Regularity Lemma and its applications

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The PLAN (?)

- Generalized Random Graph Sequences
- ε-regular pairs
- Generalized Quasi-Random Sequences

- The Szemerédi Regularity Lemma
- Why do we like Szemerédi Regularity Lemma?
- Cluster graph

- Applications: **RT**$(n, K_4) \leq \frac{1}{8}n^2 + o(n^2)$
- Removal Lemma
- Ruzsa-Szemerédi Theorem, and its importance

- **The plan was** to prove $r_k(n) = o(n^2)$, using Extremal Hypergraph Theory
- but Tim Gowers cam!

Extensions

- Sparse Regularity Lemmas: KOHAYAKAWA-RÖDL
- Weak Hypergraph Regularity Lemmas: FRANKL-RÖDL
- Strong Hypergraph Regularity Lemmas:
  RÖDL-NAGLE-SKOKAN-SCHACHT TIM GOWERS
- and some newer ones, TAO ...

Skipping, among others:

- Algorithmic aspects
- Connections to Property testing
- Weak Regularity Lemma FRIEZE-KANNAN, . . .

Szemerédi Regularity Lemma

- Origins/connections to the existence of arithmetic progressions in dense sequences
- Connection to the quantitative ERDŐS-STONE theorem
- First graph theoretic applications (RUZSA-SZEMERÉDI theorem, Ramsey-Turán problems)
- **Counting lemma**, **removal lemma**, coloured regularity lemma

Regularity Lemma, Regular Partitions

Given G, with X and Y, the edge-density between X and Y is

$$d(X,Y) := \frac{e(X,Y)}{|X||Y|}.$$

Regular pairs

Regular pairs are highly uniform bipartite graphs, namely ones in which the density of any reasonably sized subgraph is about the same as the overall density of the graph.

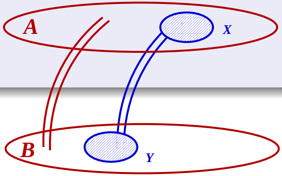
Definition (ε-regular set-pairs)

Let $\varepsilon > 0$. Given a graph G and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair (A, B) is ε-regular if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \text{ and } |Y| > \varepsilon|B|$$

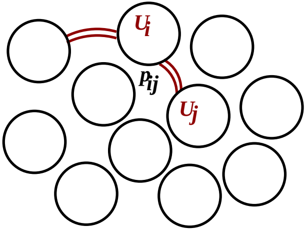
we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$



In random graphs this holds for large disjoint vertex sets

Generalized random graphs



Given a probability matrix $A := (p_{ij})_{r\times r}$ and integer $n_1, \ldots, n_r$.

- We choose the subsets $U_1, \ldots, U_r$ and join $x \in U_i$ to $y \in U_j$ with probability p_{ij} **independently**.

- Regularity Lemma: the graphs can be approximated by generalized random graphs well.

The Regularity Lemma

The Regularity Lemma says that

- every dense graph can be partitioned into a small number of regular pairs and a few leftover edges.
- Since regular pairs behave as random bipartite graphs in many ways, the R.L. provides us with an approximation of an arbitrary dense graph with the union of a constant number of random-looking bipartite graphs.

Regularity Lemma

Theorem (Szemerédi, 1978)

For every $\varepsilon > 0$ and m there are $M(\varepsilon, m)$ and $N(\varepsilon, m)$ with the following property: for every graph G with $n \geq N(\varepsilon, m)$ vertices there is a partition of the vertex set into k classes

$$V = V_1 + V_2 + \ldots + V_k$$

such that

- $m \leq k \leq M(\varepsilon, m)$,
- $||V_i| - |V_j|| < 1$, $(1 \leq i < j \leq k)$
- *all but at most* εk^2, *of the pairs* (V_i, V_j) *are* ε*-regular.*

See

→ SzemRegu, KomSim

The role of m

is to make the classes V_i sufficiently small, so that the number of edges inside those classes are negligible. Hence, the following is an alternative form of the R.L.

