A phase transition for the corank of random band matrices over $\mathbb{F}_p$

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Universality phenomenon in classical random matrix theory

GOE random matrix: $H_n = \frac{1}{\sqrt{2n}}\left(G_n + G_n^T\right)$, where G_n is an $n \times n$ matrix with i.i.d. standard normal entries.
The eigenvalues of H_n have joint density

$$\frac{1}{Z}\exp\left(-\frac{1}{4}\sum_{k=1}^{n}\lambda_k^2\right)\prod_{i<j}|\lambda_i - \lambda_j|.$$

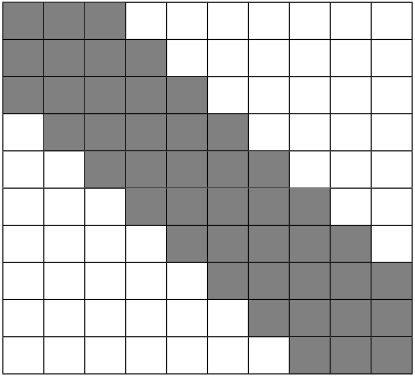
Universality results:

Let W_n be a symmetric matrix, with independent entries (up to symmetry) such that the entries have sufficiently well-behaved moments. Then, asymptotically, the spectrum of W_n behaves the same way as that of H_n.

Universality beyond independent entries: Adjacency matrices of random regular graphs.

The breakdown of universality for band matrices

Given a bandwidth w, let B_n be obtained from H_n by setting $H_n(i,j)$ to be 0, whenever $|i-j| > w$.



The breakdown of universality for band matrices

A conjectured metal/insulator phase transition

	Eigenvalues	Eigenvectors
$w \gg \sqrt{n}$	GOE	delocalized
$w \ll \sqrt{n}$	Poisson	localized

State of art results

- GOE eigenvalue statistics and delocalization for $w \gg n^{3/4}$.
- Localization for $w \ll n^{1/4}$.
- Poisson eigenvalue statistics for constant w.

(Erdős, Yau, Bourgade, Schenker, Knowles, Yin ...)

Universality of the mod p corank of random matrices

Theorem (Wood)

Let M_n be an $n \times n$ random matrix over $\mathbb{F}_p$, where the entries are i.i.d. copies of a given non-constant random variable. Then

$$\lim_{n\to\infty} \mathbb{P}(\dim\ker M_n = k) = p^{-k^2} \prod_{i=1}^{k} \left(1 - p^{-i}\right)^{-2} \prod_{i=1}^{\infty} \left(1 - p^{-i}\right).$$

Beyond independent entries: Reduced Laplacian of random d-regular directed graphs (M.).

More general context

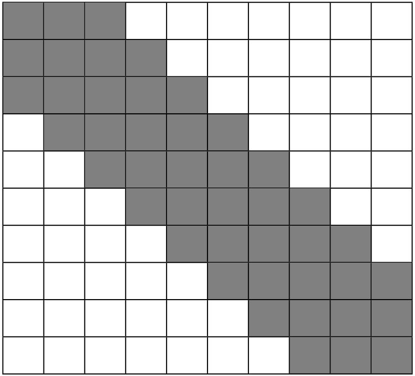
Cokernels of matrices over $\mathbb{Z}$ or $\mathbb{Z}_p$.
Connections to Cohen-Lenstra heuristics for the class groups of quadratic number fields, arithmetic statistics, homology of random simplicial complexes.

Spectrum $\approx$ Cokernel

A phase transition for the corank of band matrices over $\mathbb{F}_p$

Let B_n be a uniform random element of the set

$$\left\{ M \in \mathbb{F}_p^{n\times n} \ :\ M(i,j) = 0 \text{ for all } i,j \text{ such that } |i-j| > w_n \right\}.$$



Gray entries are i.i.d. uniform elements of $\mathbb{F}_p$. The white enrties are set to be 0.

