ON RADICALS OF POLYNOMIAL RINGS

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Abstract. In this paper we investigate connections in the behaviour of a ring and the polynomial rings over it with respect to a given radical.

1. Introduction

In this paper all rings are associative, not necessarily with an identity. Let A be a ring and X be a (possibly infinite) set of commuting indeterminates over A. We will consider the polynomial ring $A[X]$ over A; if $X = \{x\}$ then we write $A[x]$ in place of $A[\{x\}]$. Marks [8] called a ring **NI** if the set of its nilpotent elements is an ideal. Smoktunowicz [10] constructed an **NI** ring over which the polynomial ring is not **NI**. Han, Lee and Yang [6] called a ring polynomial **NI** if $R[X]$ is **NI** for every finite set X of commuting indeterminates, and investigated **NI** and polynomial **NI** rings. Our aim in the present paper is to extend a part of their results from **N** to an arbitrary radical **R** in the sense of Kurosh and Amitsur.

For undefined notions and basic results in radical theory we refer to [4]. The semisimple class of a radical class **R** will be denoted by $\mathcal{S}\mathbf{R}$.

2. Definitions and Examples

DEFINITION 1. For an arbitrary radical $\mathbf{R}$, a ring A is said to be $\mathbf{R}I$ if $\mathbf{R}(A)$ contains all subrings $S \subseteq A$ such that $S \in \mathbf{R}$, and $\mathbf{R}$-*reduced* if it has no non-zero subring S such that $S \in \mathbf{R}$. (If $\mathbf{R}$ is the nil radical then the $\mathbf{R}$-reduced rings are exactly the reduced rings.) Denote by $\mathbf{R}^*(A)$ the sum of all subrings $S \subseteq A$ such that $S \in \mathbf{R}$.

The following can be considered as a reformulation of an observation of McConnell [9, Proposition 1.2] (see conditions (ii) and (iii) there), so we give it here without proof.

2000 Mathemathics Subject Classification: primary 16N80.

Key words and phrases. polynomial ring, radical, Amitsur property.

Research supported by the National Foundation for Scientific Research (OTKA) grants no. K101515 and K84233 and by the exchange program between the Hungarian and the Mongolian Academies of Sciences. The second author thanks also the Professors' team of the University of Mongolia for their support.

Typeset by $\mathcal{AMS}$-TEX

PROPOSITION 1 (cf. [9, Proposition 1.2]). *For a ring A and a radical* $\mathbf{R}$*, the following are equivalent.*

(i) A *is an* $\mathbf{R}I$ *ring.*

(ii) $\mathbf{R}(A) = \mathbf{R}^*(A)$.

(iii) $A/\mathbf{R}(A)$ *is an* $\mathbf{R}$*-reduced ring.* □

DEFINITION 2. A ring A is said to be *polynomial* $\mathbf{R}I$ if $A[X]$ is $\mathbf{R}$I for every finite set X of commuting indeterminates.

Clearly, if $A[X]$ is $\mathbf{R}$I for a finite set X and Y is a subset of X then $A[Y]$ is also $\mathbf{R}$I. In particular, if a ring A is polynomial $\mathbf{R}$I for a radical $\mathbf{R}$ then A is $\mathbf{R}$I.

Next we present examples of $\mathbf{R}$I rings and polynomial $\mathbf{R}$I rings.

NOTATION. The following symbols will be used:

$\mathbf{B}$ is the Baer (prime) radical,
$\mathbf{L}$ is the Levitzki radical,
$\mathbf{N}$ is the Köthe (nil) radical,
$\mathbf{J}$ is the Jacobson radical,
$\mathbf{G}$ is the Brown–McCoy radical.

It is well known that $\mathbf{B} \subset \mathbf{L} \subset \mathbf{N} \subset \mathbf{J} \subset \mathbf{G}$, where all inclusions are strict.

Example 1. Every zero ring A^0 is polynomial $\mathbf{R}$I for any radical $\mathbf{R}$.

