DISCRETE MATHEMATICS 7 (1974) 349–376. © North-Holland Publishing Company

EXTREMAL GRAPH PROBLEMS WITH SYMMETRICAL EXTREMAL GRAPHS. ADDITIONAL CHROMATIC CONDITIONS

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Received 12 September 1973*

Abstract. The main result of this paper is that for a special, but rather wide class of "sample graphs", the extremal graphs, i.e. the graphs of n vertices without subgraphs isomorphic to the sample graph and having maximum number of edges under this condition, have very simple and symmetric structure. This result remains valid even in the case when the condition "the graph does not contain the sample graph" is replaced by the condition "the graph does not contain the sample graph and its chromatic number is greater than t, where t is a fixed integer". The results of this paper have a lot of different applications, a few of which are listed in Section 3.

0. Notations

The graphs, considered in this paper, do not contain loops or multiple edges. They will be denoted by capital letters, the upper indices will always denote the number of vertices. The vertices of a graph will be denoted by $x, \ldots$, the edges by $(x, y), \ldots$; $v(G)$, $e(G)$ and $\chi(G)$ denote the number of vertices, edges and the chromatic number of the graph G, respectively. If $x \in G$, then $\operatorname{st} x$ denotes the star of x, i.e., the set of vertices joined to x. The number of these vertices, i.e. the valence (degree) of x, will be denoted by $\sigma(x)$; if E is a set, then $|E|$ denotes its cardinality.

To simplify the definitions of graphs we use the following *operations*:

(a) $G = \Sigma\, G_i$ if G_i are spanned subgraphs of G no two of which have vertices in common and vertices of different G_i's are never joined.

(b) $G = \mathsf{X} G_i$ if G_i's are spanned subgraphs of G_i no two of which have vertices in common and vertices of G belonging to different G_i's are always adjacent.

(c) If G_1 is a subgraph of G or a set of vertices of it, then $G - G_1$ is

* Original version received 30 March 1972.

the graph obtained from G by omitting all the vertices of G_1 from it and also all the edges at least one endpoint of which is omitted.

Special graphs. $K_d(r_1, ..., r_d) = \times G^{r_p}$, where $e(G^{r_p}) = 0$ (the complete d-chromatic graph with r_p vertices in its pth class). $K_d(1, ..., 1) = K_d$ is the complete graph of d vertices. P^l and C^l denote the path and circuit of l vertices, respectively.

Constants will be denoted by $c_0, ..., c_m, ...$ and will always be supposed positive. $N_k, ...$ will denote large but fixed positive integers.

Important! If we have to distinguish between the following two relations:
(a) G_1 is a subgraph of G;
(b) G contains a subgraph, isomorphic to G_1;
then we use $G_1 \subseteq G$ in the first case and $G_1 \subset G$ in the second one.

1. Introduction

A well-known theorem of Turán [14] states that: Let p and $n \geq p$ be given integers. If we consider all the graphs of n vertices not containing K_p, then there exists exactly one among them having maximum number of edges. If this graph is denoted by S^n, then S^n is defined by

$$S^n = K_{p-1}(n_1, ..., n_p),$$

where $\Sigma n_i = n$ and $|n_i - n/(p-1)| < 1$ $(i = 1, ..., p-1)$.

It is natural to replace K_p by other graphs or families of graphs in Turán's problem. Thus we obtain:

Problem $(L_1, ..., L_\lambda)$. Let $L_1, ..., L_\lambda$ be given graphs. How many edges can a graph G^n have if it does not contain any L_i?

Here the graphs L_i will be called sample graphs, the graphs attaining the maximum will be called extremal graphs and the maximum itself will be denoted by $f(n; L_1, ..., L_\lambda)$.

The most important theorem of [7] asserts that $f(n; L_1, ..., L_\lambda)$ depends above all on

(1) $$d = \min \chi(L_i) - 1$$

if n is large. More exactly,

(2) $$\lim_{n\to\infty} f(n; L_1, ..., L_\lambda)/\binom{n}{2} = 1 - 1/d\,.$$

P. Erdös and the author tried to modify the original proof of (2) to show that not only the asymptotic number of edges but the "asymptotic structure" of the extremal graphs are also determined by d. The main result, obtained by Erdös and the author independently [4,5,12] can be summarized in the following two theorems. (Throughout this paper, d is defined by (1)!)

Theorem A. *Let S^n be an extremal graph for $L_1, ..., L_\lambda$. Let $c = \{\max v(L_i)\}^{-1} > 0$. Then S^n can be obtained from a product $\times_{p\le d} G^{n_p}$ by omitting $O(n^{2-c})$ edges, where*

$$e(G^{n_p}) = O(n^{1-c}), \qquad n_p - n/d = O(n^{1-c}) \text{ when } n \to \infty\,.$$

The minimal valence of vertices in S^n is $n(1-1/d) + O(n^{1-c})$. The number of vertices of G^{n_p} joined to at least ϵn vertices of G^{n_p} is $O_\epsilon(1)$.

Theorem B. *For every $\epsilon > 0$, there exists a $\delta > 0$ (depending also on $L_1, ..., L_\lambda$) such that if n is sufficiently large and*

$$e(G^n) \geq \binom{n}{2}(1-(1/d)-\delta)$$

and $L_i \not\subset G^n$ $(i = 1, ..., \lambda)$, then G^n can be obtained by omitting at most ϵn^2 edges from a product $\times_{p\le d} G^{n_p}$, where

$$|n_p - n/d| < \epsilon n, \qquad e(G^{n_p}) < \epsilon n^2, \quad p = 1, ..., d\,.$$

(Theorem B is a stability theorem: it expresses that if G^n is almost extremal, then G^n has almost the same structure as the extremal graphs.)

Theorems A, B are valid without any restrictions on the graphs $L_1, ..., L_\lambda$. If we wish to get some further information on the structure of the extremal and almost extremal graphs, we need to restrict our investigations to special cases. In this paper we deal with the case when there exists an integer τ such that

(3) $L_1 \subset P^\tau \times K_{d-1}(\tau, ..., \tau)$.

Without any loss of generality we may assume that

(4) $\tau = \max v(L_i)$.

(3) *and* (4) *are assumed from now on.*

These conditions mean that if the minimum of the chromatic number of sample graphs is $d+1$, then at least one sample graph of chromatic number $d+1$ is almost d-chromatic in the sense that it can be coloured by $d+1$ colours, so that e.g. the subgraph spanned by the vertices of the first and second colours is a path or a subgraph of a path.

We shall see (Section 3) that our results on this special case have a lot of different applications, i.e., the class of problems considered here is fairly wide.

To formulate our theorems, we have to define some classes of "very symmetrical graphs".

Definition 1.1. *Symmetric subgraphs.* Let T_1 and T_2 be connected, spanned subgraphs of G. They are called symmetric (in G) if either $T_1 = T_2$ or:

(i) $T_1 \cap T_2 = \emptyset$; and

(ii) $(x, y) \notin G$ if $x \in T_1$, $y \in T_2$; and

(iii) there exists an isomorphism $\psi_2 : T_1 \to T_2$ such that for every $x \in T_1$ and $u \in G - T_1 - T_2$, x is joined to u if and only if $\psi_2(x)$ is joined to u.

