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SPANNING RETRACTS OF A PARTIALLY ORDERED SET

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Two general kinds of subsets of a partially ordered set P are always retracts of P: (1) every maximal chain of P is a retract; (2) in P, every isometric, spanning subset of length one with no crowns is a retract. It follows that in a partially ordered set P with the fixed point property, every maximal chain of P is complete and every isometric, spanning fence of P is finite.

1. Introduction

It is a problem of long-standing to characterize those partially ordered sets P with the *fixed point property: every order-preserving map f of P to P has a fixed point*, that is, $f(a)=a$ for some $a \in P$. Apart from the well-known result of A. Tarski [14] (cf. [9]) and A.C. Davis [5] that *a lattice has the fixed point property if and only if it is complete*, little is known. Efforts to solve the problem have, in the past, invariably concerned partially ordered sets P satisfying some "completeness" condition that requires certain distinguished subsets of P to have a supremum or infimum (cf. [1, 2, 6, 8, 15]). Recent combinatorial investigations [6, 7, 13] have highlighted the importance of *retracts* for the fixed point problem: *a partially ordered set P has the fixed point property if and only if each retract of P has the fixed point property.* (Recall, that a subset Q of P is called a *retract* of P if there is an order-preserving map f of P to P such that $f(P)=Q$ and $f \mid Q$ is the identity map of Q; in this case we call f a *retraction* of P onto Q.)

The purpose of this paper is to show that two kinds of subsets of a partially ordered set are always retracts. This information we use to derive some immediate consequences for the fixed point problem.

Theorem. *For any partially ordered set P*

(1) *every maximal chain of P is a retract and*

(2) *in P, every isometric, spanning subset of length one with no crowns is a retract.*
Therefore, if P has the fixed point property, then every maximal chain of P is complete and every isometric, spanning fence of P is finite.

2. Chains

In this section we shall establish

Theorem 1. *Every maximal chain of a partially ordered set P is a retract of P.*

Proof. Let P be a partially ordered set and let C be a maximal chain of P. For each $x \in P$ set

$$N_x = \{c \in C \mid x \text{ is noncomparable with } c\}.$$

Evidently, $N_x = \emptyset$ if $x \in C$. Conversely, if $N_x = \emptyset$, then $C \cup \{x\}$ is a chain, whence by the maximality of C, $x \in C$.

Let α denote a well ordering of the set C. Define a map f of P to C by $f(x) = x$ if $x \in C$ and, if $N_x \neq \emptyset$, $f(x)$ is the least member of N_x with respect to the well ordering α. To verify that the map f establishes C as a retract of P we need only check that f is order-preserving.

Let $u \leq v$ in P. If $u \in C$ and $v \in P - C$, then $u \leq c$ for all $c \in N_v$; in particular, $f(u) = u \leq f(v)$ since $f(v) \in N_v$. Similarly, $f(u) \leq f(v)$ if $u \in P - C$ and $v \in C$. Let $u \in P - C$ and $v \in P - C$. If $f(u) \leq v$, then $f(u) \leq c$ for all $c \in N_v$ so $f(u) \leq f(v)$. If $u \leq f(v)$, then $c \leq f(v)$ for all $c \in N_u$ so $f(u) \leq f(v)$. Therefore, we may assume that $f(u) \in N_u \cap N_v$ and $f(v) \in N_u \cap N_v$. If $f(u)$ precedes $f(v)$ with respect to α then $f(v)$ cannot be the least member of N_v with respect to α. Similarly, $f(v)$ cannot precede $f(u)$ with respect to α. It follows that $f(u) = f(v)$.

It follows easily from Theorem 1 that if Q is a retract of P and C is a maximal chain in Q, then C is a retract of P; in particular, if C is a maximal chain in an interval $[x, y]$ of P, then C is a retract of P.

3. Spanning retracts of length one

A *fence* F in a partially ordered set P is a subset $\{x_0, x_1, x_2, \ldots\}$ of P in which either

$$x_0 < x_1,\ x_1 > x_2, \ldots, x_{2m-1} > x_{2m},\ x_{2m} < x_{2m+1}, \ldots$$

or

$$x_0 > x_1,\ x_1 < x_2, \ldots, x_{2m-1} < x_{2n},\ x_{2m} > x_{2m+1}, \ldots$$

are the only comparability relations. Call *P connected* if for each $x, y \in P$ there is a finite fence containing both x and y, and define the *distance* $d_P(x, y)$ from x to y in P by

$$d_P(x, y) = \inf(|F| - 1 \mid F \subseteq P,\ F \text{ is a fence, and } x, y \in F).$$

Several facts concerning the distance function are obtained in [13]. We require the following observation. Let P and Q be partially ordered sets and let f be an order-preserving map of P onto Q. If x and y are contained in a fence in P then $f(x)$ and $f(y)$ are contained in a fence in Q and $d_Q(f(x), f(y)) \leq d_P(x, y)$; in particular, if P is connected, then Q is connected.