Indeed, let $S^0 \subseteq A^0$, $S^0 \in \mathbf{R}$. Since $S^0 \triangleleft A^0$, we have $S^0 \subseteq \mathbf{R}(A^0)$. Thus we obtain $\mathbf{R}^*(A^0) \subseteq \mathbf{R}(A^0)$, so A^0 is an $\mathbf{R}$I ring by Proposition 1. Next, for any finite set X and any natural number n, $A^0[X]$ is also a zero ring, hence it is an $\mathbf{R}$I ring, and thus A^0 is a polynomial $\mathbf{R}$I ring.

Recall that a radical $\mathbf{R}$ is said to be *strict* if, for every ring A, $\mathbf{R}(A)$ contains all subrings $S \subseteq A$ such that $\mathbf{R}(S) = S$. Clearly, a radical $\mathbf{R}$ is strict if and only if every ring A is $\mathbf{R}$I.

Example 2. For a strict radical $\mathbf{R}$, every ring A is polynomial $\mathbf{R}$I. In particular, this holds for any A-radical $\mathbf{R}$ in the sense of Gardner [3].

Indeed, if X is a finite set of commuting indeterminates and $\mathbf{R}(S) = S \subseteq A[X]$, then $S \subseteq \mathbf{R}(A[X])$ by the strictness of $\mathbf{R}$.

Example 3. Let $\mathbb{Q}$ be the rational number field and $\mathcal{U}(\mathbb{Q})$ be the upper radical of $\mathbb{Q}$ (the largest radical for which $\mathbb{Q}$ is semisimple). Let $\mathbf{R}$ be any radical such that $\mathbf{J} \subseteq \mathbf{R} \subseteq \mathcal{U}(\mathbb{Q})$. Then $\mathbb{Q}$ is not an $\mathbf{R}$I ring.

Indeed, take the set J of all rational numbers with even numerator and odd denominator. J is obviously a ring and, for any $a = \frac{2k}{2m+1} \in J$, it is straightforward to check that $b = \frac{a}{a-1} = \frac{2k}{2(k-m)-1)} \in J$, and b is a solution of the equation $a \circ b =: a + b - ab = 0$. Hence $(J, \circ)$ is a group, that is, J is a Jacobson radical ring. Thus $J \in \mathbf{J} \subseteq \mathbf{R} \subseteq \mathcal{U}(\mathbb{Q})$. Hence $0 \neq J = \mathbf{R}(J)$, and $\mathbf{R}(\mathbb{Q}) = 0$. Therefore $\mathbb{Q}$ is not an $\mathbf{R}$I ring.

Example 4. $\mathbb{Q}$ is a polynomial $\mathbf{R}$I ring for any radical $\mathbf{R}$ such that $\mathbf{B} \subseteq \mathbf{R} \subseteq \mathbf{N}$.

Clearly, $\mathbf{N}(\mathbb{Q}[X]) = 0$ because $\mathbb{Q}[X]$ is a reduced ring. Therefore $\mathbf{R}(\mathbb{Q}[X]) = 0$. And since $\mathbb{Q}[X]$ has no non-zero nilpotent elements, if $S = \mathbf{R}(S) \in \mathbf{N}$ for a subring S of $\mathbb{Q}[X]$ then $S = 0$. Thus $\mathbb{Q}$ is a polynomial **RI** ring.

The following is clear.

Example 5. The matrix ring $M_n(F)$ over an arbitrary field F ($n \geq 2$) is not a polynomial **RI** ring for any radical $\mathbf{R}$ such that $\mathbf{B} \subseteq \mathbf{R} \subseteq \mathcal{U}(M_n(F))$.

The next two examples are taken from [6].