$T_1, ..., T_\gamma$ are symmetric if for every $1 \leq i < j \leq \gamma$, T_i and T_j are symmetric.

Remark 1.2. The transitivity of our relation is the consequence of the connectedness of the considered subgraph.

If we speak about a set of symmetric subgraphs, we suppose that the isomorphisms $\psi_i : T_1 \to T_i$ are fixed even in those cases when they are not uniquely determined.

Definition 1.3. $\mathbf{G}(n, r, d)$ is the class of graphs G^n having the following properties:

(i) It is possible to omit $\leq r$ vertices of G^n, so that the remaining graph $G^{\hat{n}}$ is a product

$$G^{\hat{n}} = \mathop{\times}_{p \le d} G^{m_p}, \qquad \text{where } |m_p - n/d| \le r\,.$$

(ii) The graphs G^{m_p} are the disjoint unions of small symmetric graphs: For every $p \le d$, there exist connected graphs $H_{p,j} \subseteq G^{m_p}$ and isomorphisms $\psi_{p,j} : H_{p,1} \to H_{p,j}$ such that $v(H_{p,j}) \le r$ and $H_{p,j}$ ($j = 1, ..., \nu_p$) are symmetric subgraphs of G^n ($G^{m_p} = \Sigma H_{p,j}$).

In (ii), it is very important that $H_{p,j}$ ($j = 1, ..., \nu_p$) are symmetric subgraphs of G^n and not only of G^{m_p}.

Now we formulate the main result of this paper.

Theorem 1.a. *Let $L_1, ..., L_\lambda$ be given graphs and let*

$$d = \min \chi(L_i) - 1$$

and $L_1 \subset P^\tau \times K_{d-1}(\tau, ..., \tau)$. There exists a constant r (depending only on τ) such that for every n, $\mathsf{G}(n, r, d)$ contains at least one extremal graph for $L_1, ..., L_\lambda$.

This result remains valid under much more general conditions too. More exactly, we shall define the concept of "chromatic conditions" and an operation D^m on graphs and prove:

Theorem 1. *Let $L_1, ..., L_\lambda$ be given graphs, d and τ be defined according to* (1) *and* (4) *respectively, and let* (3) *also hold*

$$L_1 \subset P^\tau \times K_{d-1}(\tau, ..., \tau).$$

Let A be a chromatic condition and let us consider those graphs of n vertices, which satisfy A and do not contain any L_i (if n is large enough, such graphs do exist!). The graphs, having maximum number of edges among the considered ones, will be called extremal graphs for $(L_1, ..., L_\lambda; \mathsf{A})$. There exists an $r = r(\tau, \mathsf{A})$ such that for every n, large enough, $\mathsf{G}(n, r, d)$ contains an extremal graph for $(L_1, ..., L_\lambda; \mathsf{A})$.

Theorem 1 is an existence theorem; however, in many cases it makes possible to find some extremal graphs for given concrete problems relatively easily. Sometimes there are very many extremal graphs, sometimes only one. Our next theorem asserts that one can decide, whether there exists only one extremal graph or not considering only $\mathsf{G}(n, r, d)$.

Theorem 2. *Using the notations of Theorem 1. There exists a constant r_0 such that if for every sufficiently large n, $\mathsf{G}(n, r_0, d)$ contains only one extremal graph (for $(L_1, ..., L_\lambda; \mathsf{A})$), then there exist no other extremal graphs.*

In the case when there are many extremal graphs, they may be characterized by the help of:

Theorem 3. *Using the notations of Theorem 1. There exists an n_0 and a finite set of extremal graphs, denoted by S, such that if $n > n_0$, then S^n is an extremal graph (for $(L_1, ..., L_\lambda; \mathsf{A})$, of course) if and only if $S^n \in \mathsf{D}^m(S)$ for some $S \in \mathsf{S}$ and integer m selested in a suitable way.*

(Here the operator D^m is a multivalent operator, thus sometimes we may consider it as a family of graphs and use the notation $G \in \mathsf{D}^m(S)$.)

Now we define the concept of "chromatic condition" and the operator D^m.

Definition 1.4. *Symmetrization.* Let T be a connected subgraph of a graph G and let each vertex of G belong to another graph $\hat{G}$. We say that $\hat{G}$ is obtained from G by symmetrizing $x_1, ..., x_m$ to $T \subseteq G$ if:

(i) $\hat{G} - \{x_1, ..., x_m\} = G - \{x_1, ..., x_m\} \supseteq T$;

(ii) the subgraphs T_j spanned by $x_{jv+1}, ..., x_{(j+1)v}$ are symmetric to T in $\hat{G}$, where $v = v(T)$, $j = 1, ..., m/v$, m is divisible by v.

Definition 1.5. *Chromatic conditions.* Let us suppose that A is a condition such that (using the expression A-graph instead of writing "a graph, satisfying the condition A"):

(i) If G is an A-graph and H contains G, then H is also an A-graph.

(ii) For every ω, there exists an A-graph each circuit of which is longer than ω.[1]

(iii) There exists a constant ρ such that if $T_1, ..., T_\rho$ are symmetric subgraphs of an A-graph G, then $G - T_\rho$ is also an A-graph.

[1] As a matter of fact, this condition can be omitted. However, in this case some classes of graphs considered by us will become empty.

Examples. We do not distinguish between a chromatic condition and the family of graphs, satisfying this condition.

(1) Let A be the family of at least t-chromatic graphs. Then A is a chromatic condition. (For the proof of (iii) see the Appendix, (ii) is proved in [1, 10].

(2) If t and u are given integers and we consider the family of graphs from which omitting any u vertices we obtain a graph of chromatic number $\geq t$, then we obtain a chromatic condition. (iii) is proved in the Appendix, (ii) follows from [1. 10].

(3) Let A be the family of graphs each vertex of which has valence greater than t. This is also a chromatic condition. However, it is interesting to notice that because of Theorem A if n is large enough, there is no difference between $(L_1, ..., L_\lambda; \mathsf{A})$ and $(L_1, ..., L_\lambda)$, unless $d = 1$.

(4) Similar is the situation with the chromatic condition "G is non-planar".

(5) If A and B are chromatic conditions, then $\mathsf{A} \cap \mathsf{B}$ and $\mathsf{A} \cup \mathsf{B}$ are also chromatic conditions.

Remark 1.6. Our theorems remain valid even if (iii) is replaced by (iii)*: there exists a sequence ρ_k such that if $T_1, ..., T_{\rho k}$ are symmetric subgraphs of k vertices in an A -graph G, then $G - T_1 \in \mathsf{A}$, too.

