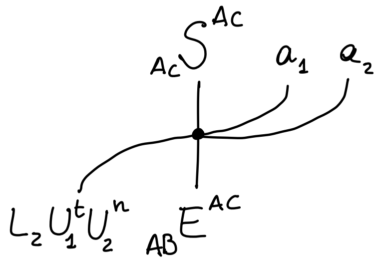
and 

for $t,n \geq 0$

I think I forgot two δ_3':



and 

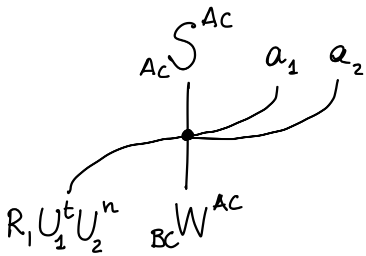
•) 

if $(a_1,a_2) \in$

$\{(U_1^{n+1},U_2^t),(R_1U_1^n,L_1U_2^t),(L_2U_1^{n+1},R_2U_2^{t-1})\}$ if $0 \leq n < t$

$\{(U_2^t,U_1^{n+1}),(R_1U_2^t,L_1U_1^n),(L_2U_2^{t-1},R_2U_1^{n+1})\}$ if $1 \leq t \leq n$

•)

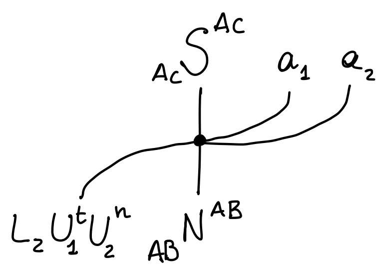


if $(a_1, a_2) \in$

$\{(U_2^{t+1}, U_1^n), (L_2U_2^t, R_2U_1^n), (R_1U_2^{t+1}, L_1U_1^{n-1})\}$ if $0 \leq t < n$

$\{(U_1^n, U_2^{t+1}), (L_2U_1^n, R_2U_2^t), (R_1U_1^{n-1}, L_1U_2^{t+1})\}$ if $1 \leq n \leq t$

•)



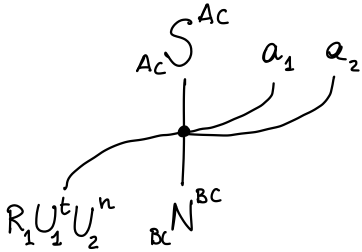
if $(a_1, a_2) \in$

$\{(U_1^{n+1}, L_2U_2^t), (R_1U_1^n, L_1L_2U_2^t), (L_2U_1^{n+1}, U_2^t)\}$ if $0 \leq n < t$

$\{(L_2U_2^t, U_1^{n+1}), (U_2^t, L_2U_1^{n+1}), (R_1U_2^t, L_1L_2U_1^n)\}$ if $1 \leq t \leq n$

$\{(L_2, U_1^{n+1})\}$ if $0 = t \leq n$

•)



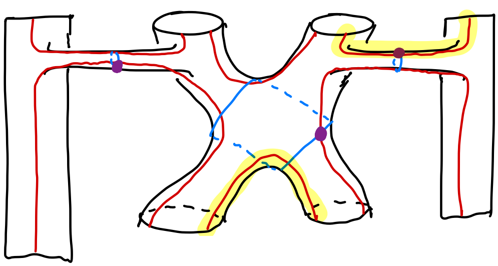
if $(a_1, a_2) \in$

$$\{(U_2^{t+1}, R_1U_1^n), (L_2U_2^t, R_2R_1U_1^n), (R_1U_2^{t+1}, U_1^n)\} \quad \text{if} \quad 0 \leq t < n$$

$$\{(R_1U_1^n, U_2^{t+1}), (U_1^n, R_1U_2^{t+1}), (L_2U_1^n, R_2R_1U_2^t)\} \quad \text{if} \quad 1 \leq n \leq t$$

