NOTE ON A HYPERGRAPH EXTREMAL PROBLEM

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Introduction. We shall consider 3-uniform hypergraphs "without loops or multiple edges." This means that we shall consider a set \( X \) which will be called the vertex-set of the hypergraph and a set of unordered triples from \( X \) called the triples of the hypergraph. The expression "without loops" means that each triple has 3 different elements and the expression "without multiple edges" means that each triple can occur at most once in the set of edges.

Problem. Let \( L \) be a family of hypergraphs. What is the maximum number of triples a hypergraph on \( n \) vertices can have if it does not contain a subhypergraph isomorphic to some members of \( L \)? For a given finite or infinite family \( L \), the problem asked above will be called an extremal problem; the maximum will be denoted by \( \text{ext}(n ; L) \); the members of \( L \) will be called sample hypergraphs, and the hypergraphs attaining the maximum generally are called extremal hypergraphs.

Definition. The \( r \)-pyramid \( L_r^t \) based on a polygon of \( t \) vertices is the hypergraph defined on the \( r + t \) vertices \( x_1, \ldots, x_r; y_1, \ldots, y_{t-1}, y_t, y_{t+1} = y_1 \) and having the triples
\[
(x_i, y_j, y_j+1) \quad (i=1, \ldots, r; j=1, \ldots, t)
\]
Further, \( L_r \) is the family of all the \( r \)-pyramids \( L_r^t \), \( t = 2, 3, \ldots \).

Theorem. \( \text{ext}(n ; L_3) > \left( \frac{1}{6} + O(1) \right) n^{8/3} \).

(This means that there are hypergraphs with \( n \) vertices and almost \( \frac{1}{6} n^{8/3} \) triples which do not contain any 3-pyramid.)

Remarks. 1) In \([1]\) we needed and proved the following lemma:

\[
(1) \quad \text{ext}(n ; L_r) = \Theta(n^3 - \frac{1}{r}), \quad r = 1, 2, 3, \ldots
\]

This lemma was needed to prove that a 4-colour-critical hypergraph cannot have too many independent vertices. As a matter of fact, we needed this result only for \( r = 2 \). At the same time W. Brown, P. Erdos and V. T. Sos, \([1]\) among some other hypergraph extremal theorems proved that if \( T \) is the family of sample hypergraphs
obtained from the triangulations of the 3-sphere, then

\[ \text{ext}(n ; T) = 0(n^3 - \frac{1}{2}) \]

They used the fact that \( T \) contains \( L_2 \), proved (1) for \( r = 2 \), from which (2) followed trivially. Using a finite geometrical construction, they also proved that (2) is sharp, i.e. the exponent is the best possible. We used a so-called probabilistic argument to prove the weaker assertion that (1) is sharp for \( r = 2 \). The main purpose of this paper is to prove that (1) is sharp for \( r = 3 \) as well.

2) There is a result of Kovary, Turan and T. Sos [3] asserting that if \( K_2(p, q) \) is the complete bipartite graph with \( p \) and \( q \) vertices in its first and second classes respectively, then

\[ \text{ext}(n ; K_2(p, q)) = 0(n^2 - \frac{1}{p}) \quad (p \leq q) \]

It can be conjectured that this result is the best possible; however, this is not proved except for \( p = 1, 2, \) and \( 3 \). If we can prove that (1) is sharp for \( r = 4 \), then it will follow that (3) is also sharp for \( p = 4 \). This suggests that it will be difficult to prove the sharpness of (1) for \( r = 4 \).

The construction. The construction given below will be based on the construction of W. G. Brown [4], showing that (3) is sharp for \( p = 3 \). In view of the second remark, this is not "surprising" at all. First we define the graph of Brown. Let \( p \) be an odd prime and the vertices of our graph be the points in the 3-dimensional affine space over the field \( GF(p) \), i.e. over the field of residues mod \( p \).