Definition 1.7. *The operator* D^m. The operator defined here is a multivalent one and it depends also on two parameters N_1 and ρ. Let us suppose that the graph G contains a family $T_{p.i,j}$ of subgraphs, $p = 1, ..., d$, $i = 1, ..., \xi_p, j = 1, ..., \rho$.

Let us suppose also that no two of them have vertices in common and if we fix p and i, then the obtained subfamily be a set of symmetric subgraphs of G. Further, if $x \in T_{p,i,j}$ and $y \in T_{p',i',j'}$, then $(x, y) \in S^n$ iff $p \neq p'$. Let us suppose that $\nu_{p,i}$, $p = 1, ..., d$, $i = 1, ..., \xi_p$, are given integers such that $\Sigma_i \nu_{p,i} = (N_0!) = N_1$ for $p = 1, ..., d$. Let $v(T_{p,i,j}) \leq N_0$ for every considered subgraph. Now we symmetrize $\nu_{p,i}$ new vertices to $T_{p,i,1}$ for every considered pair (p, i).[2] If ρ and N_0 are fixed, let $\mathsf{D}(G)$ denote the family of graphs obtainable in this manner. $\mathsf{D}^m(G)$ is defined recursively: $\mathsf{D}^m(G)$ is the family of graphs obtained by applying D to the graphs of $\mathsf{D}^{m-1}(G)$ ($\mathsf{D}^m(G)$ can be considered as a family of graphs and as an operator as well).

[2] Here we implicitely supposed that $\nu_{p,i}$ is divisible by $v(T_{p,i,j})$.

As we shall see in Lemma 3.4.1, if a graph does not contain any sample graphs and we apply the operator D to this graph with a suitably large ρ, then the obtained graph will neither contain any sample graph; on the other hand, if the original graph was an A-graph, then (because of Definition 1.5(iii) and (i)) the new graph will be an A-graph as well. Therefore, if we apply this operation to an extremal graph, the obtained graph will have at most as many edges as the corresponding extremal graph. Theorem 3 asserts that if n_0 is large enough, it will have exactly the same number of edges.

The Appendix includes a theorem, showing that our theorems are the best possible in a certain sense.

2. Applications

(A) Let

$$H(n, d, s) =: K_{s-1} \times K_d(m_1, \ldots, m_d), \tag{5}$$

where $|m_i - (n-s+1)/d| < 1$ and $\Sigma m_i = n-s+1$.

For many families of sample graphs, $H(n, d, s)$ is the only extremal graph if n is large enough. E.g:

Theorem 2.1 (Moon [11]). *There exists an $n(d, s)$ such that if $n > n(d,s)$ and L is the sum of $s \subset K_{d+1}$, then $H(n, d, s)$ is the only extremal graph for L.*

For $d = 1$, the theorem was proved by Erdös and Gallai [6] first. My Ph.D. Thesis [13] contains a generalization of this theorem for the case when K_{d+1} is replaced by an arbitrary $d+1$-chromatic graph at least one edge of which is colour-critical (an edge is called colour-critical if by its omitting the chromatic number of the graph decreases). This result is a very special case of the next theorem.

Theorem 2.2. *Let $L_1, \ldots, L_\lambda$ be given graphs,* $\min \chi(L_i) = d+1$. *If omitting any $s-1$ vertices of any L_i we obtain a $\geq d+1$-chromatic graph but omitting s suitable edges of L_1 we get a d-chromatic graph, then $H(n, d, s)$ is the only extremal graph whenever n is sufficiently large. Further, for every chromatic condition A, there exists an integer g(A) such that*

(6) $f_{\mathsf{A}}(n; L_1, ..., L_\lambda) = f(n; L_1, ..., L_\lambda) - (n/d)g(\mathsf{A}) + O(1).$

This theorem is an almost trivial consequence of Theorems 1,2. An interesting special case of it is:

Theorem 2.3. *$H(n, d, 1)$ (= the extremal graph of Turán's theorem) is the extremal graph for $L_1, ..., L_\lambda$ for every sufficiently large n if and only if* (1) *holds and a $d+1$-chromatic L_i contains a critical edge.*

(B) A problem of P. Turán, published by Erdös [3] asks: Let L be the graph, determined by the vertices and edges of a regular polyhedron. How many edges can G^n have if $L \not\subset G^n$?

In the case of the tetrahedron $L = K_4$, thus Turán's theorem gives a complete answer. If L is the octahedron-graph, the results of Erdös and the author give a fairly good description of the extremal graphs [9]. (In this case, $L = K_3(2,2,2)$.) In the case of the cube, we have some upper and lower bounds [8], but they are different powers of n (it seems to me that the lower one is too weak).

Let D^{20} and I^{12} denote the dodecahedron- and the icosahedron-graphs, respectively.

Theorem 2.4. *If n is large enough, $H(n, 2, 6)$ is the only extremal graph for D^{20}. Every almost extremal graph is very similar to $H(n, 2, 6)$ in the following sense:*

If $D^{20} \not\subset G^n$ and

(7) $e(G^n) > e(H(n, 2, 6)) - \frac{1}{2}n + M_2$,

where M_2 is a suitable constant, then we can omit 5 vertices of G^n, so that the remaining graph is 2-chromatic.

Theorem 2.5. *If n is large enough, $H(n, 3, 3)$ is the only extremal graph for I^{12}.*

Remark 2.6. Let A be the chromatic condition, "it is impossible to omit 5 vertices of G to obtain a 2-chromatic graph". It can be proved that $D^{20} \in \mathsf{A}$, but it is possible to omit 6 edges from it so that the obtained graph is 2-chromatic. Therefore:

(a) Theorem 2.4 is the special case of Theorem 2.3 apart from the fact that $g(A) = 1$. Thus, if we wish to prove Theorem 2.4, we can deduce it from Theorem 1 directly, or to prove in Theorem 2.3 that $g(A) \geq 1$ if $H(n, d, s)$ is not an A-graph for every sufficiently large n. Trivially, this implies Theorem 2.4.

(b) In Theorem 2.4, we asserted that the almost extremal graphs are similar to $H(n, 2, 6)$ in the sense that we can omit 5 vertices from them so that the remaining graph be 2-chromatic. Does this property express a real similarity?

Yes, this and (7) imply that there exist an m_1 and an m_2 such that G^n can be obtained from $K_5 \times K_2(m_1, m_2)$ by omitting $O(n)$ edges, where $m_i - \frac{1}{2}n = O(\sqrt{n})$. (If we do not omit edges and $|m_i - \frac{1}{2}(n-5)| < 1$, then the obtained graph is just $H(n, d, s)$.)

(c) (7) is the best possible; M_2 can not be replaced by a small negative integer.

(d) Theorem 2.5 is essentially deeper than Theorem 2.4, it cannot be deduced from Theorem 2.3. Its proof, based on Theorems 1, 2, will be published later. No assertion analogous to the second part of Theorem 2.4 holds for the icosahedron.

(C) A theorem of Erdös, Gallai and Andrásfai states that: If G^n does not contain K_3 and is not 2-chromatic, then

$$e(G^n) \leq f(n; K_3) - \tfrac{1}{2}m + O(1) \tag{8}$$

(see [2]).