$$\{(R_1, U_2^{t+1})\} \quad \text{if} \quad 0 = n \leq t$$

Motivation

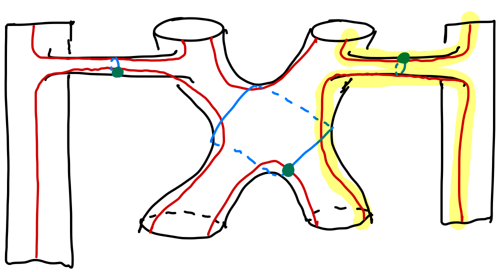


Partial Kauffman state ${}_B E^C$

→ E at the crossing

→ incoming idempotent C

→ outgoing idempotent B



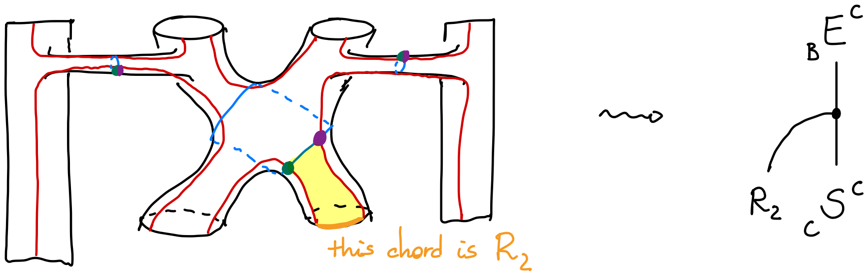
Partial Kauffman state ${}_C S^C$

→ S at the crossing

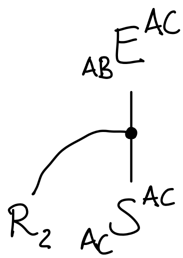
→ incoming idempotent C

→ outgoing idempotent C

There is a holomorphic disc

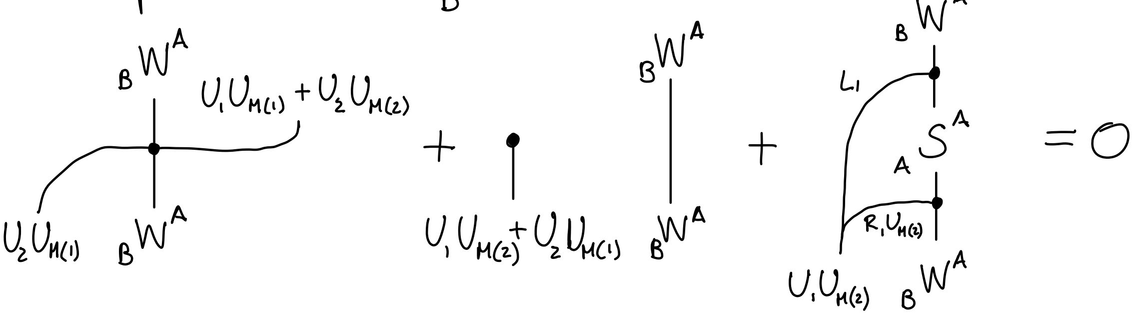


<u>RK</u>: the same hol. disc, with different choice of intersection point on the leftmost β curve, also gives the "stabilised" δ_1^1 contribution:

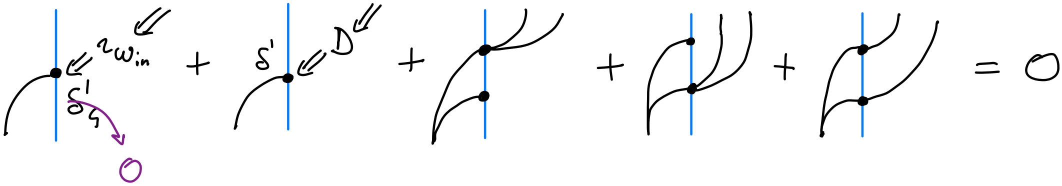


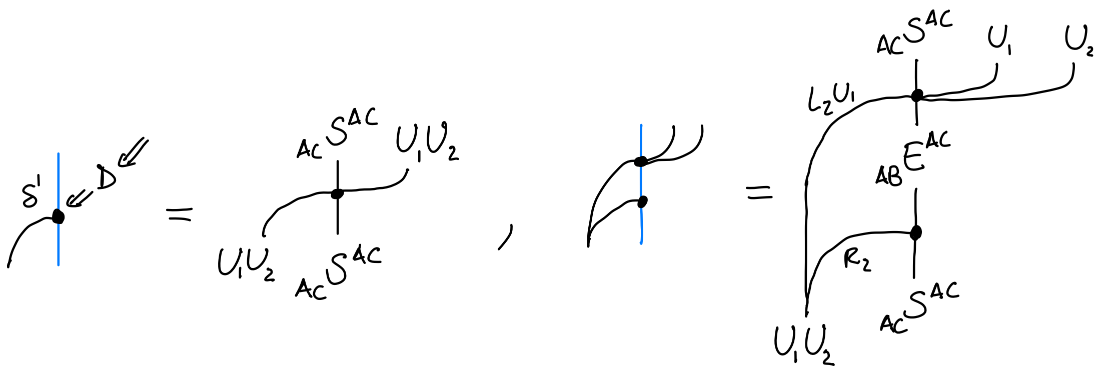
<u>Let's check some DA relations</u>

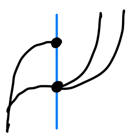
0-input relation for ${}_BW^A$:



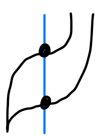
2-input relation on $({}_{AC}S^{AC}, U_1, U_2)$:







$= 0$ [because $\delta^1_3 \neq 0$ only on ${}_{AC}S^{AC}$, and there are no δ^1_1 maps ${}_{AC}S^{AC} \to {}_{AC}S^{AC}$]



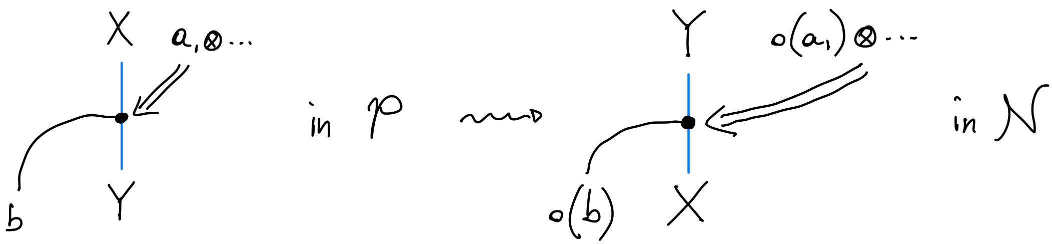
$= 0$ [the only non-trivial δ^1_1 on ${}_{AC}S^{AC}$ is $1 \otimes 1$, which is (U_1U_2)-equiv., but not U_1-equiv. or U_2-equiv.]

Thus, the sum of all terms is 0.

Negative crossing

The negative crossing $\mathcal{N}$ is spanned by the same generators as $\mathcal{P}$ over $\mathbb{F}$, and with same idempotents.

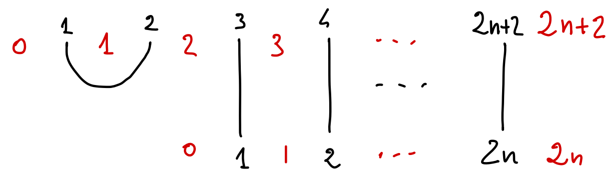
For the structure map:



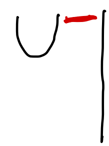
where σ :
$$\begin{aligned} R_i &\longmapsto L_i \\ L_i &\longmapsto R_i \\ U_i &\longmapsto U_i \end{aligned}$$

⑦ Local minimum bimodule

We consider the case when the local minimum is on the far left:



For all partial Kauffman states, $I^{in} \cap \{0,1,2\} = \{2\}$
$I^{out} \cap \{0\} = \phi$



(This is because the outer region must be unmarked)

$\mathcal{B}_1 = \mathcal{B}_*(n+1, M)$ $\qquad\qquad$ $\mathcal{B}_2 = \mathcal{B}_*(n, \widehat{M})$

If $M(s) = 1$ and $M(2) = t$, define

$$\widehat{M}(i) := \begin{cases} t-2 & \text{if } i = s-2 \\ M(i+2)-2 & \text{if } i \neq s-2 \end{cases}$$

$\mho$ is a module generated over $\mathbb{F}_2$ by partial Kauffman states, which are pairs of idempotents $(\underline{x}, \psi(\underline{x}))$ such that

- $\underline{x} \cap \{0,1,2\} = \{2\}$, $\underline{x} \not\ni 2n+2$, and $|\underline{x}| = n+1$;
- if $\underline{x} = (2, x_2, x_3, \dots, x_{n+1})$, $\psi(\underline{x}) = (x_2-2, x_3-2, \dots, x_{n+1}-2)$.

To define the structure map, we need an extra definition.

Def: An <u>ADMISSIBLE SEQUENCE</u> is a sequence $a_1, \dots, a_{2K-1}$ of algebra elements in $\mathcal{B}_*(n+1, M)$ of the form:

$$
\begin{aligned}
a_1 &= L_2 \mu_1 \\
a_2 &= U_1 \mu_2 \\
a_3 &= U_2 \mu_3 \\
a_4 &= U_1 \mu_4 \\
&\vdots \\
a_{2K-2} &= U_1 \mu_{2K-2} \\
a_{2K-1} &= R_2 \mu_{2K-1}
\end{aligned}
$$

where each μ_{2i} is a monomial in $U_1, U_3, U_4, \dots, U_{2n+2}$ and each μ_{2i+1} is a monomial in $U_2, U_3, U_4, \dots, U_{2n+2}$.