It is natural to call a connected subset Q of P *isometric* in P if, for each $x, y \in Q$, $d_Q(x, y) = d_P(x, y)$. The importance of this concept to the study of retracts stems from the following observation: *every connected retract of a partially ordered set P is isometric in P.*

A connected subset S of a partially ordered set P is called *spanning* in P if $|S| > 1$ and if the maximal (minimal) elements of S are maximal (minimal) elements of P. For instance, every maximal chain of P is spanning in P. Of particular concern to us shall be spanning subsets of P of *length one*, that is, connected subsets of P consisting only of maximal and minimal elements of P.

There is one final item of terminology: for $n \geq 4$, call a subset $\{c_1, c_2, \ldots, c_n\}$ of a partially ordered set P a *crown* provided that $c_1 < c_n$ and $c_1 < c_2$, $c_2 > c_3, \ldots, c_{n-2} > c_{n-1}$, $c_{n-1} < c_n$ are the only comparability relations and, in the case $n = 4$, there is no $a \in P$ such that $c_1 < a < c_2$, $c_3 < a < c_4$.

Theorem 2. *In a partially ordered set every isometric, spanning subset of length one with no crowns is a retract.*

The proof of Theorem 2 rests on a graph-theoretic result due to R. Nowakowski and I. Rival [12, Theorem 5]:

Let $G = (V_0 \cup V_1, E)$ be a bipartite graph and let $H = (W_0 \cup W_1, F)$ be a connected, isometric subgraph of G without cycles satisfying $W_0 \subseteq V_0$, $W_1 \subseteq V_1$, and $|W_0 \cup W_1| > 1$. Then H is a retract of G. Moreover, there is a retraction f [edge-preserving] *of the vertices of G to the vertices of H satisfying $f(V_0) \subseteq W_0$ and $f(V_1) \subseteq W_1$.*

With a partially ordered set P of length one we may associate a bipartite graph $G = (V_0 \cup V_1, E)$ whose vertices $V = V_0 \cup V_1$ consist of the elements of P, V_0 corresponding to the maximal elements of P and V_1 to the remaining elements, and, in which vertices x and y are adjacent if $x < y$ or $x > y$. It is now immediate that *if P is a connected partially ordered set of length one and if R is an isometric, spanning subset of P without crowns, then there is a retraction map f of P onto R such that* $f(\max(P)) \subseteq \max(R)$ *and* $f(\min(P)) \subseteq \min(R)$.

Proof of Theorem 2. Let P be a partially ordered set and let R be an isometric, spanning subset of P which is of length one and which contains no crowns. Let us assume that P and (hence) R are connected.

Since R is isometric in P, R is isometric in $\max(P) \cup \min(P)$; therefore, there is a retraction f of $\max(P) \cup \min(P)$ onto R such that $f(\max(P)) \subseteq \max(R)$ and $f(\min(P)) \subseteq \min(R)$. For each $x \in P$ set

$$U_x = \{u \in P \mid u \in \max(P),\ u \geq x\},$$
$$L_x = \{v \in P \mid v \in \min(P),\ v \leq x\}.$$

Assume that for each $x \in P$, $U_x \neq \emptyset \neq L_x$: the extension of f to P is accomplished easily. As $u \in U_x$, $v \in L_x$ imply that $f(u) > f(v)$ in R and R contains no four-crowns, either $|f(U_x)| = 1$ or $|f(L_x)| = 1$. Define a map f' of P onto R by

$$f'(x) = \begin{cases} f(x) & \text{if } x \in \max(P) \cup \min(P), \\ f(u) & \text{if } |f(U_x)| = 1 \quad \text{and} \quad u \in U_x, \\ f(v) & \text{if } |f(U_x)| > 1 \quad \text{and} \quad v \in L_x, \end{cases}$$

for $x \in P$. It is straightforward to verify that f' is a retraction of P onto R.

We must deal with the possibility that $U_x = \emptyset$ or $L_x = \emptyset$ for some $x \in P - R$. A simple artifice resolves the difficulty.