In connection with this result Erdös asked:

Problem. What is the maximum number of edges, a graph of n vertices and chromatic number $\geq t$ can have if it does not contain K_3?

I showed [13] that:

Theorem 2.7. *Let $f_t(n; K_3)$ denote the maximum in the problem above. Then*

$$f_t(n; K_3) = \tfrac{1}{4}n^2 - g_3(t)\tfrac{1}{2}n + O(1), \tag{9}$$

where $g_3(t)$ is the largest integer m such that for any graph G not containing K_3 and having chromatic number $\geq t$, at least m vertices of G must be omitted to get a 2-chromatic graph.

Remark 2.8. (a) According to [10], there exist graphs of chromatic number $\geq t$ and not containing circuits shorter than ζ. If G is such a graph and G_1 is a subgraph of it of $< \zeta$ vertices, then G_1 is a tree, i.e., a 2-chromatic graph. Therefore the definition of $\hat{g}_3(t)$ is legitimate. Comparing $\hat{g}_3$ and g_3 of [1], one can easily prove that

$$(10) \qquad c_1 t^2 \log t / \log\log t < \hat{g}_3(t) < c_2 t^2 (\log t)^2 .$$

(b) Theorem 2.7 would also be a very special case of Theorem 2.3 if the corresponding $g(\mathsf{A})$ were known.

(c) Replacing K_3 by K_4, we get an essentially more difficult problem the exact solution of which is unknown to me.

(d) In the original form of (8), O(1) was replaced by the best possible constant. Later I generalized this theorem for every K_p (see [12]) determining the exact constants and the extremal graphs as well.

Now we turn to the proofs of Theorem 1,2,3.

3. Proofs of Theorems 1,2,3

3.1. *A general lemma*

If we prove any of the Theorems 1, 3, to prove the others will be an essentially simpler task. Actually, we will prove Theorem 3 first and deduce the other two theorems from it. The proof of Theorem 3 consists of two parts, a general and a special one. Here the expression "general part" means that if we replace the condition (3) by

$$(11) \qquad L_1 \subseteq T \times K_{d-1}(\tau, \ldots, \tau),$$

where T is a 2-chromatic graph, then the estimations $O(n^{2-c})$ and $O(n^{1-c})$ can be replaced in Theorem A by

$$O(f(n; T)) \quad \text{and} \quad O(f(n; T)/n) ,$$

respectively. The proof of this assertion needs no new ideas but a slight modification of the proofs in [4,5,12] (see also [13]). In our case, T is a path and therefore a tree and one can easily prove that if T is a tree, then $f(n;T) = O(n)$.

This leads us to Lemma 3.1.1, the proof of which will be sketched in the Appendix.

Lemma 3.1.1. *Let us suppose that* (10) *holds with a tree T. Let K be a fixed integer and G^n be a graph not containing any L_i and satisfying the condition*

(12) $$e(G^n) \geq f(n; L_1, ..., L_\lambda) - Kn\ .$$

If we colour the vertices of G^n by d colours so that the number of edges joining vertices of the same colour be minimal, and denote by G^{np} the subgraph of G^n determined by the pth colour at this colouring, then:

(i) *G^n can be obtained from the product $\mathsf{X}_{p \leq d} G^{np}$ by omitting $O(n)$ edges. (In other words, at most $O(n)$ pairs (x, y) of vertices, where $x \in G^{np}$ and $y \in G^{nq}$, $p \neq q$, are not joined in G^n. These pairs will be called "missing edges".)*

(ii) $e(G^{np}) = O(n)$.

(iii) $n_p - n/d = O(\sqrt{n})$.

(iv) *Let $\alpha(x)$ and $\beta(x)$ denote the number of vertices*

(a) *of G^{np} joined to $x \in G^{np}$, and*

(b) *of $G^n - G^{np}$ not joined to $x \in G^{np}$, respectively.*

Then for every $\epsilon > 0$, the number of vertices such that either $\alpha(x) > \epsilon n$ or $\beta(x) > \epsilon n$ is $O_\epsilon(1)$.

(v) *Let A_p be the class of vertices $x \in G^{np}$ such that $a(x) \leq \epsilon n$, $\beta(x) \leq \epsilon n$. If $\epsilon < 1/2\tau$ and n is large enough, then $T \not\subset G(A_p)$. (If E is a subset of the set of vertices of a graph G, then G(E) denotes the subgraph of G spanned by E.)*

As we can see, (i)–(iv) are improvements of the corresponding assertions of Theorem A. (v) is needed to prove (ii) and this is also the point where we shall apply our main condition (3). The proof – as we have already remarked – is outlined in the Appendix.

3.2. *Graphs not containing P^l*

Here we shall investigate the structure of graphs not containing P^l

and prove that they have a fairly simple and symmetric structure for every fixed l. This result is needed to guarantee that the graphs $G(A_p)$ of Lemma 3.1.1(v) cannot be to complicated when $T = P^l$.

Lemma 3.2.1. *Let l be a given integer, $\epsilon > 0$ and let $\{m_k\}$ be an arbitrary sequence of positive integers. There exists an integer N_0 such that if $n > N_0$ and $P^l \not\subset G^n$, then G^n contains a family of subgraphs $T_{i,j}$, $i = 1, 2, \ldots, j = 1, 2, \ldots$, with the following properties:*

(i) *No two of them have vertices in common.*

(ii) *For every fixed i, $T_{i,j}$'s are symmetrical subgraphs of G^n. If $v(T_{i,j}) = k_i$, then $j = 1, 2, \ldots, m_{k_i}$.*

(iii) $\Sigma\,\Sigma\, v(T_{i,j}) > n - \epsilon n$.

Proof. The lemma holds for $l = 2$ trivially. We apply induction on l. Let us suppose that the lemma holds for $l-1$.

According to the hypothesis, we determine $\hat{N}_0$ corresponding to $l-1$, $\frac{1}{2}\epsilon$ and $\hat{m}_k = m_k\, t\, 2^{k(l-1)}$. Now if G^n is a graph not containing P^l and U_ν's are the connected components of it, we divide the components into three classes. The first one contains the components of $> \hat{N}_0$ vertices not containing P^{l-1}, the second class contains the components of $\leq \hat{N}_0$ vertices and the third one contains the other components. If μ denotes the number of graphs of $\leq \hat{N}_0$ vertices and n_γ the number of vertices of components in the γth class ($\gamma = 1, 2, 3$), then at least

$$n_2 - \hat{N}_0 \cdot \mu \max_{k \leq N_0} m_k = n_2 - O(1)$$

vertices of components from the second class belong to symmetric graphs of the desired type: we select m_{k_1} isomorphic components of k_1 vertices from the second class (if we can), then m_{k_2} isomorphic components of k_2 vertices and so on. If finally we cannot select m_{k_t} isomorphic components of k_t vertices from the second class, then there remain less than $\mu \max_{k \leq \hat{N}_0} m_k$ components. Now the only thing we have to notice is that isomorphic components are symmetric subgraphs.

