Véges geometriák szerepe az Extremális Gráfelméletben

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Angol-Magyar előadás-kiegészítő: Az eredeti "abstract" szövege magyar, a további hozzáírások angolul. Ezek nem slide-ok, ez csak egy kiegészítő!

References

[1] M. Ajtai, J. Komlós, and E. Szemerédi: 'An $O(n \log n)$ sorting network', Proc. 15th Ann. ACM Symp. on Theory of Computing (ACM, New York, USA, 1983) 1–9.

[2] C. Berge, M. Simonovits: The colouring numbers of the direct product of two hypergraphs, Lecture Notes in Mathematics, vol. 411, Hypergraph Seminar, Columbus, OH, 1972 (1974), pp. 21–33.

[3] J.A. Bondy and M. Simonovits: Cycles of even length in graphs, J. Combin. Theory Ser. B **16** (1974) 97–105. **MR**0340095

[4] William G. Brown,: On graphs that do not contain a Thomsen graph, Can. Math. Bull. **9** (1966), 281–285. **MR**0200182

[5] W.G. Brown, P. Erdős, and V.T. Sós: 'On the existence of triangulated spheres in 3-graphs and related problems', Period. Math. Hungar. **3** (1973) 221–228.

[6] W.G. Brown, P. Erdős, and V.T. Sós: Some extremal problems on r-graphs, in "New Directions in the Theory of Graphs, Proc. 3rd Ann Arbor Conf. on Graph Theory, University of Michigan, 1971," pp. 53–63, Academic Press, New York, (1973).

[7] P. Erdős, A. Rényi, and V.T. Sós: On a problem of graph theory. Stud. Sci. Math. Hung. **1** (1966) 215–235 (Also see [8], 201–221) **MR**0223262

[8] P. Erdős: The Art of Counting. Selected writings of P. Erdős, (J. Spencer ed.) Cambridge, The MIT Press (1973). **MR**0532670

[9] Z. Füredi and M. Simonovits: The history of degenerate (bipartite) extremal graph problems, in Erdős Centennial, Bolyai Soc. Math. Stud., **25,** János Bolyai Math. Soc., Budapest, (2013) 169–264. arXiv:1306.5167. **MR**3203598

[10] Tamás Kővári, V.T. Sós, and P. Turán: On a problem of Zarankiewicz, Colloq Math. 3 (1954), 50–57. MR 0065617

[11] A. Lubotzky, Ralph Phillips, and P. Sarnak: Ramanujan Conjecture and explicit construction of expanders (Extended Abstract), Proceedings of the STOC (1986), pp. 240–246.

[12] A. Lubotzky, R. Phillips, and P. Sarnak: Ramanujan graphs, Combinatorica **8** (1988), 261–277. MR 0963118

[13] G.A. Margulis: Explicit constructions of graphs without short cycles and low density codes, Combinatorica **2** (1982), 71–78. MR 0671147

[14] István Reiman: Über ein Problem von K. Zarankiewicz, Acta Math. Acad. Sci. Hungar. **9** (3-4) (1958), 269–273. MR 0101250

[15] I. Reiman: An extremal problem in graph theory, (Hungarian). Mat. Lapok 12 (1961), 44–53.

[16] M. Simonovits: Note on a hypergraph extremal problem, in: C. Berge, D. Ray-Chaudury (Eds.), Hypergraph Seminar, Columbus, Ohio, USA, (1972), Lecture Notes in Mathematics, Vol. 411, Springer, Berlin, (1974), pp. 147–151. MR 0366740

1. Introduction

A véges geometriai konstrukciók alapvető fontosságúak az extremális gráfelméletben. Az előadásban elsősorban a Turán típusú extremális problémákhoz tartozó konstrukciókról fogok beszélni, a szokásosnál informálisabban, bizonyos megoldatlan problémákkal kapcsolatban.