Theorem (M. '24)

As $n \to \infty$, $\dim\ker B_n$ has Cohen-Lenstra limiting distribution if and only if

$$\lim_{n\to\infty} w_n - \log_p(n) = \infty.$$

Reminder

The Cohen-Lenstra distribution ν_p is defined by

$$\nu_p(k) = p^{-k^2} \prod_{i=1}^{k} \left(1 - p^{-i}\right)^{-2} \prod_{i=1}^{\infty} \left(1 - p^{-i}\right).$$

Small band widths – The non-Cohen-Lenstra phase

Let $v \in \mathbb{F}_p^n$ such that $a = \min \operatorname{supp}(v)$, $b = \max \operatorname{supp}(v)$, $a < b$.

$$\operatorname{supp}(B_n v) \subset [a - w_n, b + w_n]$$

Thus, $B_n v$ is a uniform random element of some subspace of

$$\{r \in \mathbb{F}_p^n \,:\, \operatorname{supp}(r) \subset [a - w_n, b + w_n]\}.$$

So

$$\mathbb{P}(B_n v = 0) \geq \frac{1}{p^{(b-a+1+2w_n)}}.$$

$$\mathbb{E}|\{v \in \ker B_n \,:\, a = \min \operatorname{supp}(v), b = \max \operatorname{supp}(v)\}|$$

$$\geq \frac{(p-1)^2 p^{b-a-1}}{p^{(b-a+1+2w_n)}} = \Omega\left(p^{-2w_n}\right).$$

For $[c, c+m-1] \subset [1, n]$, we have

$$\mathbb{E}|\{v \in \ker B_n \,:\, \emptyset \neq \operatorname{supp}(v) \subset [c, c+m-1]\}| = \Omega(m^2 p^{-2w_n}).$$

Small band widths – The non-Cohen-Lenstra phase

For $[c, c+m-1] \subset [1, n]$, we have

$$\mathbb{E}|\{v \in \ker B_n \,:\, \emptyset \neq \operatorname{supp}(v) \subset [c, c+m-1]\}| = \Omega(m^2 p^{-2w_n}).$$

Assuming that $m \leq p^{w_n}$ by a second moment argument, we get

$$\mathbb{P}(\text{There is a } 0 \neq v \in \ker B_n \,:\, \operatorname{supp}(v) \subset [c, c+m-1]) = \Omega(m^4 p^{-4w_n})$$

Small band widths – The non-Cohen-Lenstra phase

For $m \leq p^{w_n}$ and $[c, c+m-1] \subset [1, n]$,

$\mathbb{P}(\text{There is a } 0 \neq v \in \ker B_n : \text{supp}(v) \subset [c, c+m-1]) = \Omega(m^4 p^{-4w_n})$

Assuming that $\lim_{n\to\infty} w_n - \log_p(n) \neq \infty$, for infinitely many n,

$$n \geq \alpha p^{w_n} \text{ some fixed } 0 < \alpha < 1.$$

Given L, let $m = \left\lfloor \frac{\alpha}{L} p^{w_n} \right\rfloor$.
Subdivide $[1, Lm]$ into L intervals

$$I_i = [(i-1)m + 1, im], \qquad i = 1, 2, \dots, L$$

Consider the event that for all $i = 1, \dots, L$, there is a $0 \neq v \in \ker B_n$ such that $\text{supp}(v) \subset I_i$.
This event has probability at least

$$\left(\Omega(m^4 p^{-4w_n})\right)^L \geq \left(\frac{C_\alpha}{L}\right)^{4L}.$$

Small band widths – The non-Cohen-Lenstra phase

$$\mathbb{P}(\dim\ker B_n \geq L) \geq \left(\frac{C_\alpha}{L}\right)^{4L} = \exp(-O(L\log L)).$$

For the Cohen-Lenstra distribution ν_p, we have

$$\nu_p(k) = p^{-k^2}\prod_{i=1}^{k}\left(1-p^{-i}\right)^{-2}\prod_{i=1}^{\infty}\left(1-p^{-i}\right) = \Omega(p^{-k^2}).$$

