Cycles of Even Length in Graphs

J. A. Bondy

University of Waterloo, Waterloo, Ontario, Canada

AND

M. Simonovits

Eötvös Loránd University, Budapest, Hungary

Communicated by W. T. Tutte

Received February 21, 1973

In this paper we solve a conjecture of P. Erdős by showing that if a graph $G^n$ has $n$ vertices and at least $100kn^{1+1/k}$ edges, then $G$ contains a cycle $C_{2l}$ of length $2l$ for every integer $l \in [k, kn^{1/k}]$. Apart from the value of the constant this result is best possible. It is obtained from a more general theorem which also yields corresponding results in the case where $G^n$ has only $cn(\log n)^{1/n}$ edges ($c > 1$).

0. Notation

The graphs considered in this paper are finite and have neither loops nor multiple edges. The number of edges of a graph $G$ will be denoted by $e(G)$. The number of vertices will be either denoted by $v(G)$ or indicated by a superscript; thus $G^n$ is always a graph on $n$ vertices. $C^k$ denotes the cycle of length $k$.

1. Introduction

P. Erdős, in [4], published without proof the following

Theorem. There exists a $c_k$ and an $n_0(k)$ such that, if

$$e(G^n) > c_k n^{1+1/k} \quad \text{and} \quad n > n_0,$$

then

$$C^{2k} \subseteq G^n.$$
Later, Erdős asked whether (1) implies $C^{2l} \subseteq G^n$ for every integer $l \in [k, n^{1/k}]$.

(In [5] he proved a weaker form of this conjecture for $k = 2$). We shall prove

**Theorem 1.** If

$$e(G^n) > 100kn^{1+1/k}, \quad (2)$$

then $C^{2l} \subseteq G^n$ for every integer $l \in [k, n^{1/k}]$.

**Remark 1.** It is reasonable to conjecture the existence of a function $f$ such that, for all sufficiently large $n$, there is a graph $S^n$ with $[f(k)n^{1+1/k}]$ edges that does not contain a $C^{2k}$; this is known to be the case for $k = 2, 3,$ and $5$ ([3], [7], [1], [8]). Therefore (at least for these values of $k$), condition (2) cannot be replaced by

$$e(G^n) > f(k)n^{1+1/k}.$$ 

In this sense our theorem is sharp.

On the other hand, if $Z^n$ is the union of approximately $(1/k)n^{1-1/k}$ complete graphs on $[kn^{1/k}]$ vertices, then

$$e(Z^n) \approx kn^{1+1/k};$$

but $Z^n$ contains no cycle of length greater than $kn^{1/k}$. Therefore, if $e(G) \approx kn^{1+1/k}$, the existence of a $C^{2l}$ in $G^n$ for $l = [kn^{1/k}]$ cannot be ensured, and this again shows the sharpness of our theorem.

**Remark 2.** In particular (for the case $k = 2$), Theorem 1 tells us that the order of magnitude of $e(G^n)$ which forces $G^n$ to contain a $C^4$ also forces $G^n$ to contain all the even cycles $C^{2l}$, $l = 2, 3, ..., 2n^{1/2}$. A similar phenomenon is established in a paper of J. A. Bondy [2], where it is shown that, if $G^n$ has enough edges to force a triangle (that is, if $e(G^n) > (n^2/4)$), then $G^n$ must contain all cycles $C^l$, $l = 3, 4, ..., (n + 3)/2$.

Theorem 1 is an easy consequence of a slightly more general theorem.

**Theorem 1*. Let $E = e(G^n)$. Then $C^{2l} \subseteq G^n$ for every integer $l \geq 2$ satisfying

$$l \leq \frac{E}{100n}, \quad ln^{1/l} \leq \frac{E}{10n}.$$ 

Besides Theorem 1, another consequence of Theorem 1* is
Theorem 2. There exists a function $g$ such that, if
\[ e(G^n) \geq g(\epsilon) n(\log n)^{1+\epsilon}, \]
then
\[ C^{2t} \subset G^n \quad \text{for every integer } l \in \left[\frac{\log n}{\epsilon \log \log n}, (\log n)^{1+\epsilon}\right]. \]  

2. Basic Lemmas

A coloring (not necessarily proper) of the vertices of a graph $G$ is $t$-periodic if the end-vertices of any (simple) path of length $t$ in $G$ have the same color.