Probléma 1 (Turán típusú problémák)**.** *Adott a kizárt gráfok egy $\mathcal{L}$ osztálya, határozzuk meg, vagy legalábbis becsüljük meg $\mathbf{ex}(n,\mathcal{L})$-et.*

Theorem 1.1 (Kővári–T. Sós–Turán, [10])**.** *Let $K_{a,b}$ denote the complete bipartite graph with a and b vertices in its color-classes. Then*

$$\mathbf{ex}(n, K_{a,b}) \le \frac{1}{2}\sqrt[a]{b-1}\cdot n^{2-(1/a)} + O(n). \tag{1}$$

Conjecture **1.** *The exponent in (1) is sharp.*

Az extrém gráfoknál általában szükségünk van egy felső becslésre, illetve egy alsó becslésre, ahol a ”degenerált” esetben véletlen, vagy geometriai vagy algebrai konstrukciókat használunk.

The “random constructions” do not help in such cases.

CONSTRUCTION **1** (E. Klein/ Erdős).

$$\mathbf{ex}(n, C_4) > cn\sqrt{n}.$$

THEOREM **1** (Brown [4]/ Erdős-Rényi-Sós [7]).

$$\mathbf{ex}(n, C_4) = \left(\frac{1}{2} + o(1)\right) n\sqrt{n}.$$

1.1. Connection to Unit distance problems

Lenz Construction. *In $\mathbb{E}^4$ there is a $K(\infty, \infty)$.*

2. Brown Construction

Reiman Construction. *Sharp constructions.*

CLAIM **1** (Erdős: Unit distances in $\mathbb{E}^2$, $\mathbb{E}^3$).

CONSTRUCTION **2** (W. G. Brown [4]).

$$\mathbf{ex}(n, K(3,3)) > \left(\frac{1}{2} - o(1)\right) n^{2-\frac{1}{3}}.$$

THEOREM **2** (Füredi).

$$\mathbf{ex}(n, K(3,3)) = \left(\frac{1}{2} + o(1)\right) n^{2-\frac{1}{3}}.$$

2.1. Kollár-Rónyai-Szabó-Alon

THEOREM **3** (Alon-Rónyai-Szabó). *If $b > (a-1)!$, then*

$$\mathbf{ex}(n, K(a,b)) \geq c_a n^{2-\frac{1}{a}}.$$

3. Benson Construction

Probléma 2 (C_8 problémája)**.** *Igaz-e* $\mathbf{ex}(n, C_8) > c_1 n^{1+\frac{1}{4}}$ *valamilyen* $c_1 > 0$*-ra?*

Probléma 3 (C_{2k} problémája)**.** *Igaz-e* $\mathbf{ex}(n, C_{2k}) > c_k n^{1+\frac{1}{k}}$ *valamilyen* $c_k > 0$*-ra?*

4. Rational exponents

Conjecture **2** (Rational exponents)**.** *Is it true that for any Degenerate extremal problem for* $\mathcal{L}$ *there exist a rational* α *such that*

$$\frac{\mathbf{ex}(n, \mathcal{L})}{n^{1+\alpha}} \to c_{\mathcal{L}} > 0$$

for some $c_{\mathcal{L}} > 0$*?*

Motivation. *If there is a geometric construction, where the vertices are from a finite geometry and the connections are according to some polynomials, then the exponent will be "rational".*

Remark 1. *The "rational exponent" conjecture does not hold for hypergraphs. The Ruzsa-Szemerédi theorem is a counter-example.*

5. Compactness problems

Conjecture **3** (Erdős)**.**

$$\mathbf{ex}(n, \{C_3, C_4\}) = \left(\left(\frac{1}{2}\right)^{3/2} + o(1)\right) n^{3/2}.$$

Theorem **4** (Erdős)**.**

$$\mathbf{ex}(n, \{C_5, C_4\}) = \left(\left(\frac{1}{2}\right)^{3/2} + o(1)\right) n^{3/2}.$$

5.1. Asymmetric versions?

Consider bipartite (RED-BLUE) graphs $G(A, B)$. We may exclude only one position, where the RED vertices of L are in A, or we may exclude the case, when both positions are excluded.

$\mathbf{ex}_B(n, L)$ means that we exclude only one of the two positions.

