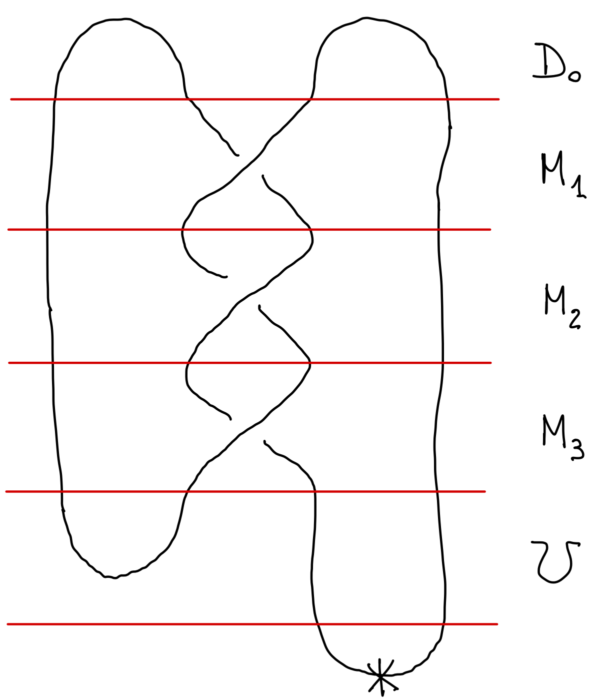
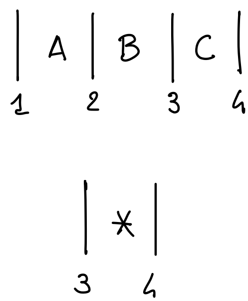
Lecture 5: computing HFK of the right trefoil



Labelling conventions



D_0 is the curved type-D structure defined as follows:

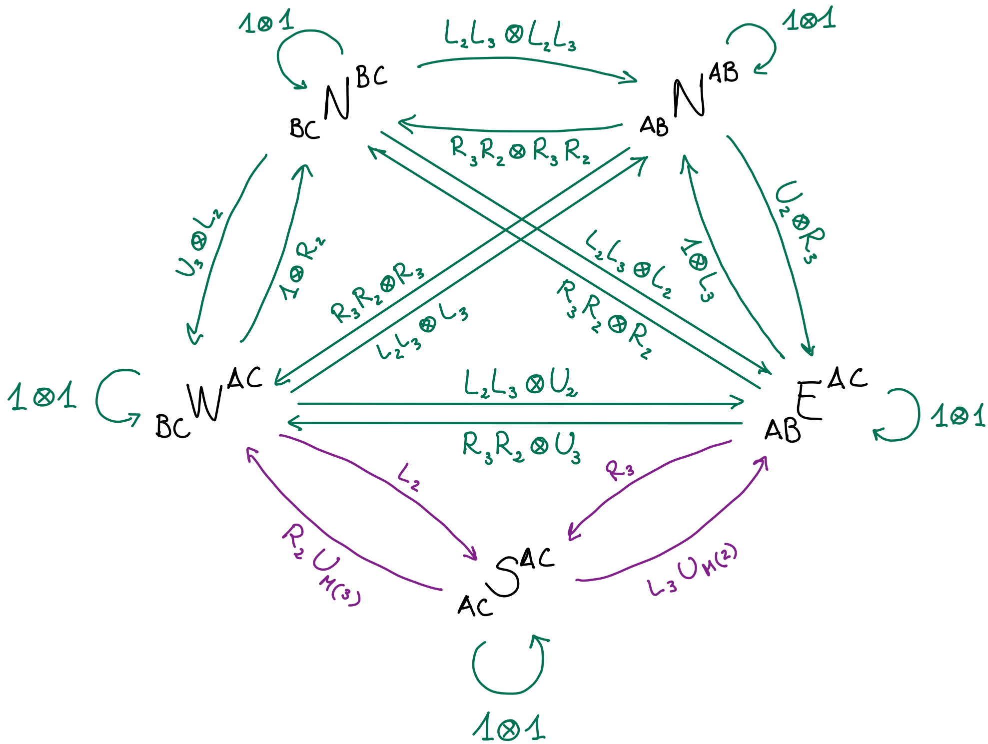
*) $D_0 = \mathbb{F}_2\langle {}_{AC}I \rangle$

*) $\mathcal{I}(2)$ acts as $AC \cdot {}_{AC}I = {}_{AC}I$, others $\cdot {}_{AC}I = 0$.

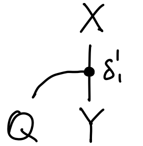
*) $\delta^1 \equiv 0$.

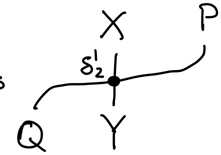
Curvature: $\mu_0^{D_0} = U_1 U_2 + U_3 U_4$.