With each element x of $P - R$ associate a pair of distinct elements x_*, x^* with the aim of adjoining x_*, x^* to P prescribing $x_* < x < x^*$. More precisely, let $P^* = \{x^* \mid x \in P - R\}$, and $P_* = \{x_* \mid x \in P - R\}$ where $P \cap P^*$, $P \cap P_*$, $P^* \cap P_*$ are empty and x_*, x^*, y_*, y^* are pairwise distinct for $x \neq y$ in $P - R$. Let $P' = P \cup P_* \cup P^*$ be partially ordered by P and the comparabilities induced by the requirement $x_* < x < x^*$. Note that R is isometric and spanning in P', and for each $x \in P'$, $\{u \in P' \mid u \in \max(P'), u \geq x\} \neq \emptyset \neq \{v \in P' \mid v \in \min(P'), v \leq x\}$. Hence, as above, R is a retract of P'. *A fortiori*, R is a retract of P.

Since the preceding argument can be applied to each connected component of P, there was no loss in generality in taking P to be connected.

Which partially ordered sets R satisfy the following "universal" retract property?

If R is isomorphic to a subset of a partially ordered set P, then R is a retract of P. In fact, the answer is well-known: R satisfies this "universal" retract property if and only if R is a complete lattice [4] (cf. [3], [6]). Theorem 2 suggests a related question:

For which partially ordered sets R of length one is it true that R is a retract of P whenever R is isomorphic to an isometric, spanning subset of P?

The answer is close at hand: *precisely* those partially ordered sets R which contain no crowns. It shall suffice to illustrate this fact by some examples. In Fig. 1 we have illustrated the case in which $R = \{c_1, c_2, \ldots, c_n\}$ is a crown; in each instance, R is not a retract.

4. Fixed points

Theorems 1 and 2 yield necessary conditions for the fixed point property in an arbitrary partially ordered set.

Theorem 3. *Let P be a partially ordered set with the fixed point property. Then every maximal chain of P is complete and every isometric, spanning fence of P is finite.*