Conjecture **4.** *Let L be a bipartite excluded graph. We know that*

$$\mathbf{ex}_B(n, L) \geq\sim \mathbf{ex}(n, L).$$

Is it true that

$$\mathbf{ex}_B(n, L) \leq c_B \mathbf{ex}(n, L)?$$

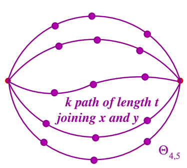
5.2. Győri Lemma

Theorem **5** (Győri)**.** *Any C_6-free graph G_n contains a $\{C_4, C_6\}$-free subgraph H_n with $e(H_n) > \frac{1}{2}e(G_n)$.*

The general conjecture:

Theorem **6.** *Any C_{2k}-free graph G_n contains a $\{C_4, C_6, \ldots, C_{2k}\}$-free subgraph H_n with $e(H_n) > c_k e(G_n)$, for some $c_k > 0$.*

5.3. $\Theta_{k,\ell}$-free graphs



5.4. Cube-free graphs

Theorem **7** (Erdős-Sim)**.**

$$\mathbf{ex}(n, Q_8) = O(n^{8/5}).$$

Conjecture **5** (Erdős-Sim)**.**

$$\mathbf{ex}(n, Q_8) > c_Q n^{8/5}.$$

No reasonable result for the 4-dimensional cube.

6. Hypergraphs and Finite Geometries

PROBLEM **1** (Erdős). *Is it true that if G_n is 4-critical, then $d_{\min}(G_n) = o(n)$?*

Toft, Simonovits, double cone, triple cone? [16]

6.1. Brown-Erdős-Sós papers

[5] [6]
r-uniform hypergraphs, $\mathcal{L}_{6,3}$
Independently, 2 important papers of Brown, Erdős, and Sós: []

7. Girth problems, Ramanujan graphs

Amennyiben időm engedi, kitérek egy ide lazán kapcsolódó konstrukció-csoportról is, a "Kis-Margulis" konstrukcióról, illetve ennek messzemenő általánosításáról, a Margulis-Lubotzky-Philips-Sarnak konstrukciókról: itt nem geometriákat használunk, hanem Cayley gráfokat.

THEOREM **8** (Lemma). *If $d_{\min}(G_n) \geq 3$, then* $\textbf{girth}(G_n) = O\left(\frac{\log n}{\log(d-1)}\right)$.

THEOREM **9** (Erdős-Pósa). *If G_n does not contain $k+1$ independent cycles, then the cycles can be represented by $ck \log k$ vertices.*

CONSTRUCTION **3** (Margulis [13]). *Consider the $\mod p$ 2×2 matrices with determinant 1. There exist two "independent" 2×2 matrices, consider the corresponding Cayley graph. It has p^3 vertices and is 4-regular.*

Here the shortest cycle has length $c \log n$.

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Let X denote the set of all 2×2 matrices with integer entries and with determinant 1. Pick the following two matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It is known that they are independent in the sense that there is no non-trivial multiplicative relation between them. So, if we take the 4 matrices A, B, A^{-1} and B^{-1}, they generate an infinite Cayley graph which is a 4-regular tree. If we take everything mod p, then it is easy to see that the tree collapses into a graph of $n \approx p^3$ vertices, in which the shortest cycle has length at least $c \log p$ for some constant $c > 0$. This yields a sequence of 4–regular graphs X_n with girth $\approx c^* \log n$ for some $c^* \approx 0.91...$

Theorem 7.1 (Margulis [?]). *For every $\varepsilon > 0$ we have infinitely many values of r, and for each of them an infinity of regular graphs X_j of degree $2r$ with girth*

$$g(X_j) > \left(\frac{4}{9} - \varepsilon\right) \frac{\log v(X_j)}{\log r}.$$

Margulis also explained, how the above graphs can be used in constructing certain (explicit) error-correcting codes and generalized this construction (in the same paper) to arbitrary even degrees.

7.1. Why are the Margulis-Lubotzky-Phillips-Sarnak graphs important?

Often we can replace constructions by "Random Constructions", however sometimes it may be important to replace the Random Constructions by Deterministic Constructions. In such cases the Expander Graphs are often very useful. See e.g. Ajtai-Komlós-Füredi Sorting network [1].

The random almost-regular graphs with degree $\approx d$ have a second eigenvalue aproximately $\sqrt{d}$.

Remark **1** (The Ramanujan graphs).

Lubotzky-Phillips-Sarnak [11] [12] results were needed to construct Ramanujan graphs. As it is mentioned in [12], Alon pointed out that they are important in Extremal Problems, too.