In the case of the first class, we apply the induction hypothesis. The graph spanned by the components of the first class contains a family of subgraphs $T_{i,j}$ satisfying (i)–(iii), where, of course, n must be replaced by n_1.

The essential step of the proof is made only in the case of the third class. If U is a component of the third class having u vertices, then it contains a P^{l-1}. If $U-P^{l-1}$ contained an other P^{l-1}, then U would contain a P^l. Therefore $U-P^{l-1}$ does not contain a P^{l-1} and we can apply the induction hypothesis. Let $T_{i,j}$ be symmetrical subgraphs of $U-P^{l-1}$ for every fixed i and $j = 1, ..., \hat{m}_{k_i}$. Sometimes they are not symmetric in G^n since they are joined to P^{l-1} in different ways. However, among any $2^{k_i(l-1)}\, m_{k_i}$ subgraphs $T_{i,j}$ (for fixed i), there exist m_{k_i} subgraphs joined to P^{l-1} in the same way. These subgraphs are symmetric in G^n as well. Therefore one can find at least $2^{k_i(l-1)}t\, m_{k_l}$ subgraphs among the considered $2^{k_i(l-1)}t\, m_{k_l}$ ones forming $2^{k_i(p-1)}(t-1)$ sets of m_{k_i} symmetric subgraphs of U. For $i = 1, 2, ...,$ the total number of vertices of these graphs will be

$$\geq (1 - \tfrac{1}{2}\epsilon)\,(u-l+1)\,(1 - 1/t) > (1-\epsilon)u$$

if ϵ is small enough, $t = 3/\epsilon$ and N_0 is large enough. Therefore the obtained systems of symmetric graphs contain at least

$$(1-\epsilon)\,n_3 + (1-\epsilon)n_1 + n_2 - \mathrm{O}(1) \geq (1-\epsilon)n$$

vertices together if n is large enough. Further, these symmetric graphs have $\mathrm{O}(1)$ vertices.

Let us notice now that if $\epsilon < \frac{1}{2}\tau$ (the sample graphs and the chromatic condition are supposed to be fixed), then Lemma 3.1.1 is applicable to every extremal graph. Indeed, there exists an A-graph G^h not containing circuits shorter than $\tau + 1$. Let $H^n = G^h + K_d(m_1, ..., m_d)$, where $|m_p - (n-h)/d| < 1$ $(p = 1, ..., d)$. Now

$$e(H^n) = \tfrac{1}{2}n^2(1-1/d) - nh\,(1-1/d) + \mathrm{O}(1).$$

Clearly, H^n is an A-graph and each subgraph of $\leq \tau$ vertices is $\leq d$-chromatic in it. The number of edges of the extremal graphs are greater than that of H^n, i.e., greater than

$$\tfrac{1}{2}n^2(1-1/d) - \mathrm{O}(n).$$

Therefore, Lemma 3.1.1 is applicable to the extremal graphs. Now we fix an extremal graph S^n and define the graphs G^{n_p} according to Lemma 3.1.1. We know that $P^l \not\subset G(A_p)$, $p = 1, ..., d$. Hence $G(A_p)$ contains a lot of symmetric subgraphs. If we knew that these graphs are symmetric in G^n as well, then we could prove Theorem 3 easily. However, generally these graphs are not symmetric in G^n. Our aim is to prove that it is possible to select some of them, symmetrical in G^n too.

3.3. *Symmetric subgraphs of the extremal graphs*

Notation. $\alpha(x)$ and $\beta(x)$ were defined in Lemma 3.1.1(v). If T is a subgraph of G^{n_p}, let

$$\alpha(T) = \sum_{x \in T} \alpha(x), \qquad \beta(T) = \sum_{x \in T} \beta(x).$$

($\alpha(T)$ expresses the number of edges of G^{n_p} at least one endpoint of which belongs to T.)

Lemma 3.3.1. *Let ζ be a positive integer. There exists a constant $c_\zeta > 0$ such that if $T_1, \ldots, T_\gamma$ are symmetric subgraphs of $G(A_p)$ and $v(T_j) \le \zeta$, $\beta(T_j) \le \zeta$ for $j = 1, \ldots, \gamma$, then at least $c_\zeta \gamma$ of them are symmetric in G^n too.*

Lemma 3.3.2. *For every sequence $\{m_k\}$ of positive integers and $\delta > 0$, there exists a constant N_0 such that if $n > N_0$, then $G(A_p)$ contains a family of subgraphs $T_{p,i,j}$ symmetric for $j = 1, \ldots, m_{k_{p,i}}$ (where $p \le d$ and $i = 1, \ldots$ are fixed, $k_{p,i} = v(T_{p,i,j})$). Further,*

$$\sum_{i} \sum_{j} v(T_{p,i,j}) \ge (1-\delta) n_p, \qquad v(T_{p,i,j}) \le N_0.$$

Proof of Lemma 3.3.1. If $c_\zeta > 0$ is fixed and $c_\zeta \gamma < 1$, then Lemma 3.3.1 is trivial. Thus we shall consider only the case when γ is large enough. We define an algorithm of selecting such that in the rth step of the algorithm we have a set of integers (denoted by I_r) and a set of vertices $y_1, \ldots, y_r \in S^n - B_0 - A_1$, where B_0 is the set of exceptional vertices, i.e., the set of vertices of $S^n - \cup_p A_p$.

Let $B_r = B_0 \cup \{y_1, \ldots, y_r\}$. The algorithm will ensure that $\{T_i : i \in I_r\}$ are symmetric in $G(A_1 \cup B_r)$ and for every y_s ($s \le r$) and T_i ($i \in I_r$), at least one vertex of T_i is not joined to y_s.

(a) If $r = 0$, $T_1, \ldots, T_\gamma$ are symmetric subgraphs of $G(A_1)$ and $v(T_i) = k$, then because of $|B_0| = O(1)$ we can find an M such that $|B_0| \le M$. At least $\gamma \cdot 2^{-\mathfrak{t}M}$ subgraphs among $T_1, \ldots, T_\gamma$ are symmetric in $G(A_1 \cup B_0)$ too, being joined to B_0 in the same way.