M_1 is a DA bimodule. We only need the submodule w/ local weight 2, which is spanned by ${}_{BC}N^{BC}$, ${}_{AB}N^{AB}$, ${}_{BC}W^{AC}$, ${}_{AC}S^{AC}$, ${}_{AB}E^{AC}$.
The actions δ^1_1 and δ^1_2 are shown in the pentagon below.
δ^1_3 can be checked from the notes of Lecture 4.
$\delta^1_n = 0$ if $n > 3$.



How to read the pentagon:

$X \xrightarrow{Q} Y$ is ,

$X \xrightarrow{Q \otimes P} Y$ is 

① Computing $D_1 = M_1 \boxtimes D_0$

⊛ Module structure : $M_1 \underset{I(2)}{\otimes} D_0$

The only pairs of generators w/ matching idempotents are $({}_{BC}W^{AC}, {}_{AC}I)$, $({}_{AC}S^{AC}, {}_{AC}I)$, $({}_{AB}E^{AC}, {}_{AC}I)$.

simply denote by ${}_{BC}W$

$\rightsquigarrow M_1 = \mathbb{F}\langle {}_{BC}W \,,\, {}_{AC}S \,,\, {}_{AB}E \rangle$

⊛ $$\delta^1_{D_1} = \delta^1_1 \circ id_{D_0} + \delta^1_2 \circ \underbrace{\delta^1_{D_0}}_{0} + \delta^1_3 \circ \underbrace{\delta^2_{D_0}}_{0} + \underbrace{\ldots}_{0}$$

Thus, $\delta^1_{D_1}$ is given simply by the δ^1_1 action on M_1:

$$\begin{tikzcd}
{}_{BC}W \arrow[dr, bend left, "L_2"] & & {}_{AB}E^{AC} \arrow[dl, bend right, "R_3"'] \\
& {}_{AC}S^{AC} \arrow[ul, bend left, "R_2U_4"] \arrow[ur, bend right, "L_3U_1"'] &
\end{tikzcd}$$

<u>RK</u>: Note that $M(3) = 4$ and $M(2) = 1$ here.

<u>RK</u>: D_1 satisfies the curved DA relations w/

$$\mu_0^{D_1} = U_1U_3 + U_2U_4 \quad \text{(new curvature)}$$

② Computing $D_2 = M_2 \boxtimes D_1$

⊛ Module structure

In $M_2 \otimes_{\mathcal{I}(2)} D_1$ the only generators with matching idempotents are ${}_{BC}N^{BC}{}_{BC}W$, ${}_{BC}W^{AC}{}_{AC}S$, ${}_{AC}S^{AC}{}_{AC}S$, ${}_{AB}E^{AC}{}_{AC}S$, ${}_{AB}N^{AB}{}_{AB}E$.

$$\Rightarrow D_2 = \mathbb{F}_2\left\langle {}_{BC}N^{BC}{}_{BC}W,\ {}_{BC}W^{AC}{}_{AC}S,\ {}_{AC}S^{AC}{}_{AC}S,\ {}_{AB}E^{AC}{}_{AC}S,\ {}_{AB}N^{AB}{}_{AB}E \right\rangle$$

⊛ $$\delta^1_{D_2} = \boxed{\delta^1_1 \circ id_{D_1}} + \boxed{\delta^1_2 \circ \delta^1_{D_1}} + \boxed{\delta^1_3 \circ \delta^2_{D_2}} + \underbrace{\ldots}_{0}$$

$$\begin{tikzcd}
{}_{BC}N^{BC}{}_{BC}W \arrow[d, shift right, "U_3"'] \arrow[drr, "L_2L_3"] & & {}_{AB}N^{AB}{}_{AB}E \arrow[d, shift left, "U_2"] \arrow[dll, "R_3R_2"'] \\
{}_{BC}W^{AC}{}_{AC}S \arrow[u, shift right, "U_4"'] \arrow[urr, "L_2L_3U_1"'] \arrow[dr, bend left, "L_2"] & & {}_{AB}E^{AC}{}_{AC}S \arrow[u, shift left, "U_1"] \arrow[ull, "R_3R_2U_4"] \arrow[dl, bend right, "R_3"'] \\
& {}_{AC}S^{AC}{}_{AC}S \arrow[ul, bend left, "R_2U_1"] \arrow[ur, bend right, "L_3U_4"'] &
\end{tikzcd}$$