Theorem (Regularity Lemma – alternative form)

For every $\varepsilon > 0$ there exists an $M(\varepsilon)$ such that the vertex set of any n-graph G can be partitioned into k sets $V_1, \ldots, V_k$, for some $k \leq M(\varepsilon)$, so that

- $|V_i| \leq \lceil \varepsilon n \rceil$ *for every i,*
- $||V_i| - |V_j|| \leq 1$ *for all i, j,*
- (V_i, V_j) *is ε-regular in G for all but at most εk^2 pairs (i, j).*

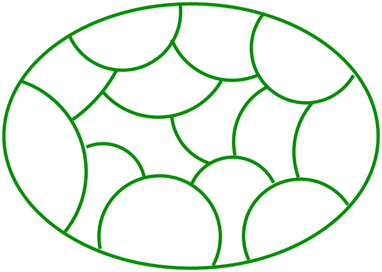
For $e(G_n) = o(n^2)$, the Regularity Lemma becomes trivial.

How to prove Regularity Lemma?

- Use the Defect form of Cauchy-Schwarz.
- Index:

$$I(\mathcal{P}) = \frac{1}{k^2} \sum d(V_i, V_j)^2 < \frac{1}{2}.$$

- Improving the partition



Defect form of the Cauchy-Schwarz

Lemma (Improved Cauchy-Schwarz inequality)

If for the integers $0 < m < n$,

$$\sum_{k=1}^{m} X_k = \frac{m}{n} \sum_{k=1}^{n} X_k + \delta,$$

then

$$\sum_{k=1}^{n} X_k^2 \geq \frac{1}{n} \left(\sum_{k=1}^{n} X_k \right)^2 + \frac{\delta^2 n}{m(n-m)}.$$

Coloured Regularity Lemma

- If we have several colours, say, Black, Blue, Red, then we have a Szemerédi partition good for each colour simultaneously.
- How to apply this?

Counting Lemma

Through a simplified example:

- If the Generalized Random Graph corresponding to G_n contains many copies of L, then G_n also contains many (approximately thesame number of copies of L)
- If the reduced graph contains an L then G_n contains at least $cn^{v(L)}$ copies of L.

Clusters, Reduced Graph

The classes V_i will be called groups or clusters.
Given an arbitrary graph $G = (V, E)$, a partition P of the vertex-set V into $V_1, \dots, V_k$, and two parameters ε, d, we define the Reduced Graph (or Cluster Graph) R as follows: its vertices are the clusters $V_1, \dots, V_k$ and V_i is joined to V_j if (V_i, V_j) is ε-regular with density more than d.

Most applications of the Regularity Lemma use Reduced Graphs, and they depend upon the fact that many properties of R are inherited by G.

Inheritance

G_n inherits the properties of the cluster graph H_k.

- sometimes in an improved form!

Through a simplified example:

- If H_k contains a C_7 then G_n contains many: cn^7.

Ramsey-Turán problems

Theorem (Szemerédi)

→ SzemRT *If* G_n *does not contain* K_4 *and* $\alpha(G_n) = o(n)$ *then*

$$e(G_n) = \frac{n^2}{8} + o(n^2).$$

How to prove this?

- Use Regularity Lemma
- Show that the reduced graph does not contain K_3.
- Show that the reduced graph does not contain