(b) Now we define the rth step. Let us suppose first that there is an $(x, y) \notin S^n$, such that $x \in T_1, y \notin A_1 \cup B_{r-1}$. Adding (x, y) to S^n, we obtain a G^n and this G^n contains at least one $L^* \simeq L_{t_0}$ since S^n is an extremal graph and G^n is an A-graph. Of course, $L^* \nsubseteq S^n$ and $S^n - \{x\} = G^n - \{x\}$, thus $x \in L^*$. Permutating the indices, we can achieve that $1, \ldots, \tau \in I_r$ and if $i > \tau$, then $L^* \cap T_i = \emptyset$. (T_i's have no common vertices and L^* has at most τ vertices!) We show that the number of T_i's such that

$$(13) \qquad T_i \times (L^* - A_1 - B_{r-1}) \subset S^n \quad \text{and} \quad i > \tau$$

is at most τ. ((13) means that every vertex of T_i is joined to every one of $L^* - A_1 - B_{r-1}$.) Otherwise, we could suppose that for $i = \tau+1, \ldots, 2\tau$, (13) holds. Let ψ_i be the isomorphisms in the definition of symmetry of $T_1, \ldots, T_\gamma$. Replacing each $\psi_i(z) \in T_i \cap L^*$ by the corresponding vertex $\psi_{i+\tau}(z) \in T_{i+\tau}$ $(i \leq \tau)$, we should obtain an $L^{**} \simeq L^* \simeq L_{t_0}$ in G^n since

(i) $\psi_{i+\tau}(z)$ is joined to all the vertices of $L^* - T_i - T_{i+\tau}$ which are joined to $\psi_i(z)$;

(ii) if $z_1, z_2 \in T_1$, then $\psi_{i_1}(z_1)$ and $\psi_{i_2}(z_2)$ are joined if and only if $\psi_{i_1+\tau}(z_1)$ and $\psi_{i_2+\tau}(z_2)$ are joined.

Since L^* and L^{**} have no vertices in common, $x \notin L^{**}$ and consequently $L^{**} \subseteq S^n$. This contradiction proves that (13) holds for less than τ indices $i \in I_{r-1}$. Therefore there exist an $y_r \in L^* - A_1 - B_{r-1}$ and at least

$$\frac{|I_{r-1}| - 2\tau}{\tau}$$

subgraphs T_i, so that y_r is not joined to each vertex of T_i. At least

$$\frac{|I_{r-1}| - 2\tau}{\tau} 2^{-\mathfrak{t}}$$

T_i are joined to y_r in the same way. Let $I_r \subset I_{r-1}$ be the corresponding set of indices. Clearly, $\{T_i \colon i \in I_r\}$ are symmetric in $G(A_1 \cup B_0 \cup \{y_1, \ldots, y_r\})$ and no y_s $(s \leq r)$ is joined to each vertex of T_i $(i \in I_r)$.

The algorithm stops (by definition) if we cannot find a suitable "missing edge" (x, y). In this case, $\{T_i : i \in I_{r-1}\}$ are symmetric not only in $G(A_1 \cup B_{r-1})$ but also in S^n since each vertex of each remaining T_i is joined to each vertex of $S^n - (A_1 \cup B_r)$. Thus $\{T_i : i \in I_r\}$ are symmetric in S^n, what implies the lemma if the algorithm stops fairly soon. Clearly, $\beta(T_i) \geq r$ for every $i \in I_r$; thus the algorithm stops in at most ζ steps. An easy calculation gives that if $\gamma > n_0$, then we have at least

$$2^{-M\zeta}(2\tau)^{-\zeta}\, 2^{-\zeta^2}\, \gamma = \hat{c}_\zeta \gamma$$

symmetric subgraphs at the end. To get rid of n_0, we observe that if

$$c_\zeta = \min(\hat{c}_\zeta, 1/n_0)\,,$$

then Lemma 3.3.1 holds for every γ.

Proof of Lemma 3.3.2. Let M be a constant such that

$$\beta(G(A_p)) < M n_p$$

(see Lemma 3.1.1(i)). Let $(1-\eta)^4 = 1-\delta$, $\zeta_k = 4Mk\eta^{-2}$, and let $\hat{m}_k = 2m_k(\eta c_{\zeta k})^{-1}$. We apply Lemma 3.2.1 to $G(A_1)$, η and $\hat{m}_k$. Thus we obtain a family $\{U_{r,s}\}$ of subgraphs symmetric in $G(A_1)$ for $s = 1, \ldots, \hat{m}_{k_r}$ ($k_r = v(U_{r,s})$ and r is fixed). Further,

$$\sum_r \sum_s v(U_{r,s}) \geq (1-\eta)n_1, \qquad v(U_{r,s}) = k_r \leq M_1\ .$$

Now we omit some of these subgraphs in two steps. First we omit all the graphs $\Sigma_s U_{r,s}$ satisfying

$$\beta\Big(\sum_s U_{r,s}\Big) \geq 2M\eta^{-1}\, v\Big(\sum U_{r,s}\Big)\ .$$

Thus we omitted at most ηn_1 vertices, otherwise

$$\beta(G(A_1)) > \eta\, n_1\, 2M\eta^{-1} = 2M\, n_1$$

would hold. In the second step we omit all the remaining subgraphs $U_{r,s}$ such that

(14) $\beta(U_{r,s}) \geq 4M\eta^{-2}\, v(U_{r,s})$.

Thus we omitted for each r at most $\eta\, v(\Sigma_s U_{r,s})$ vertices. (The total number of vertices of the remaining graphs is at least $(1-2\eta)(1-\eta)n_1$.)

Let us consider now the remaining components $U_{r,s}$ of a graph $\Sigma_s U_{r,s}$ for a fixed r. Let C_r denote the number of these components. Clearly, $C_r \geq (1-\eta)\hat{m}_{k_r}$. Since $\zeta_{k_r} > k_r$ and $\zeta_{k_r} > \beta(U_{r,s})$, we can apply Lemma 3.3.1 to any $c_{\zeta_{k_r}}^{-1} \cdot m_{k_r}$ of these components. We apply Lemma 3.3.1 repeatedly: having obtained some sets of m_{k_r} symmetric subgraphs of k_r vertices we apply the lemma to the other components. Thus we obtain at least

$$C_r - c_{\zeta_{k_r}}^{-1} \cdot m_{k_r} \geq (1-\eta)\, C_r \qquad (\eta < \tfrac{1}{2})$$

components, collected into sets of m_{k_r} symmetric subgraphs. If we take the union of these families of sets of symmetric graphs for every r, then we get a family $T_{i,j}$ of subgraphs, symmetric in S^n for every fixed i, while $j = 1, ..., m_{k_i}$, $k_i = v(T_{i,j})$. Further

$$\frac{1}{n_1} \cdot \sum \sum v(T_{i,j}) \geq (1-\eta)^2\,(1-2\eta) \sim (1-\eta)^4 = 1-\delta\ .$$

3.4. *Symmetrization and extremal graph problems*

The last lemma we need expresses that symmetrization is an operation which can be used to solve some extremal problems.

Lemma 3.4.1. *Let $T_1, ..., T_\gamma \subseteq G$ be symmetric subgraphs and let L be an arbitrary graph such that $v(L) \leq \gamma$. If $\hat{G}$ is obtained from G by symmetrizing*

$$x_1, ..., x_m \in G - \bigcup T_i$$

to T_1 (where m is divisible by $v(T_1)$) and $L \not\subseteq G$, then $L \not\subseteq \hat{G}$.