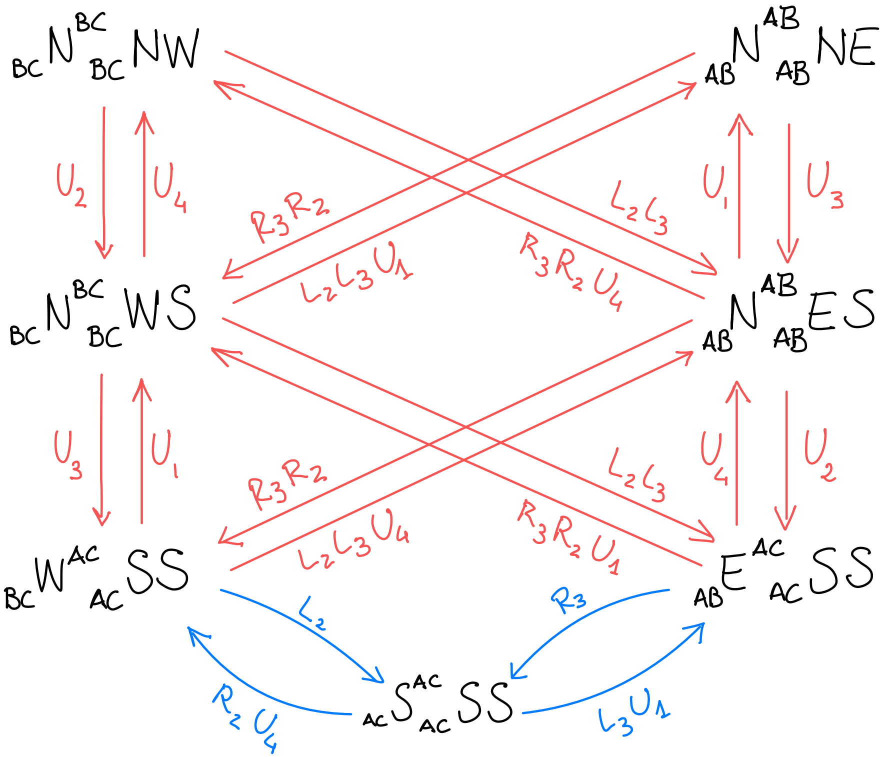
Note that
$M(3) = 1$ and
$M(2) = 4$ now.

<u>Note</u>: the contribution from $\delta^1_3 \circ \delta^2_{D_1}$ is zero.

Moreover, since $\delta^1_i \equiv 0 \ \forall i \geq 4$, the higher terms vanish too.

<u>RK</u>: this is a curved type-D str., $\mu_0^{D_2} = U_1U_2 + U_3U_4$

③ Computing $D_3 = M_3 \boxtimes D_2$



$$\delta^1_{D_2} = \boxed{\delta^1_1 \circ id_{D_1}} + \boxed{\delta^1_2 \circ \delta^1_{D_1}} + \boxed{\delta^1_3 \circ \delta^2_{D_2}} + \cancel{\cdots} \to 0$$

<u>Rem:</u> 1) We still have $\delta^1_3 \circ \delta^2_{D_2} \equiv 0$.

2) We used $1 \otimes 1$ at ${}_{BC}N^{BC}$ and ${}_{AB}N^{AB}$.

Note that it transforms U_2 into U_3 and U_3 into U_2.

3) Now $M(3) = 4$ and $M(2) = 1$.

4) This is a curved type D structure with $\mu_0^{D_3} = U_1U_3 + U_2U_4$.

4 Computing $D_{\mho} = \mho \boxtimes D_3$

idempotents

As a module, $\mho = \mathbb{F}_2\langle {}_{*}I^{BC} \rangle$.

The δ maps are complicated (see Lecture 4)

$$\Rightarrow D_{\mho} = \mathbb{F}_2\langle NNW, NWS, WSS \rangle .$$

$$\delta^1_{D_{\mho}} = \boxed{\delta^1_2 \circ \delta^1_{D_3}} + \boxed{\delta^1_4 \circ \delta^3_{D_3}} + \cdots$$

$$\begin{tikzcd}
NNW \arrow[d, shift right, "U_3"'] \arrow[d, bend right=60, "U_3"'] \\
NWS \arrow[u, shift right, "U_4"'] \arrow[d, shift right, "U_3"'] \\
WSS \arrow[u, shift right, "U_3U_4"']
\end{tikzcd}$$

Notes:

- (*) δ^1_2 eats U_2 and returns $U_{M(1)} = U_3$.
- (*) $\delta^1_2(U_1) = 0$, which is why it disappears.
- (*) The two U_3 arrows from NNW cancel out w/ each other
- (*) $U_3U_4 = 0$ on idempotent $*$.

(*) The two preferred sequences in $\delta^1_4 \circ \delta^3_{D_3}$ are:

$$NNW \xrightarrow{L_2L_3} NES \xrightarrow{U_1} NNE \xrightarrow{R_3R_2} NWS$$

$$WSS \xrightarrow{L_2L_3U_4} NES \xrightarrow{U_1} NNE \xrightarrow{R_3R_2} NWS$$

Finally, we get $\mathrm{NNW} \xleftarrow{U_4} \mathrm{NWS} \xrightarrow{U_3} \mathrm{WSS}$.

Usually, this is written as

$$\begin{tikzcd} a \bullet & \bullet\, b \arrow[l, "U"'] \arrow[d, "V"] \\ & \bullet\, c \end{tikzcd}$$

<u>RK</u>: OSz's method recover the $UV=0$ version of CFK (so you need to quotient it by that).

This is not relevant here because we do not have any UV.