Example 6. Let F be a field, $\mathbb{Z}$ be the ring of integers and $\{t_n \mid n \in \mathbb{Z}\}$ be commuting indeterminates over F. Set

$$A = F[\{t_n\}_{n\in\mathbb{Z}}]/(\{t_{n_1}t_{n_2}t_{n_3} \mid n_3 - n_2 = n_2 - n_1 > 0\})$$

and $R = A[x, \sigma]$, the skew polynomial ring in one indeterminate x over A, where σ is the F-automorphism of A satisfying $\sigma(t_n) = t_{n+1}$ for all $n \in \mathbb{Z}$. Then R is polynomial **NI**.

Example 7. Smoktunowicz [10, Theorem 12] constructed a ring R (in fact, an algebra over an arbitrary countable field) such that A is nil but the polynomial ring $A[x, y]$ in two commuting indeterminates is not nil. Hence A is **NI** but not polynomial **NI**. (If we want a ring with identity with the same property then we can take the Dorroh extension of A with $\mathbb{Z}$.) On the other hand, by Example 2 above, A is polynomial **RI** for any strict radical $\mathbf{R}$.

DEFINITION 3. Let $\mathbf{R}$ be an arbitrary radical and $x_1, x_2, \ldots, x_n, \ldots$ be commuting indeterminates. Put $\mathbf{R}_n = \{A \mid A[x_1, \ldots, x_n] \in \mathbf{R}\}$. Clearly, $\mathbf{R} = \mathbf{R}_0 \supseteq \mathbf{R}_1 \supseteq \cdots \supseteq \mathbf{R}_n \supseteq \ldots$. Gardner [2] proved that each $\mathbf{R}_n$ ($n = 0, 1, 2, \ldots$) is a radical.

DEFINITION 4. For an arbitrary radical $\mathbf{R}$, a ring A is said to be an *absolute* $\mathbf{R}$*-ring* if $A[x_1, \ldots, x_n] \in \mathbf{R}$ for all $n \geq 0$, hence for the class $\overline{\mathbf{R}}$ of all absolute $\mathbf{R}$-rings we have $\overline{\mathbf{R}} = \bigcap_{n\in\mathbb{N}} \mathbf{R}_n$, and $\overline{\mathbf{R}}$ is a radical class.

DEFINITION 5. A class $\mathcal{M}$ of rings is said to be *polynomially extensible* if $A[x] \in \mathcal{M}$ for all rings $A \in \mathcal{M}$.

The following notion was introduced in [15].

DEFINITION 6. Let $\mathbf{R}$ be a radical, κ be a cardinal number and X be a set of commuting indeterminates of cardinality κ. To indicate the latter, we write X_κ for X; to allow a unified treatment, we also write X_0 for the empty set. We say that $\mathbf{R}$ has the *κ-Amitsur property* if, for all rings A,

$$\mathbf{R}(A[X_\kappa]) = (A \cap \mathbf{R}(A[X_\kappa]))[X_\kappa].$$

For $\kappa = 1$ we say that $\mathbf{R}$ has the *Amitsur property.*

By [15, Proposition 2.6], if a radical $\mathbf{R}$ has the κ-Amitsur property for some cardinal κ then it has the λ-Amitsur property for all λ with $\kappa \leq \lambda$.

3. Results

We start with a result on strict radicals.

THEOREM 2. *For a strict radical* $\mathbf{R}$*, the following are equivalent.*
(i) $\mathbf{R}(A[x]) = \mathbf{R}(A)[x]$ *for every ring* A.
(ii) $\mathbf{R}$ *has the Amitsur property.*
(iii) $\mathcal{S}\mathbf{R}$ *is polynomially extensible.*

Proof. (i) $\Longrightarrow$ (ii): Krempa [7, Theorem 1] observed that $\mathbf{R}$ has the Amitsur property if and only if $(A \cap \mathbf{R}(A[x])) = 0$ implies $\mathbf{R}(A[x]) = 0$. Now, $\mathbf{R}(A)$ is a radical subring of $\mathbf{R}(A)[x]$ hence, by condition (i), also of $\mathbf{R}(A[x])$, so $\mathbf{R}(A) \subseteq \mathbf{R}(A[x])$ since $\mathbf{R}$ is strict. Therefore $\mathbf{R}(A) \subseteq A \cap \mathbf{R}(A[x])$, hence if the latter is zero then also $\mathbf{R}(A) = 0$, and then $\mathbf{R}(A[x]) = \mathbf{R}(A)[x] = 0$ as well.