$$d(V_i, V_j) > \frac{1}{2} + \varepsilon$$

Szemerédi-Ruzsa

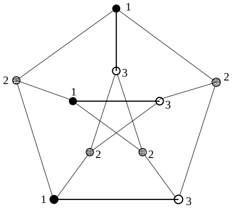
$f(n, 6, 3)$

Removal Lemma

Originally for K_3, Ruzsa-Szemerédi

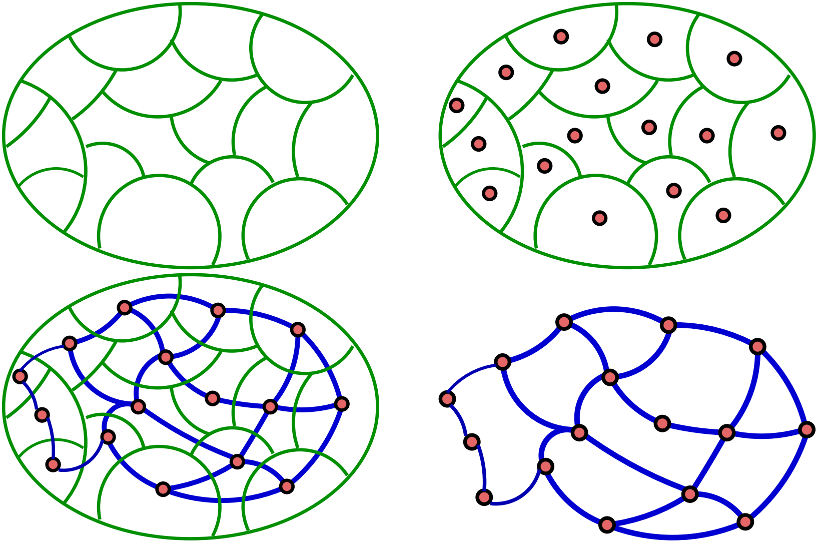
Generaly: through a simplified example:

For every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) \to 0$, $\delta > 0$:
If a G_n does not contain δn^{10} copies of the Petersen graph, then we can delete εn^2 edges to destroy all the Petersen subgraphs.



- something similar is applicable in Property testing.

The Cluster graph, illustrated:



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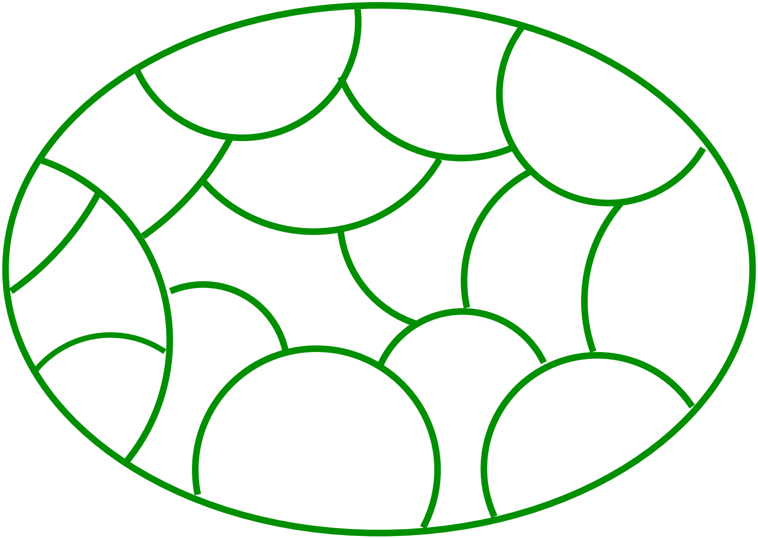

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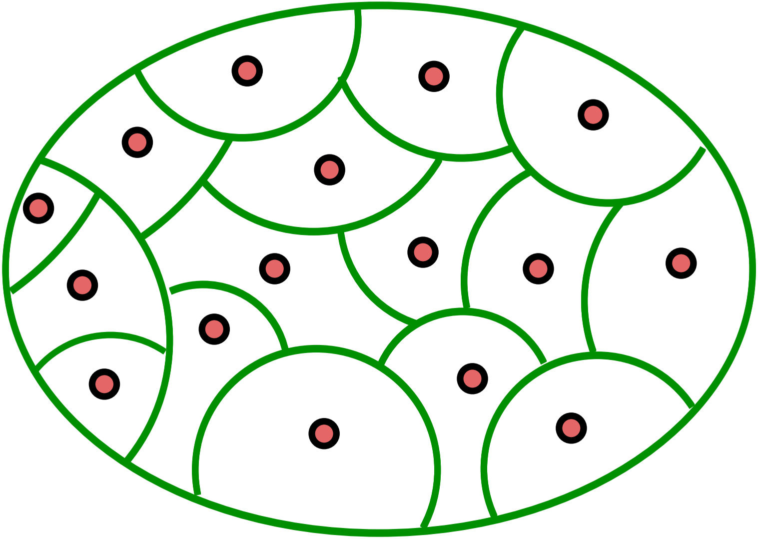