Let us join two points \( x \) and \( y \) by an edge if

\[ \sum_{i=1}^{3} (x_i - y_i)^2 = a \]

where \( a \) is a quadratic residue or non-residue depending on whether \( p \) has the form \( 4s + 3 \) or \( 4s + 1 \), \( a \neq 0 \) and is constant for a given graph. According to a well-known theorem of Lebesgue [5, p. 325] the valences of each vertex will be \( p^2 - p \) and, as Brown proves, this graph never contains a \( K_2(3,3) \). Let us join 3
points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in the graph of Brown by a triple if

$$
3 \sum_{i=1}^{3} (x_i + y_i + z_i)^2 = a.
$$

This hypergraph contains some 3-pyramids of very special "position". We omit a few triples and prove that the obtained hypergraph does not contain 3-pyramids. Let us denote the graph of Brown by $\mathcal{B}$, the hypergraph defined above by $\mathcal{A}$ and let $U$ be the hypergraph obtained from $\mathcal{A}$ by omitting all those vertices $\mathbf{x} = (x_1, x_2, x_3)$ for which

$$
\text{at least one of } x_1, x_2, x_3 \text{ vanishes}.
$$

Of course we omit all the triples at least one vertex of which has already been omitted. Since in $U$ each coordinate can be chosen in $p - 1$ different ways, $U$ has

$$
(p - 1)^3
$$

vertices. For every edge $(\mathbf{x}, \mathbf{c})$ of $\mathcal{B}$, there are $\binom{p^3}{2}$ pairs $(\mathbf{y}, \mathbf{z})$ such that $\mathbf{y} + \mathbf{z} = \mathbf{c}$, $\mathbf{y} \neq \mathbf{z}$, $\mathbf{y} \neq \mathbf{x}$, $\mathbf{z} \neq \mathbf{x}$. These $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ triples will belong to $\mathcal{A}$ and each triple of $\mathcal{A}$ can be counted only 3 times this way, so $\mathcal{A}$ has $\binom{p^3}{2} = \binom{p^3}{1}$ triples and each vertex of $\mathcal{A}$ has the valence $\binom{p^3}{2} + O(p^5)$. Since we omitted only $O(p^2)$ vertices, the number of triples in $U$ is

$$
\binom{p^3}{2} + O(p^7)
$$

We shall prove that $U$ does not contain 3-pyramids.

Let us suppose that the vertices $x_i$ ($i = 1, 2, 3$) and $y_j$ ($j = 1, \ldots, k$) define an $L_3$ in $U$. The triples of this $L_3$ in $U$ are the triples $(x_i, y_j, y_{j+1})$. Let $w_j = - (y_j + y_{j+1})$. According to (4) and (5) the vertices $x_i$ are joined to the vertices $w_j$ in the graph $\mathcal{B}$. We know that the vertices $x_i$ are all different and that the graph $\mathcal{B}$ does not contain a complete bipartite graph with 3 vertices in each class. Hence there are at most $2^3$ different vertices among $w_1, \ldots, w_k$. On the other hand

$$
-(w_j - w_j - 1) = y_j + y_{j+1} - y_j - 1 - w_j = y_j + 1 - y_j - 1 - 0.
$$
This gives that the set \( \{ x_j : j = 1, \ldots, k \} \) has exactly 2 elements and every second element of it is the same. Therefore

\[ k = 2m \quad \text{and} \quad w_{2j} = u, \quad w_{2j-1} = v \quad (j = 1, \ldots, m), \quad u \neq v. \]

Let us notice that from (9), (10) and \( X_{2m} + 1 = Y_1 \) follows

\[ 0 = (x_1 - x_3) + (x_3 - x_5) + \cdots + (x_{2m} - 3 - x_{2m - 1}) + (x_{2m - 1} - x_{2m + 1}) = m(u - v) \]

and, since \( u \neq v \), \( m \) must be a multiple of \( p \). Until now we considered the larger hypergraph \( A \). It is easy to see that \( A \) can contain 3-pyramids. However, \( U \) cannot contain these subgraphs. Indeed, the vertices \( y_{2i+1} \) form an arithmetic progression of vectors with increment \( u - v \). Hence at least one coordinate, e.g. the first of \( u - v \), is different from 0; therefore, the first coordinates of the vectors \( y_{2j+1} \) form an arithmetic progression of residues mod \( p \). Since the number of elements in this progression is at least \( p \), at least one of them must be 0; hence the corresponding \( y_{2j+1} \) does not belong to \( U \). Thus, (7) and (8) complete the proof of our theorem if \( n = (p - 1)^3 \). Since \( \text{ext}(n; I) \) is monotone increasing in \( n \) and the primes are fairly dense among the integers (i.e. for every \( \varepsilon > 0 \) the interval \( (n - \varepsilon n, n) \) contains a prime if \( n \) is large enough), our theorem follows for every \( n \).
REFERENCES