So

$$\sum_{k=L}^{\infty}\nu_p(k) = O(p^{-L^2}).$$

Localization of the kernel $\overset{?}{\longleftrightarrow}$ heavier than Cohen-Lenstra tail of the cokernel

Large band widths – The Cohen-Lenstra phase

The moment method of Wood

Let M_n be a sequence of $n \times n$ random matrices over $\mathbb{F}_p$. Assume that for all d, we have

$$\lim_{n\to\infty} \sum \mathbb{P}(v_1, v_2, \dots, v_d \in \ker M_n) = 1,$$

where the sum is over all d-tuples $(v_1, v_2, ..v_d) \in \left(\mathbb{F}_p^n\right)^d$ of linearly independent vectors.
Then $\dim \ker M_n$ has Cohen-Lenstra limiting distribution.

This is a very special case of a more general result on random abelian p-groups.
In this talk, we only check that above condition for $d = 1$, that is,

$$\lim_{n\to\infty} \sum_{0 \neq v \in \mathbb{F}_p^n} \mathbb{P}(v \in \ker B_n) = 1.$$

Large band widths – The Cohen-Lenstra phase

Given a vector $v \in \mathbb{F}_p^n$, what is $\mathbb{P}(v \in \ker B_n)$?
Let

$$v'_i = \begin{cases} 1 & \text{if } [i - w_n, i + w_n] \cap \operatorname{supp}(v) \neq \emptyset, \\ 0 & \text{if } [i - w_n, i + w_n] \cap \operatorname{supp}(v) = \emptyset. \end{cases}$$

So

$$\operatorname{supp}(v') = \cup_{i \in \operatorname{supp}(v)} [i - w_n, i + w_n].$$

v	0	0	0	0	2	0	1	0	0	0	0	0	0	2	0	0	0	0	0	0	1
v'	0	0	1	1	1	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1

$B_n v$ is a uniform random element of

$$\{r \in \mathbb{F}_p^n \,:\, \operatorname{supp}(r) \subset \operatorname{supp}(v')\}.$$

Thus,

$$\mathbb{P}(v \in \ker B_n) = p^{-|\operatorname{supp}(v')|}.$$

Large band widths – The Cohen-Lenstra phase

v	0	0	0	0	2	0	1	0	0	0	0	0	0	2	0	0	0	0	0	0	1
v'	0	0	1	1	1	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1

$$\mathbb{P}(v \in \ker B_n) = p^{-|\operatorname{supp}(v')|}.$$

Each island of 1's of v'

- has size at least $2w_n + 1$, if it has 0's on both sides.
- has size at least $w_n + 1$, if it has a 0 on one side.

Given v' as above, what can be v?

v	0	0	0	0	*	*	*	0	0	0	0	0	0	*	0	0	0	0	0	0	*
v'	0	0	1	1	1	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1

$$|\{u \in \mathbb{F}_p^n \ : \ u' = v'\}| \leq p^{|\operatorname{supp}(v')| - s(v')w_n}$$

$$s(v') = |\{i \ : \ v'_i \neq v'_{i+1}\}|.$$

Large band widths – The Cohen-Lenstra phase

$$\mathbb{P}(v \in \ker B_n) = p^{-|\operatorname{supp}(v')|}.$$

$$|\{u \in \mathbb{F}_p^n \;:\; u' = v'\}| \leq p^{|\operatorname{supp}(v')| - s(v')w_n}$$

$$s(v') = |\{i \;:\; v'_i \neq v'_{i+1}\}|.$$

$$\sum_{\substack{u \in \mathbb{F}_p^n \\ u' = v'}} \mathbb{P}(u \in \ker B_n) \leq p^{|\operatorname{supp}(v')| - s(v')w_n} p^{-|\operatorname{supp}(v')|} = p^{-s(v')w_n}.$$