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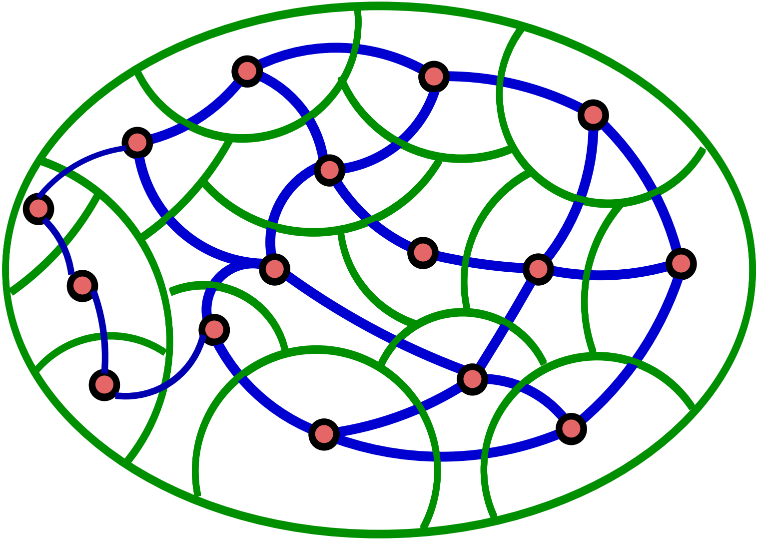

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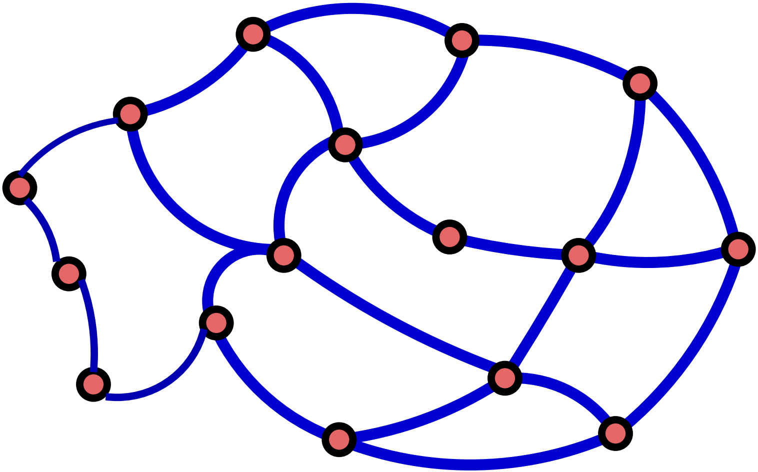
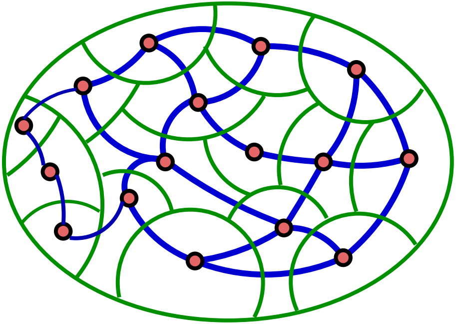


Encoding? Logarithm? Generating function?