(ii) $\Longrightarrow$ (i): As we have seen just before, $\mathbf{R}(A) \subseteq A \cap \mathbf{R}(A[x])$ and, since $\mathbf{R}$ is strict, $\mathbf{R}(A)[x] \subseteq \mathbf{R}(A[x])$. By the Amitsur property, $(A \cap \mathbf{R}(A[x]))[x] = \mathbf{R}(A[x]) \in \mathbf{R}$, and then also $A \cap \mathbf{R}(A[x]) \in \mathbf{R}$, being a homomorphic image of $(A \cap \mathbf{R}(A[x]))[x]$. Clearly, $A \cap \mathbf{R}(A[x])$ is an ideal of A, whence $A \cap \mathbf{R}(A[x]) \subseteq \mathbf{R}(A)$. So we have

$$\mathbf{R}(A)[x] \subseteq \mathbf{R}(A[x]) = (A \cap \mathbf{R}(A[x]))[x] \subseteq \mathbf{R}(A)[x]$$

which yields $\mathbf{R}(A[x]) = \mathbf{R}(A)[x]$.

(ii) $\Longleftrightarrow$ (iii): Stewart [12, Proposition 3.1] proved that every strict radical is polynomially extensible, and by [14, Theorem 3.6] a radical $\mathbf{R}$ is polynomially extensible and has the Amitsur property if and only if both $\mathbf{R}$ and $\mathcal{S}\mathbf{R}$ are polynomially extensible, which gives the equivalence of conditions (ii) and (iii). □

Remark. Stewart [12] constructed a strict radical $\mathbf{R}$ such that $\mathbf{R}(A[x]) \neq \mathbf{R}(A)[x]$ for some ring A, so not every strict radical has the Amitsur property.

For what comes next, the following observation of Divinsky and Suliński will be needed. Notice that the ring $\mathbb{Z}[X_\kappa]$ of polynomials with integer coefficients operates on $A[X_\kappa]$ by multiplication in the obvious way.

PROPOSITION 3 (cf. [1, Theorem])**.** *Let* $\mathbf{R}$ *be a radical and* X_κ *be a set of commuting indeterminates. For any polynomial* $f \in \mathbb{Z}[X_\kappa]$*, we have* $f\mathbf{R}(A[X_\kappa]) \subseteq \mathbf{R}(A[X_\kappa])$. □

THEOREM 4. *Let* $\mathbf{R}$ *be a radical with the Amitsur property, and* A *be any ring. The following conditions are equivalent:*
(i) A *is polynomial* $\mathbf{R}$*I.*
(ii) For every natural number $n \geq 0$, $\mathbf{R}(A[X_n]) = \mathbf{R}^*(A[X_n]) = \mathbf{R}(A)[X_n] = \mathbf{R}^*(A)[X_n]$.
(iii) For every natural number $n \geq 0$, $A[X_n]/\mathbf{R}(A[X_n])$ *is* $\mathbf{R}$*-reduced.*
(iv) For every natural number $n \geq 0$, $A[X_n]$ *is* $\mathbf{R}$*I.*
(v) $\mathbf{R}(A)$ *is an absolute* $\mathbf{R}$*-ring and, for every natural number* $n \geq 0$, $\frac{A}{\mathbf{R}(A)}[X_n]$ *is* $\mathbf{R}$*-reduced.*