Def: Given an admissible sequence $a_1, \dots, a_{2K-1}$, define

$$
b := \prod_{i=1}^{2K-1} \mu_i \Bigg|_{\substack{U_1 \mapsto U_{t-2} \\ U_2 \mapsto U_{s-2} \\ U_3 \mapsto U_1 \\ U_4 \mapsto U_2 \\ \vdots \\ U_{2n+2} \mapsto U_{2n}}} \in \mathcal{B}_*(n, \widehat{M})
$$

For each partial Kauff. state $I_{\underline{x}} = (\underline{x}, \psi(\underline{x}))$ and for each admissible sequence $a_1, \ldots, a_{2K-1}$, there is a contribution to δ^1_{2K}:

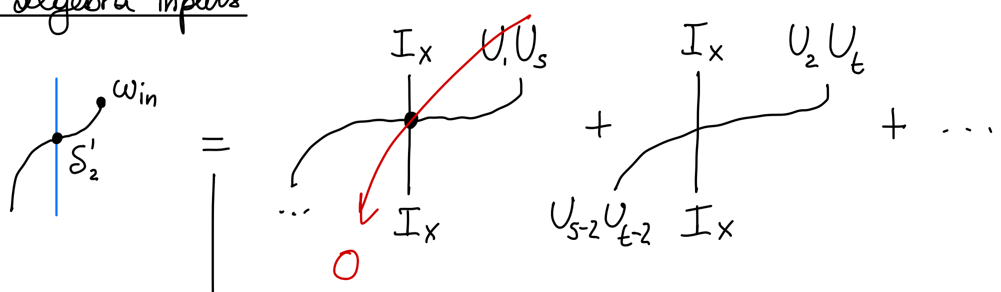
$$\delta^1_{2K}(I_x, a_1, \ldots, a_{2K-1}) = b \otimes I_x$$

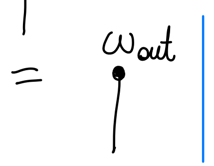
Moreover, for every I_x, $m \geq 0$, and μ monomial in $U_3, \ldots, U_{2n+2}$ there is a contribution to δ^1_2:

$$\delta^1_2(I_x, U_2^m \mu) = \left(U_s^m \mu\right)\Big|_{\substack{U_3 \mapsto U_1 \\ \vdots}} \otimes I_x$$

<u>Let's check some curved DA relations</u>

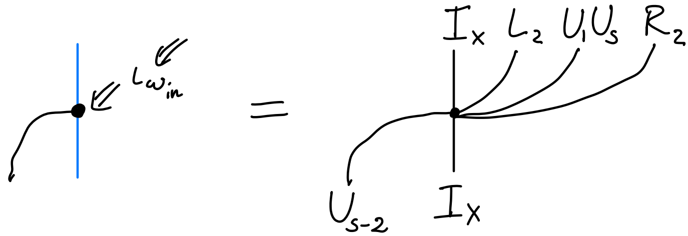
<u>0 algebra inputs</u>



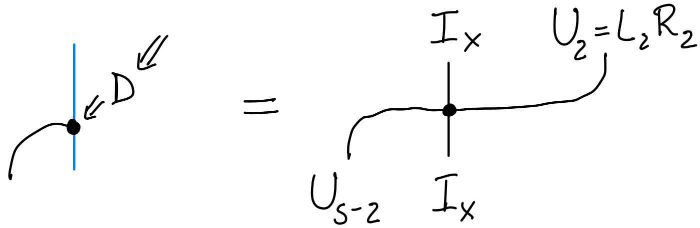


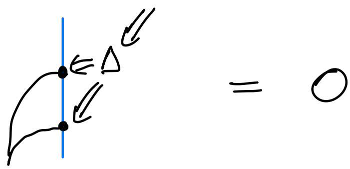
$\Rightarrow$ curved DA relation satisfied

2 algebra input relation on (I_x, L_2, R_2)



(only non-trivial summand)



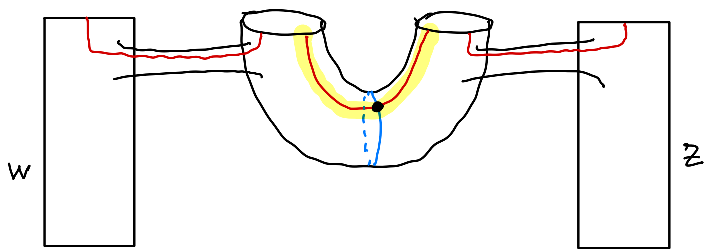


Again, the curved DA relation is satisfied.

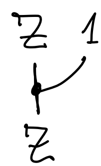
⑧ Terminal A_∞-module for the global minimum

Depending on the version of the terminal A_∞-module we choose, we get different versions of CFK.

The simplest case is $\widehat{CFK}$, where we disallow discs from crossing any of the two basepoints w & z.



There is a single lower Kauffman state Z, with incoming idempotent $I_Z = \{1\} \subseteq \{0,1,2\}$, generating $\widehat{t\mho}$ over $\mathbb{F}_2$.

The only non-trivial relation is .

More explicitly: *) $m_i \equiv 0$ for $i=1$ and $i \geq 3$

*) $m_2(Z, \gamma) = 0$ for every non-constant path in $\mathrm{Path}_{\mathbb{F}}(Q(2,1))$

*) $m_2(Z, 1) = Z$.