- Original Graph, satisfying some $\mathcal{P}$.
- Cluster graph H_k satisfying some $\mathcal{P}'$ and having proportionally many edges
- Solving the corresponding problem for H_k
- Translating the result for G_n

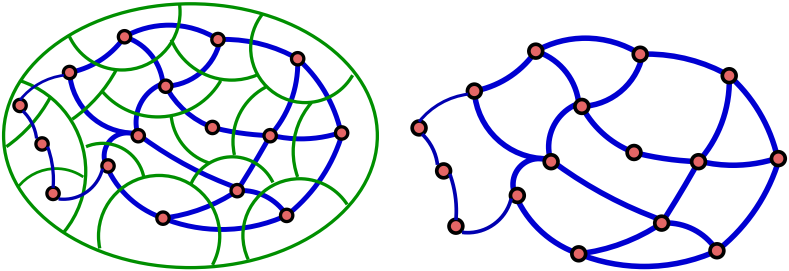
How to prove Erdős-Stone?



- No K_{p+1} in the Reduced graph H_k
- Apply Turán's theorem
- Estimate the edges of the original graph:

$$e(G_n) \leq e(H_k)m^2 + 3\varepsilon n^2.$$

How to prove Stability?



- No K_{p+1} in the Reduced graph H_k
- Apply Turán's theorem with stability (Füredi)
- Estimate the edges of the original graph

Babai-Sim.-Spencer

The largest K_3 subgraph of a Random graph is its largest bipartite subgraph

...

de Marco–Jeff Kahn

Various Regularity Lemmas

- Original Ugly
- Original Nice
- Weak Regularity Lemma
  — Frieze-Kannan
  — Connections to Statistical approach
- Weak Hypergraph Regularity
- Good Hypergraph Regularity: Rödl,... Schacht, Gowers

How to get rid of Regularity Lemma?

- and why????

- The thresholds are too large
- But Regularity Lemma often makes the things transparent

— See Luczak: Odd cycle Ramsey

Blowup Lemma

Komlós, G. Sárközy, Szemerédi: → KSSzBlowUp

Good to prove the existence of spanning subgraphs

- Pósa-Seymour conjecture,. . .

(A, B) is (ε, δ)-**super-regular** if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \text{ and } |Y| > \varepsilon|B|$$

we have

$$e(X, Y) > \delta|X||Y|,$$

and

$$deg(a) > \delta|B| \text{ for all } a \in A,$$

$$\text{and } deg(b) > \delta|A| \text{ for all } b \in B.$$

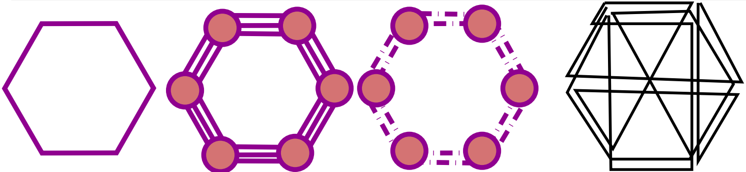
Blowup Lemma II

Theorem

Given a graph R_r and $\delta, \Delta > 0$, there exists an $\varepsilon > 0$ such that the following holds. N = arbitrary positive integer,

- *replace the vertices of R with pairwise disjoint N-sets $V_1, V_2, \dots, V_r$.*
- *Construct two graphs on the same $V = \cup V_i$. $R(N)$ is obtained by replacing all edges of R with copies of $K_{N,N}$,*
- *and a sparser graph G is constructed by replacing the edges of R with (ε, δ)-super-regular pairs.*

If H with $\Delta(H) \le \Delta$ is embeddable into $R(N)$ then it is already embeddable into G.



Other Regularity Lemmas

- FRIEZE-KANNAN

Background in statictics, more applicable in algorithms

- LOVÁSZ-B. SZEGEDY: Limit objects, continuous version
- ALON-FISCHER-KRIVELEVICH-M. SZEGEDY:

Used for property testing

- ALON-SHAPIRA: property testing is equivalent to using Regularity Lemma

Szemerédi's Lemma for the Analyst

This is the title of a paper of L. Lovász and B. Szegedy
Hilbert spaces, compactness, covering

Hypergraph regularity lemmas

- Frankl-Rödl
- Frankl-Rödl 2.
- F. Chung
- A. Steger
- Rödl, Skokan, Nagle, Schacht,. . .
- Gowers, Tao,. . .

Many thanks for your attention.