Proof. (i) $\Longrightarrow$ (ii): Since A is polynomial $\mathbf{R}$I, $A[X_n]$ is $\mathbf{R}$I, hence by Proposition 1, $\mathbf{R}(A[X_n]) = \mathbf{R}^*(A[X_n])$. Clearly, $\mathbf{R}(A) \in \mathbf{R}$ is a subring of $A[X_n]$. Since $A[X_n]$ is $\mathbf{R}$I, we have $\mathbf{R}(A) \subseteq \mathbf{R}(A[X_n])$ and also $\mathbf{R}^*(A) \subseteq \mathbf{R}(A[X_n])$. By Lemma 3, $\mathbf{R}(A)[X_n] \subseteq \mathbf{R}(A[X_n])$ and also $\mathbf{R}^*(A)[X_n] \subseteq \mathbf{R}(A[X_n])$. Now we have

$$\mathbf{R}(A)[X_n] \subseteq \mathbf{R}^*(A)[X_n] \subseteq \mathbf{R}(A[X_n]) = \mathbf{R}^*(A[X_n])\,.$$

Since $\mathbf{R}$ has the Amitsur property, it has also the n-Amitsur property, therefore $\mathbf{R}(A[X_n]) = (A \cap \mathbf{R}(A[X_n]))[X_n]$. Clearly, $A \cap \mathbf{R}(A[X_n]) \in \mathbf{R}$, so $A \cap \mathbf{R}(A[X_n]) \subseteq \mathbf{R}(A)$. Thus $\mathbf{R}(A)[X_n] \supseteq \mathbf{R}(A[X_n])$, and we have proved all the equalities in condition (ii).

(ii) $\Longrightarrow$ (iii): Since $\mathbf{R}(A[X_n]) = \mathbf{R}^*(A[X_n])$, every $\mathbf{R}$-radical subring of $A[X_n]$ is in $\mathbf{R}(A[X_n])$. Therefore $A[X_n]/\mathbf{R}(A[X_n])$ has no non-zero radical subring, as required.

(iii) $\Longrightarrow$ (iv): Since $A[X_n]/\mathbf{R}(A[X_n])$ is $\mathbf{R}$-reduced, every $\mathbf{R}$-radical subring of $A[X_n]$ is in $\mathbf{R}(A[X_n])$. Thus $A[X_n]$ is $\mathbf{R}$I.

(iv) $\Longrightarrow$ (i): Clear by definition.

(ii) $\Longrightarrow$ (v): From $\mathbf{R}(A[X_n]) = \mathbf{R}(A)[X_n]$, $\mathbf{R}(A)$ is an absolute $\mathbf{R}$-ring. From $\mathbf{R}^*(A[X_n]) = \mathbf{R}(A)[X_n]$ we have that $\frac{A[X_n]}{\mathbf{R}(A)[X_n]}$ is $\mathbf{R}$-reduced. But

$$\frac{A[X_n]}{\mathbf{R}(A)[X_n]} \cong \frac{A}{\mathbf{R}(A)}[X_n]\,,$$

whence the latter ring is also $\mathbf{R}$-reduced.

(v) $\Longrightarrow$ (i): Since $\mathbf{R}(A)$ is an absolute $\mathbf{R}$-ring, $\mathbf{R}(A)[X_n]$ is a radical ideal of $A[X_n]$. Thus $\mathbf{R}(A)[X_n] \subseteq \mathbf{R}(A[X_n])$, and then as above, $\mathbf{R}(A)[X_n] = \mathbf{R}(A[X_n])$ because $\mathbf{R}$ has the Amitsur property. Therefore

$$\frac{A[X_n]}{\mathbf{R}(A[X_n])} = \frac{A[X_n]}{\mathbf{R}(A)[X_n]} \cong \frac{A}{\mathbf{R}(A)}[X_n],$$

and the last ring is $\mathbf{R}$-reduced. □

Han, Lee and Yang [6, Proposition 1.4] gave several equivalent conditions for a ring A to be polynomial $\mathbf{N}$I, under the condition that there is a common bound for the indices of nilpotency of the nilpotent elements of A. Using Theorem 4 above, we show that several of these conditions are equivalent without any restriction on the ring A.