$$|\{v' \in \{0,1\}^n \;:\; s(v') = s\}| \leq 2n^s.$$

$$\sum_{\substack{v \in \mathbb{F}_p^n \\ s(v') = s}} \mathbb{P}(v \in \ker B_n) \leq 2p^{-sw_n} n^s.$$

$$\sum_{\substack{v \in \mathbb{F}_p^n \\ s(v') \geq 1}} \mathbb{P}(v \in \ker B_n) \leq \sum_{s=1}^{\infty} 2p^{-sw_n} n^s \to 0 \text{ provided that } p^{-w_n} n \to 0.$$

Large band widths – The Cohen-Lenstra phase

Delocalization

With probability tending to 1, for all $0 \neq v \in \ker B_n$, we have $v' \equiv 1$.

$$\lim_{n\to\infty} \sum_{0\neq v\in\mathbb{F}_p^n} \mathbb{P}(v \in \ker B_n) = \lim_{n\to\infty} \sum_{\substack{v\in\mathbb{F}_p^n \\ v'\equiv 1}} \mathbb{P}(v \in \ker B_n) \leq p^n p^{-n} \leq 1.$$

Trivially,

$$\sum_{0\neq v\in\mathbb{F}_p^n} \mathbb{P}(v \in \ker B_n) \geq (p^n - 1)p^{-n} \geq 1 - o(1).$$

Thus,

$$\lim_{n\to\infty} \sum_{0\neq v\in\mathbb{F}_p^n} \mathbb{P}(v \in \ker B_n) = 1.$$

Further questions

Conjecture

There is a one parameter family of distributions $(\nu_\alpha)_{\alpha\in\mathbb{R}}$ on $\mathbb{Z}_{0\le}$ with the following property. Let $n_1 < n_2 < \dots$ be a sequence of positive integers. Let B_i be an $n_i \times n_i$ uniform random band matrix over $\mathbb{F}_p$ with band width w_i. Let us assume that

$$\lim_{i\to\infty} w_i - \log_p(n_i) = \alpha.$$

Then the distribution of dim $ker(B_i)$ converges to ν_α.

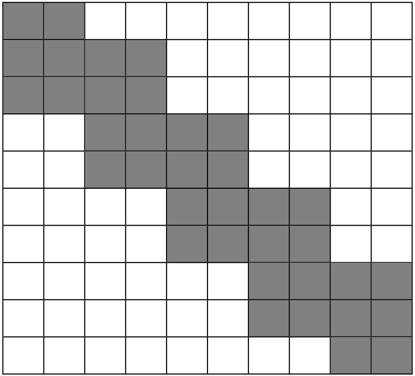
Conjecture

If

$$\lim_{i\to\infty} w_i - \log_p(n_i) = -\infty,$$

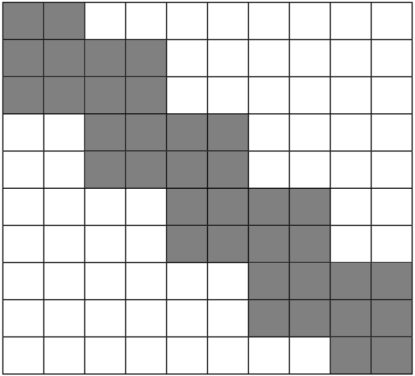
then $\dim\ker(B_i)$ has Gaussian fluctuations.

A simplified model – Truncated block bidiagonal matrices



All the cells correspond to $m \times m$ blocks. The entries of the white cells are set to be 0. The entries of the gray cells are chosen as i.i.d. uniform elements of $\mathbb{F}_p$.

A simplified model – Truncated block bidiagonal matrices



All the cells correspond to $m \times m$ blocks. The entries of the white cells are set to be 0. The entries of the gray cells are chosen as i.i.d. uniform elements of $\mathbb{F}_p$.

Thank you for your attention.

Slides are available on my webpage.