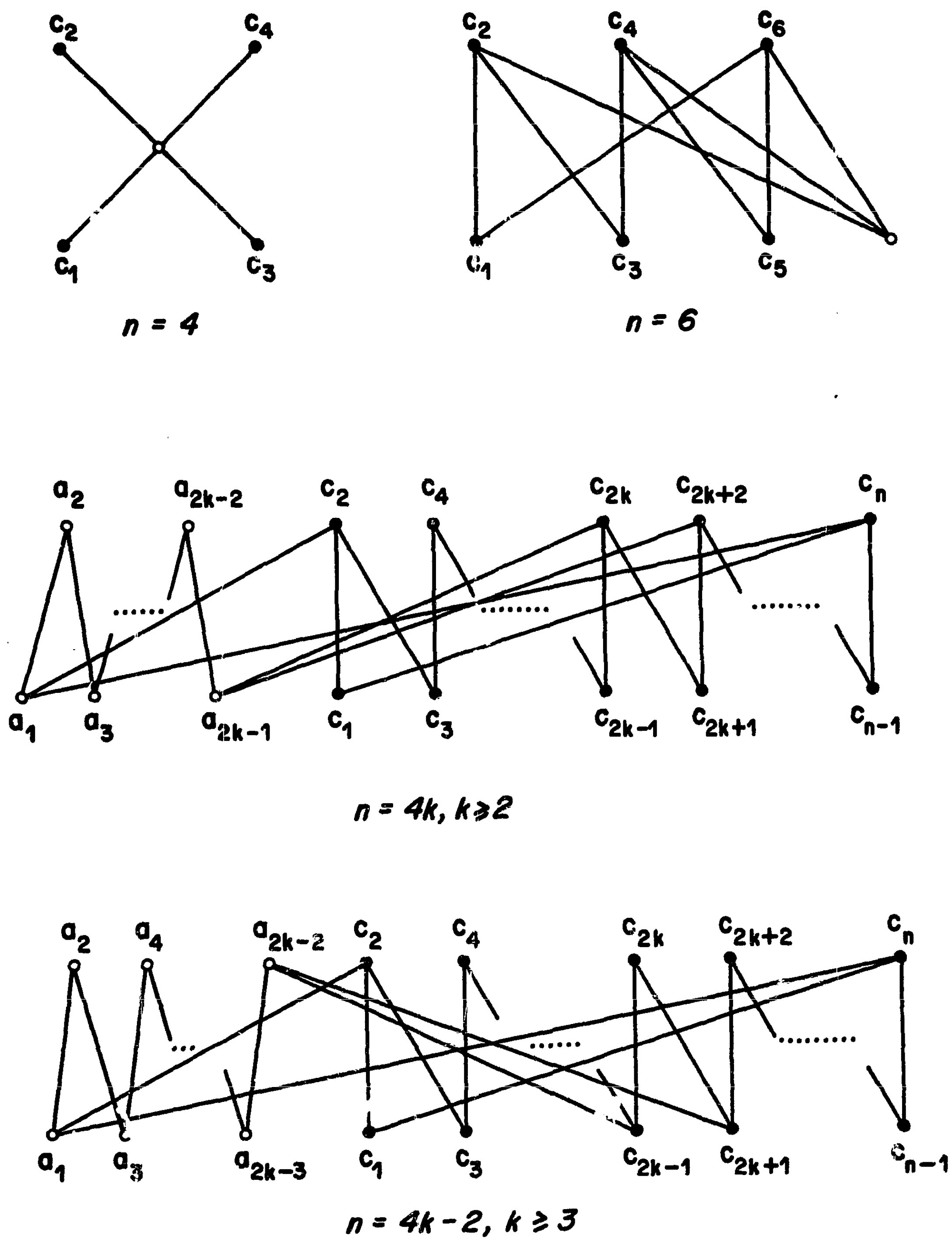

Fig. 1.

Proof. Every maximal chain of P and every isometric, spanning fence of P is a retract of P; therefore, each must have the fixed point property. As chains are lattices, chains with the fixed point property are complete. It is easily seen that infinite fences have fixed point free maps, hence, every isometric, spanning fence of P is finite.

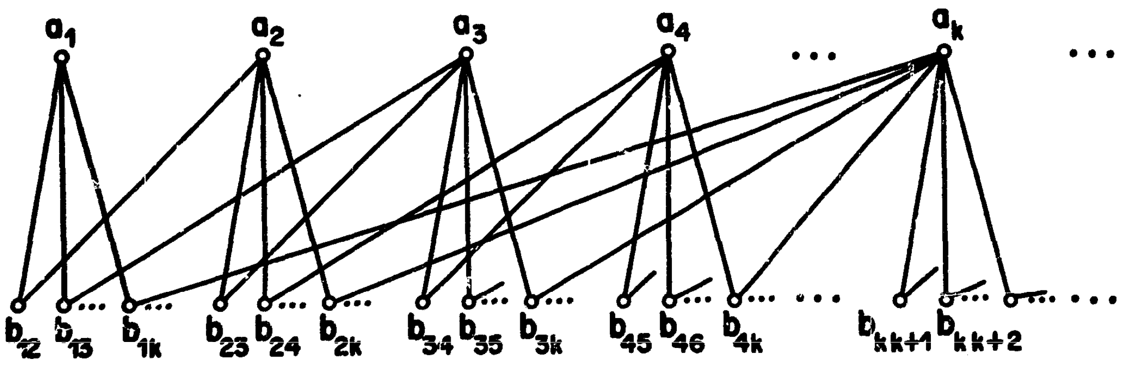

Fig. 2.

Most fixed point theorems have furnished sufficient conditions for the fixed point property (cf. [1, 2, 8, 10, 14]). Theorem 3 supplies two interesting necessary conditions, although it does not provide a solution to the fixed point problem. For instance, let us consider the partially ordered set P consisting of elements a_i and b_{ij}, where $i = 1, 2, \ldots, j = 1, 2, \ldots$, and $i < j$, with comparabilities prescribed by $a_i > b_{jk}$ if either $i = j$ or $i > j$ and $i = k$ (its diagram is illustrated schematically in Fig. 2). Then P has length one (so every maximal chain is complete) and P does contain infinite spanning fences, yet none that is isometric. In fact, the largest isometric spanning fence has only four elements. Moreover, it is easy to construct an order-preserving map of this partially ordered set to itself which fixes no element.

The partially ordered set P depicted in Fig. 3 displays an additional complication. Every maximal chain of P is complete, every isometric, spanning fence of P

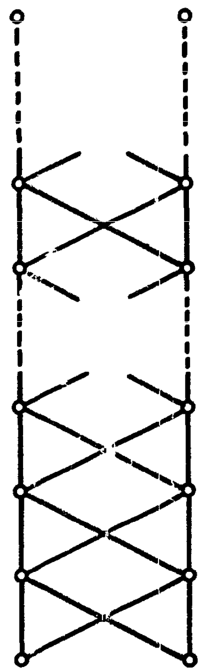

Fig. 3.

is finite, *every finite retract of P has the fixed point property* and, yet, P is fixed point free. While an approach to the fixed point problem that is based on a "compactness" result (reducing the problem to its finite case) seems plausible, such a result would not likely be an easy extension of Theorem 3.

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