Lemma 1. Let $t$ be a positive integer, and let $G$ be a connected graph for which
\[ e(G) > 2t v(G). \]  
Then the number of colors in any $t$-periodic coloring of $G$ is at most two.

Proof. (i) First we show that any graph $G$ with $e(G) > 2tv(G)$ contains two adjacent vertices joined by two vertex-disjoint paths, each of length at least $t$. The technique we use is due to Pósa.

In the case where each vertex has valence at least $2t$, we can find such a $t$-graph in the following way. Let a longest path in $G$ be $x_1 \ldots x_r$. Then $x_1$ is adjacent only to vertices of this path, say to $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$, where
\[ 2 = i_1 < i_2 < \cdots < i_r \text{ and } r \geq 2t. \]

The path $x_1x_2 \ldots x_{i_2t}$, together with the edges $x_{i_t}x_{i_{t+1}}$, form the desired $t$-graph.

The general case, when there may be vertices of valence less than $2t$, can now be proved by induction on $v(G)$. For $0 < v(G) \leq 4t$ it is trivial that (3) cannot be satisfied, and so there is nothing to prove here. If $v(G) = 4t + 1$, $G$ must be complete and clearly contains a $t$-graph of the desired type. Suppose now that every graph that satisfies (3) and has $k \geq 4t + 1$ vertices contains such a $t$-graph, and let $G$ be a graph on $k + 1$ vertices with some vertex $x$ of valence less than $2t$. Then
\[ e(G - x) > e(G) - 2t v(G) \geq 2tv(G) - 2t = 2tv(G - x). \]

Thus, by the induction hypothesis, $G - x$ contains a $t$-graph of the desired type and hence so also does $G$. 
(ii) Let the three cycles of such a $\theta$-graph be $C_1, C_2, C_3$ with lengths $l_1, l_2, l_3$, respectively. Clearly, the restrictions of the $t$-periodic coloring of $G$ to the $\theta$-graph and to each cycle $C_i$ are also $t$-periodic. Let $t_i$ be the smallest integer such that $C_i$ is $t_i$-periodic, $i = 1, 2, 3$. It is easy to see that any period on one cycle induces the same period on the other cycles and therefore

$$l_1 = l_2 = l_3;$$

also, $t_i \mid l_i, i = 1, 2, 3$. If $C_3$ is the longest of the three cycles, then

$$l_1 + l_2 - l_3 = 2.$$

Setting $t_i = t^*, i = 1, 2, 3$, we find that $t^* \mid 2$ and hence that $t^* = 1$ or $t^* = 2$. Therefore, the number of colors in the $\theta$-graph is at most two.

(iii) Because $G$ is connected, each vertex of $G$ is joined to some vertex of this $\theta$-graph by a path of length $k t_i$ for some integer $k$, and hence has the same color as this vertex. It follows that the number of colors in the whole graph $G$ is also at most two. This completes the proof of the lemma.

It is, in fact, easy to show that either $G$ is bipartite with the natural coloring (trivially a 2-periodic coloring), or else $G$ is unicolored.

**Lemma 2.** Let $G^n$ be a bipartite graph in which every vertex has valence at least $s = \max\{5n^{1/2}, 50l\}$. Then $G^n$ contains a $C^{2l}$.

**Proof.** Choose an arbitrary vertex $x$ of $G^n$ and let $V_i$ be the set of vertices at distance $i$ from $x$. Since $G^n$ is bipartite, each set $V_i$ is an independent set.

Suppose that $G^n$ contains no $C^{2l}$. We shall show that this implies that, for $1 \leq i \leq l$,

$$\frac{|V_i|}{|V_{i-1}|} \geq \frac{s}{5l} \quad (4)$$

thus leading to the contradiction that $\nu(G^n) > n$, (since $s \geq 5n^{1/2}$ and, consequently, $|V_i| \geq n^{1/2} |V_{i-1}|$).

We prove (4) by induction on $i$. It is trivial for $i = 1$ since the vertex $x$ has valence at least $s$. Suppose that it is true for $i - 1$. Let $H_1, H_2, ..., H_q$ be the components of the subgraph $H$ of $G^n$ induced by $V_{i-1} \cup V_i$, and let $W_j$ be the set of vertices of $H_j$ that are on level $i - 1$, that is, in $V_{i-1}$ (see Figure 1).