PROPOSITION 5. *The following conditions on a ring A are equivalent:*

(i) A is polynomial $\mathbf{N}I$.

(ii) $\mathbf{N}(A)$ is absolute nil and $A/\mathbf{N}(A)$ is a reduced ring.

(iii) $A/\overline{\mathbf{R}}(A)$ is an $\overline{\mathbf{R}}$-reduced ring, where $\mathbf{R}$ is any radical such that $\overline{\mathbf{N}} \subseteq \mathbf{R} \subseteq \overline{\mathbf{J}}$.

(iv) $A[X]$ is polynomial $\mathbf{N}I$ for some set X of commuting indeterminates.

Proof. As is well known, $\mathbf{N}$ has the Amitsur property, hence Theorem 4 applies.

(i) $\Longrightarrow$ (ii): By (v) of Theorem 4, $\mathbf{N}(A)$ is absolute nil, and since $(A/\mathbf{N}(A))[x_1, \ldots, x_n]$ is reduced, $A/\mathbf{N}(A)$ is also.

(ii) $\Longrightarrow$ (i): Since $A/\mathbf{N}(A)$ is reduced, $(A/\mathbf{N}(A))[X_n]$ is reduced for any n. Again by Theorem 4, our claim follows.

(ii) $\Longleftrightarrow$ (iii): Since $\mathbf{N}(A)$ is absolute nil, $\mathbf{N}(A) = \overline{\mathbf{N}}(A)$. By [15, Proposition 2.12], $\overline{\mathbf{R}}(A) = \overline{\mathbf{N}}(A) = \mathbf{N}(A)$, and since $A/\mathbf{N}(A)$ is reduced, $A/\overline{\mathbf{R}}(A)$ is also. Hence (ii) and (iii) are equivalent.

(i) $\Longleftrightarrow$ (iv) is clear. □

The following question is asked in [6]:

Question 1. Let A be a ring such that $A[x]$ is **NI**. Is then A polynomial **NI**?

Now it is natural to ask:

Question 2. Let A be a ring such that $A[x]$ is **R**I for some radical **R**. Is then A polynomial **R**I?

Let $\mathbb{P}$ denote the class of all polynomial rings in one indeterminate. For any radical $\mathbf{R}$ we consider the lower radical $\mathbf{R}^1 = \mathcal{L}(\mathbf{R} \cap \mathbb{P})$ determined by the (homomorphic closure of) the class $\mathbf{R} \cap \mathbb{P}$.

PROPOSITION 6. *Suppose that, for a radical* $\mathbf{R}$*, Question 2 has a positive answer for every ring* A*. Then* $\mathbf{R}^1 = \mathbf{R}_1 = \mathbf{R}_2 = \dots$ *.*

Proof. Let A be in $\mathbf{R}_1$, so that $A[x] \in \mathbf{R}$; then by the assumption $A[x, y] \in \mathbf{R}$, and so $A[x][y] \in \mathbf{R}$. Hence $A \in \mathbf{R}_2$, thus $\mathbf{R}_1 = \mathbf{R}_2$ and $\mathbf{R}_1 = \mathbf{R}^1$. □

COROLLARY 7. *Let* $\mathbf{R}$ *be a radical such that* $\mathbf{R}^1 \neq \mathbf{R}_1$ *or* $\mathbf{R}_1 \neq \mathbf{R}_2$*. Then Question 2 has a negative answer for some* A*.* □

COROLLARY 8. *If either* $\mathbf{R}^1 = \mathbf{R}_1$ *or* $\mathbf{R}_1 = \mathbf{R}_2$ *for a radical* $\mathbf{R}$*, then* $\mathbf{R}^1 = \mathbf{R}_1 = \mathbf{R}_2 = \dots$ *.* □