Proof. (Essentially the same as the argument used in the proof of Lemma 3.3.1). Indirectly, let $L \simeq L^* \subseteq \hat{G}$. Since $v(L) \leq \gamma$, if $T_{\gamma+1}, ..., T_{\gamma+\omega}$ are the subgraphs obtained from $x_1, ..., x_m$, then the number of T_j's $(j > \gamma)$ for which $T_j \cap L^* \neq \emptyset$ does not exceed the number of T_j's $(j \leq \gamma)$ for which $T_j \cap L^* = \emptyset$. Without loss of generality we may suppose that for $j > \gamma + \mu$, $T_j \cap L^* = \emptyset$ and for $j = \gamma + 1, ..., \gamma + \mu$, $T_j \cap L^* \neq \emptyset$ and for $j = 1, ..., \mu$, $T_j \cap L^* = \emptyset$, too. Let us replace in L^* every $x \in T_j \cap L^*$ by $\psi_{j-\gamma}(\psi_j^{-1}(x))$, $j = \gamma + 1, ..., \gamma + \mu$. Thus we obtain a new graph $L^{**} \simeq L$ in

$$\hat{G} - \bigcup_{\gamma+1}^{\gamma+\omega} T_j = G - \{x_1, ..., x_m\} \subseteq G .$$

This contradicts $L \not\subset G$.

3.5. *The background of our theorems*

If there are given some sample graphs and a chromatic condition, then, as we have seen, the extremal graphs contain a lot of symmetrical subgraphs. Let us consider two sets of symmetric subgraphs in an extremal graph S^n. If we select almost all the subgraphs of the first set but do not select at least $\tau + \rho$ of them (see Definition 1.5), then symmetrizing the vertices of the selected subgraphs to the subgraphs of the second set, we obtain a new graph S_1^n of the same number of vertices which is an A-graph because of Definition 1.5(iii) and does not contain any sample graph. Symmetrizing the same number of vertices of the second set to the subgraphs of the first set, we obtain a third graph S_2^n and, as we shall see, at least in one case the number of edges increased, or the new graphs S_1^n and S_2^n have the same number of edges as the original one. The original one was an extremal graph, therefore the number of edges remains unchanged and the two new graphs are extremal graphs. Applying this operation repeatedly one can easily prove Theorem 1. After this the other theorems will follow easily. However, here we shall prove Theorem 3 first.

3.6. *Proof of Theorem 3*

In the first part of the theorem (i.e., in (a) and (b)), we give some estimations on $\alpha(W)$ and $\beta(W)$ which will be needed when we apply Lemma 3.3.1 in (c).

(a) Let $T_1, ..., T_\rho$ be symmetric subgraphs of S^n and $W \subseteq T_1 \cap G(A_1)$, then

(15) $\quad |B_0|\, v(W)+\alpha(W)\geq \beta(W)$.

(For the definitions of B_0 and ρ see Lemma 3.1.1 and Definition 1.5.) Indeed, $S^n - W \supseteq S^n - T_1$ is an A-graph. Therefore, if we omit all the edges at least one endpoint of which belongs to W and join each edge of W to each vertex of $S^n - A_1 - B_0$, then we obtain a new A-graph $\hat{S}^n$ not containing any L_i. (We can apply here a modified version of the proof of Lemma 3.4.1: from $L_i \simeq L^* \subseteq \hat{S}^n$ would follow $L_i \simeq L^{**} \subseteq S^n$ because the vertices of $W \cap L^*$ could be replaced by some vertices of $A_1 - W - L^*$ joined to each vertex of $L^* - A_1 - B_0$. Such vertices do exist because $\epsilon < 1/2\tau$ in Lemma 3.1.1. Since the vertices of W in the new graph $\hat{S}^n$ are not joined to B_0, by this replacing of vertices we obtain an $L^{**} \simeq L^*$ in S^n!) Thus $e(S^n) \geq e(\hat{S}^n)$. This proves (15) because

$$e(S^n)-e(\hat{S}^n)\leq \alpha(W)-\beta(W)+|B_0|\, v(W) .$$

(b) We prove that if $T_1, \ldots, T_\rho$ are symmetric subgraphs of S^n and $W \subset T_1 \cap G(A_1)$, and either $W = T_1$ or $W_1, \ldots, W_\tau$, W are symmetric subgraphs of T_1, then

(16) $\quad \alpha(W) = \mathrm{O}(v(W))$.

This, (15) and $|B_0| = \mathrm{O}(1)$ will imply

(16*) $\quad \beta(W) = \mathrm{O}(v(W))$.

From $P^\tau \not\subset W$ follows that

(17) $\quad e(W) = \mathrm{O}(v(W))$.

Further, at most $\frac{1}{2}(\tau-1)$ vertices of $G(A_1)-T_1$ can be joined to T_1. Indeed, if $U_1, \ldots, U_a$ are symmetric subgraphs of a graph G and $x_1, \ldots, x_a$ are joined to U_1, then G contains a circuit

$$(u_1\, x_1\, v_1 \ldots u_2\, x_2\, v_2 \ldots u_3\, x_3\, v_3 \ldots\ldots u_a\, x_a\, v_a) ,$$

where u_j, v_{j-1} are the vertices of U_j joined by a path in $U_j (j = 1, \ldots, a$, $v_0 = v_a)$. Hence $P^{2a} \subset G$. Therefore at most $\frac{1}{2}(\tau-1)$ vertices of $G(A_1)-T_1$ can be joined to W. Similarly, at most $\frac{1}{2}(\tau-1)$ vertices of $T_1 - W$ can be joined to W. Thus

$$\alpha(W) \leq e(W) + (2(\tau-1)/2 + |B_0|)\, v(W) = O(v(W)).$$

(c) Let N_0 and ρ be given integers. A family $T_{i,j}$ of subgraphs ($j = 1, \dots, \rho, i = 1, \dots, N_1 = N_0!$) is a matrix of symmetric graphs (in S^n) if for every fixed i, $T_{i,1}, \dots, T_{i,\rho}$ are symmetric subgraphs of S^n and the graphs $T_i =: \Sigma_j T_{i,j}$ are also symmetric subgraphs apart from the fact that they are not connected.

Now we prove the existence of an $N_0 = N_0(\rho)$ such that $G(A_1)$ contains either N_1 symmetric subgraphs of $\leq N_0$ vertices or a matrix of symmetric subgraphs (if n is large enough). Let M be a constant such that in (16*), $\beta(W) < M\, v(W)$. Let $m_k = \rho\, 2^\tau\, (c_{kM})^{-1}$ and $\epsilon = \delta = 1/2\tau$. The constant N_0 is the corresponding constant of Lemma 3.2.1. Now we apply Lemma 3.3.2 to S^n and $\hat{m}_k \equiv N_1$. Thus we obtain a system U_j of symmetric graphs ($j = 1, \dots, N_1$). If $v(U_j) \leq N_0$, then there is nothing to prove. If $v(U_j) > N_0$, then we apply Lemma 3.2.1 to U_1. Thus we obtain a system $T_{1,j}$ of symmetric subgraphs of U_1, $j = 1, \dots, m_k$ ($k = v(T_{1,j})$). As we know, at most $\frac{1}{2}(\tau-1)$ vertices of $G(A_1) - U_1$ are joined to U_1 and at most $\frac{1}{2}(\tau-1)$ vertices of $U_1 - \bigcup T_{1,j}$ are joined to $\bigcup T_{1,j}$. Therefore at least $2^{-\tau} m_k$ subgraphs from $T_{1,1}, \dots, T_{1,m_k}$ are symmetric in $G(A_1)$ as well. Applying Lemma 3.3.1 (and (16*) with the constant M) to these graphs we obtain ρ of them symmetric in S^n, too. If $\psi_i \colon U_1 \to U_i$ are the isomorphisms in the definition of symmetry and $T_{1,j}$, $j = 1, \dots, \rho$, are the symmetric subgraphs of U_1, then $T_{i,j} =: \psi_i(T_{1,j})$ is trivially a matrix of symmetric graphs.