A path $x_1x_2 \cdots x_n$ in $G^n$ will be called *monotonic* if the distance between $x$ and $x_i$ is monotonic. (This means that a monotonic path crosses any level at most once.)
We shall show that $e(H_1) < 4e(H_1)$. This is trivial if $W_1$ has just one vertex, so assume that $W_1$ has at least two vertices. Let $a \in V_p$ be a vertex of $G^n$ such that

(i) there are two monotonic paths $P_1, P_2$ joining $a$ to $W_1$ which have just the vertex $a$ in common.

(ii) $p$ is the minimum subject to (i).

First we show that each vertex of $W_1$ is joined to $a$ by a monotonic path. For $y \in W_1$ is joined to $x$ by a monotonic path $P_3$ and, by the minimality of $p$, $P_3$ must intersect $P_1$ in some vertex $z$. The path consisting of the section of $P_3$ between $y$ and $z$ and the section of $P_1$ between $z$ and $a$ is a monotonic path from $y$ to $a$. This is illustrated in Figure 2.

We now assign colors red and blue to the vertices of $W_1$ in such a way
that, if two vertices have different colors, then they are joined to \( a \) by vertex-disjoint monotonic paths. This is done as follows. Each vertex of \( W_1 \) that can be joined to \( a \) by a monotonic path disjoint from \( P_2 \) is colored red; all other vertices of \( W_1 \) are colored blue. To see that this coloring has the required property, let \( x_1 \) and \( x_2 \) be vertices of \( W_1 \) colored red and blue, respectively, let \( P_1' \) be a monotonic path from \( x_1 \) to \( a \) disjoint from \( P_2 \), and let \( P_2' \) be a monotonic path from \( x_2 \) to \( a \). Moving along \( P_2' \) from \( x_2 \) towards \( a \), let \( v \) be the first vertex of \( (P_1' \cup P_2) - a \) encountered (see Figure 3). Because \( x_2 \) has the color blue, such a \( v \) exists; \( v \) cannot belong to \( P_1' \) for then the section of \( P_2' \) between \( x_2 \) and \( v \) together with the section of \( P_1' \) between \( v \) and \( a \) would constitute a monotonic path from \( x_2 \) to \( a \) disjoint from \( P_2 \), contradicting the assumption that \( x_2 \) is colored blue. But then \( v \in P_2 \) and we have a monotonic path \( x_2 P_2' v P_2 a \) disjoint from \( P_1' \).

We now color the vertices of \( H_1 \) in \( V \), green and show that this coloring of \( H_1 \) is \( t \)-periodic with \( t = 2l - i + p - 1 \). For, since \( i \) is even, if one end-vertex of a path of length \( t \) in \( H_1 \) is green, then so is the other. Also, there can be no path of length \( t \) joining a red and a blue vertex, because, if a red \( x_1 \) were joined to a blue \( x_2 \) by such a path, this path together with vertex-disjoint monotonic paths from \( x_1 \) to \( a \) and from \( x_2 \) to \( a \) would form a cycle of length \( 2l \). Therefore, the coloring of \( H_1 \) is indeed \( t \)-periodic. Since three colors are used in this coloring, Lemma 1 implies that

\[
e(H_1) < 2tv(H_1) < 4lt(H_1).
\]

Arguing similarly for \( H_2, \ldots, H_q \), we obtain

\[
e(H_j) < 4lt(H_j), \quad j = 1, \ldots, q,
\]
and, since the $H_j$ are the components of $H$,
\[ e(H) < 4l v(H). \]  
(5)

Let $H^*$ denote the subgraph of $G^n$ induced by $V_{i-2} \cup V_{i-1}$. Then, similarly,
\[ e(H^*) < 4l v(H^*), \]  
(6)
and, by the induction hypothesis,
\[ \frac{|V_{i-1}|}{|V_{i-2}|} \geq \frac{s}{5l}. \]  
(7)

But, clearly, since each vertex of $G^n$ has valence at least $s$,
\[ e(H) + e(H^*) \geq s |V_{i-1}|. \]

Therefore, by (5) and (6),
\[ 4l(|V_{i-1}| + |V_i| + |V_{i-2}| + |V_{i-1}|) = 4l(v(H) + v(H^*)) > e(H) + e(H^*) \geq s |V_{i-1}| \]
and so
\[ |V_i| > \frac{1}{4l} ((s - 8l) |V_{i-1}| - 4l |V_{i-2}|). \]

Using (7) we obtain
\[ |V_i| > \frac{1}{4l} \left(s - 8l - \frac{20l^2}{s}\right) |V_{i-1}| \]
and, therefore, since $s \geq 50l$,
\[ \frac{|V_i|}{|V_{i-1}|} > \frac{1}{4l} (s - 9l) > \frac{1}{4l} \cdot \frac{4s}{5} = \frac{s}{5l}, \]
as desired.