Example 8. Question 2 has a negative answer for the Jacobson radical $\mathbf{J}$. Indeed, by Smoktunowicz and Puczyłowski [11, Theorem 4.1], there exists a ring A such that $A[x] \in \mathbf{J} \setminus \mathbf{N}$. So $A \in \mathbf{J}_1$ but $A \notin \mathbf{J}_2$ because the latter would mean $A[x, y] \cong (A[x])[y] \in \mathbf{J}$ and, as is well known, $B[y] \in \mathbf{J}$ implies $B \in \mathbf{N}$ (see e.g. [4, Proposition 4.9.27]).

Gardner [2] asked whether the chain $\mathbf{R} = \mathbf{R}_0 \supseteq \mathbf{R}_1 \supseteq \dots \supseteq \mathbf{R}_n \supseteq \dots$ terminates for every radical class $\mathbf{R}$. In this connection, Gardner [2] gives examples of radicals which show that $\mathbf{R}_0 \not\supseteq \mathbf{R}_1 \not\supseteq \dots \not\supseteq \mathbf{R}_{n+1}$ may hold for any n. Finally, Gardner's question was answered in the negative by Tumurbat, Mendes and Mekei [13]: there exist radicals $\mathbf{R}$ such that $\mathbf{R}_0 \not\supseteq \mathbf{R}_1 \not\supseteq \dots \not\supseteq \mathbf{R}_n \not\supseteq \dots$. For such radicals Question 2 has a negative answer.

Concerning Question 1, we have:

PROPOSITION 9. *Question 1 has a positive answer for every ring* A *if and only if either* $\mathbf{N}_1 = \mathbf{N}_2$ *or* $\mathbf{N}_1 = \mathbf{N}^1$*.*

Proof. $\Longrightarrow$ follows from Proposition 6. To see $\Longleftarrow$, notice first of all that the two conditions of equality are equivalent by Corollary 9, hence it suffices to consider only one of them. Let $\mathbf{N}_1 = \mathbf{N}_2$, and take any ring A. Since $\mathbf{N}$ has the Amitsur property, we have $\mathbf{N}(A[x]) = (A \cap \mathbf{N}(A[x]))[x]$. Now, $\frac{A[x]}{\mathbf{N}(A[x])} = \frac{A[x]}{(A\cap\mathbf{N}(A[x]))[x]}$. Since $A[x]$ is **NI**, $\frac{A[x]}{\mathbf{N}(A[x])}$ is reduced, and $\mathbf{N}(A) \subseteq A \cap \mathbf{N}(A[x]) \subseteq \mathbf{N}(A)$. Therefore $\mathbf{N}(A)[x] = \mathbf{N}(A[x]) \in \mathbf{N}$, that is, $\mathbf{N}(A) \in \mathbf{N}_1$, so $\mathbf{N}(A[x_1, \dots, x_n]) \in \mathbf{N}$. Hence $\mathbf{N}(A)$ is absolute nil, and by Proposition 5 A is polynomial **NI**. □

COROLLARY 10. *Question 1 has a positive answer for every ring* A *if and only if, for every ring* B*,* $B[x]$ *nil implies* $B[x, y]$ *nil.* □

THEOREM 11. *Let $\mathbf{R}_1 \subseteq \mathbf{R}_2$ be radicals which satisfy the Amitsur property. If $\overline{\mathbf{R}_1} = \overline{\mathbf{R}_2}$ and a ring A is polynomial $\mathbf{R}_2I$, then A is also polynomial $\mathbf{R}_1I$.*

Proof. Suppose that A is a polynomial $\mathbf{R}_2$I ring. Then by Theorem 4, $\mathbf{R}_2(A)$ is an absolute $\mathbf{R}_2$-ring. Therefore

$$\mathbf{R}_2(A) = \overline{\mathbf{R}_2}(A) = \overline{\mathbf{R}_1}(A) \subseteq \mathbf{R}_1(A) \subseteq \mathbf{R}_2(A)\,.$$