3. **Main Theorem**

We are now in a position to prove Theorem 1*. First we recall its statement.

**Theorem 1*. Let $E = e(G^n)$. Then $C^{2l} \subseteq G^n$ for every integer $l \geq 2$ satisfying
\[ l \leq \frac{E}{100n}, \quad ln^{1/n} \leq \frac{E}{10n}. \]  
(8)
Proof (by induction on $n$). For $n = 1$ the theorem is trivial, since condition (8) cannot be satisfied in this case. We now suppose that the theorem has been proved for all graphs on $n - 1$ vertices. Let $G^n$ be a graph on $n$ vertices and let $l \geq 2$ be an integer satisfying (8).

It has been shown by Erdős [6] that any graph $G$ contains a bipartite spanning subgraph $H$ with $e(H) \geq e(G)/2$; in fact $H$ can be chosen so that each vertex has valence in $H$ at least half its valence in $G$.

So let $H^n$ be such a bipartite spanning subgraph of $G^n$. If each vertex of $H^n$ has valence at least $E/2n$ then, by Lemma 2, we have that, for every integer $l$ such that

$$\max\{5ln^{1/l}, 50l\} \leq E/2n,$$

$H^n$ contains a cycle of length $2l$. Thus, in this case, Theorem $1^*$ is proved.

So suppose now that some vertex $w$ of $H^n$ has valence less than $E/2n$. By the choice of $H^n$, $w$ has valence less than $E/n$ in $G^n$. Let $G^{n-1} = G^n - w$. By the induction hypothesis, $G^{n-1}$ contains a cycle of length $2l$ for every integer $l$ satisfying

$$l \leq \frac{e(G^{n-1})}{100(n-1)}, \quad (n-1)^{1/l} \leq \frac{e(G^{n-1})}{10(n-1)}.$$

But if $l$ satisfies (8) with $G^n$, then it also satisfies (8) with $G^{n-1}$ since,

(a) if $l \leq e(G^n)/100n$, then

$$l \leq \frac{e(G^n)}{100n} = \frac{e(G^n) - e(G^n)/n}{100(n-1)} \leq \frac{e(G^{n-1})}{100(n-1)}$$

(since $w$ has valence less than $e(G^n)/n$).

(b) if $ln^{1/l} \leq e(G^n)/10n$, then

$$(n-1)^{1/l} \leq ln^{1/l} \leq \frac{e(G^n)}{10n} = \frac{e(G^n) - e(G^n)/n}{10(n-1)} \leq \frac{e(G^{n-1})}{10(n-1)}.$$

Hence $G^{n-1}$, and therefore also $G^n$, contains a cycle of length $2l$ for every integer $l$ satisfying (8). This completes the proof.

Perhaps, by other methods, Theorem $1^*$ could be improved so as to be meaningful for

$$E \geq \frac{cn \log n}{\log \log n}.$$
However, this would then be the best possible result since, if $c^*$ is small, there exists no fixed $l$ such that every graph on $n$ vertices and with $(c^* n \log n)/(\log \log n)$ edges has a cycle of length $2l$.

**Remark 3.** One can find an $l$ satisfying (8) in Theorem 1* if and only if

$$E \geq \frac{100n \log n}{\log 10}.$$  \hspace{1cm} (9)

If (9) holds, then (8) is satisfied for all values of $l$ in an interval. The upper end of this interval is $E/100n$. The lower end can be determined in the following way:

For a fixed $n$ the function $y = xn^{1/x}$ is strictly decreasing in $(0, \log n]$. Let $\phi_n(x) = y$ denote its inverse. Then $\phi_n(E/10n)$ is the lower end of our interval. $\phi_n(E/10n)$ is a transcendental function but one can easily give good approximations for it using the iteration

$$\psi_{n,1}(y) = \frac{\log n}{\log y}, \hspace{1cm} \psi_{n,k}(y) = \frac{\log n}{\log y - \log \psi_{n,k-1}}.$$  

**REFERENCES**