Thus $\mathbf{R}_2(A) = \mathbf{R}_1(A)$, hence $\mathbf{R}_1(A)$ is an absolute $\mathbf{R}_1$-ring. Applying condition (v) in Theorem 4 to the radical $\mathbf{R}_2$, we obtain that $\frac{A}{\mathbf{R}_2(A)}[X_n]$ is $\mathbf{R}_2$-reduced, and then $\frac{A}{\mathbf{R}_1(A)}[X_n] = \frac{A}{\mathbf{R}_2(A)}[X_n]$ is an $\mathbf{R}_1$-reduced ring. Again by Theorem 4, A is a polynomial $\mathbf{R}_1$I ring. □

Remark. Without the condition $\overline{\mathbf{R}_1} = \overline{\mathbf{R}_2}$, the statement is not true. For example, consider the radicals $\mathbf{L} \subseteq \mathbf{N}$. By Golod [5], there exists a ring A such that $0 \neq A \in \overline{\mathbf{N}}$ and $\mathbf{L}(A) = 0$. Hence $\overline{\mathbf{L}} \neq \overline{\mathbf{N}}$, and the ring A is polynomial **N**I but not polynomial **L**I, not even **L**I.

COROLLARY 12. *If A is a polynomial $\mathbf{J}I$ ring then it is a polynomial $\mathbf{N}I$ ring.* □

Remark. The converse is not true. For example, $\mathbb{Q}$ is polynomial **N**I but not even **J**I. Moreover, $\mathbb{Q}$ is not **G**I.

PROPOSITION 13. *For a ring A, the following conditions are equivalent:*

(i) *A is polynomial $\mathbf{J}I$.*

(ii) *$\mathbf{J}(A)$ is absolute nil and, for every n, $\frac{A}{\mathbf{J}(A)}[X_n]$ has no non-zero subring S such that $S \in \mathbf{J}$.*

(iii) *$\mathbf{J}(A)$ is absolute nil and, for every n, every non-zero subring of $\frac{A}{\mathbf{J}(A)}[X_n]$ has a non-zero primitive homomorphic image.*

Proof. (i) and (ii) are equivalent by Theorem 4, conditions (i) and (v).

(ii) $\Longleftrightarrow$ (iii): The Jacobson radical satisfies the Amitsur property, hence $\mathbf{J}(A[X_n]) = (A\cap\mathbf{J}(A[X_n]))[X_n]$. Since $\mathbf{J}(A)$ is absolute nil, $\mathbf{J}(A)[X_n]$ is also, therefore $\mathbf{J}(A)[X_n] \subseteq \mathbf{J}(A[X_n])$. Now, for any radical $\mathbf{R}$ and any ring B we have $\mathbf{R}(B) \supseteq B \cap \mathbf{R}(B[x])$, repeating this we get $\mathbf{R}(B[x]) \supseteq B[x] \cap \mathbf{R}(B[x,y])$, hence $\mathbf{R}(B) \supseteq B \cap \mathbf{R}(B[x]) \supseteq B \cap B[x] \cap \mathbf{R}(B[x,y]) = B \cap \mathbf{R}(B[x,y])$, and similarly $\mathbf{R}(B) \supseteq B \cap \mathbf{R}(B[X_n])$. Thus in our case we have $\mathbf{J}(A) \supseteq A \cap \mathbf{J}(A[X_n])$, whence $\mathbf{J}(A)[X_n] = \mathbf{J}(A[X_n])$. Consequently,

$$\frac{A[X_n]}{\mathbf{J}(A[X_n])} = \frac{A[X_n]}{\mathbf{J}(A)[X_n]} \cong \frac{A}{\mathbf{J}(A)}[X_n].$$

The required equivalence follows now from a well-known property of the Jacobson radical: a ring belongs to **J** if and only if it has no non-zero primitive homomorphic image. □

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