Applying the result of (c), the simpler case when we obtain N_1 symmetric graphs of $\leq N_0$ vertices will always be neglected.

(d) Let be given for every p a matrix of symmetric subgraphs $T_{p,i,j}$ in $G(A_p)$. Then

$$(18) \qquad \mathop{\times}_{p} \sum_{i} \sum_{j} T_{p,i,j} \subseteq S^n .$$

Indeed, if e.g. $x \in T_{1,1,1}$ and $y \in T_{2,1,1}$ and $(x, y) \notin S^n$, then

$$N_1 = N_0! \leq \alpha(x) \leq \alpha(T_{1,1,1}),$$

contradicting (16*) if N_0 is large enough what can be assumed.

If now we omit $T_{p,i,\rho}$ for every $p = 1, \dots, d$ and $i = 1, \dots, N_1/v_p$ ($v_p = v(T_{p,i,j})$), then we obtain a graph $G^{n-N_1 d}$ such that

$$S^n \in \mathsf{D}(G^{n-dN_1}).$$

Our main purpose is to prove that G^{n-dN_1} is also an extremal graph. Because of Definition 1.5(iii) it is an A-graph and it does not contain sample graphs. Therefore, the fact that it is an extremal graph or not depends only on the number of its edges. In order to characterize $e(G^{n-dN_1})$, we introduce a new function

$$\Phi(S^n) = \sum_p \frac{\alpha(T_{p,i,j})-\beta(T_{p,i,j})}{v(T_{p,i,j})} = \sum_p \frac{\alpha_p - \beta_p}{v_p}. \tag{19}$$

(Here we have to suppose that the family $T_{p,i,j}$ is fixed!) Now

$$\begin{aligned} e(S^n)-e(G^{n-N_1 d}) = \\ = e(K_d(N_1,...,N_1))+(n-dN_1)(d-1)N_1+\Phi(S^n)N_1. \end{aligned} \tag{20}$$

To prove (20), observe that if $\alpha_p = \beta_p = 0$ for each p, then the first term is the number of edges both endpoints of which were omitted, the second term is the number of edges just one endpoint of which was omitted. In the other cases $\Phi(S^n)$ is just the "correction".

(e) First we need a definition.

Definition 1.7*. Let G^n be a given graph and $T_{p,j}$ be symmetric subgraphs of it when $j = 1, ..., \rho$ for every fixed p. Let $v(T_{p,j}) \leq N_0$ and $\mathsf{X}_p \Sigma_j T_{p,j} \subseteq G^n$. Let us symmetrize mN_1 new vertices to $T_{1,1}$ and then mN_1 new vertices to $T_{2,1}$ and so on. The obtained graph of $n+dmN_1$ vertices will be denoted by $\mathsf{D}^{*m}(G^n)$. Clearly, the operator D^{*m} is a very special case of D^m.

Without loss of generality we may restrict our consideration to the case, when an N_2 is given and $n \equiv N_2 \pmod{N_1}$. Let us fix a system $T_{p,i,j}$ in each considered S^n. Thus $N_1\Phi(S^n)$ is a well-defined bounded integer valued function. (it is bounded because of (16), (16*) and integer because N_1 is divisible by $v(T_{p,i,j})$.) Let S^h be a considered extremal graph for which it attains its maximum, $T'_{p,i,j}$ be the fixed system of matrices in S^h. Now we can apply D^{*m} to S^h and the set $T'_{p,1,j}$. Let

$$Z_m =: \mathsf{D}^{*m}(S^h).$$

Further, for the sake of simplicity, if S^{h+dmN_1} is an extremal graph, we denote it by S_m. Since Z_m is an A-graph not containing any L_i, we have

(21) $$\Delta(m) = e(S_m) - e(Z_m) \geq 0.$$

If we could prove that

(22) $$\Delta(m+1) \leq \Delta(m),$$

then

(23) $$\Delta(m+1) = \Delta(m)$$

would hold for every m large enough.

(f) Now we prove (22). The proof may seem too technical for the reader; however, it is the special case of a general method used in [12] (so called "progressive induction"). The advantage of this method is that one can apply induction even in the cases when the assertion is not true for small values of m, i.e., we do not have an "initial hypothesis".

There exists an $\hat{S}_m$ of $h+dmN_1$ vertices such that

$$S_{m+1} \in \mathrm{D}(\hat{S}_m).$$

As we have seen, $\hat{S}_m$ is an A-graph not containing any L_i. Thus

(24) $$\begin{aligned}\Delta(m+1)-\Delta(m) &= \{e(S_{m+1})-e(S_m)\}-\{e(Z_{m+1})-e(Z_m)\}\\ &\leq \{e(S_{m+1})-e(\hat{S}_m)\}+\{e(Z_{m+1})-e(Z_m)\}\\ &= N_1\{\Phi(S_{m+1})-\Phi(Z_{m+1})\}\\ &= \{\Phi(S_{m+1})-\Phi(S^h)\}N_1 \leq 0.\end{aligned}$$

(Here we applied (20) and $\mathrm{D}(Z_m) \ni Z_{m+1}$.) Thus we proved (22) and, consequently, (23) too: for $m > m_0$, $\Delta(m)$ is constant. Hence in (24), we have equality everywhere. Therefore $\hat{S}_m$ is an extremal graph. *Thus we proved that if m is large enough, every extremal graph S_{m+1} can be obtained from an extremal graph S_m by applying* D: $S_{m+1} \in \mathrm{D}(S_m)$.

Clearly, this is one half of Theorem 3. The other half is that if we apply D to an extremal graph, the obtained graph is always extremal ($m > m_0$!).

Since we have equality in (24) everywhere,

$$\Phi(S_m) = \Phi(Z_m) = \Phi(Z_0)\,,$$

i.e., $\Phi(S_m)$ is a constant if $m > m_0$. Let $S^*_{m+1} = \mathsf{D}(S_m)$. It does not contain any sample graph and is an A-graph. Further,

$$e(S^*_{m+1}) - e(S_{m+1}) = \{e(S^*_{m+1}) - e(S_m)\} - \{e(S_{m+1}) - e(\hat{S}_m)\}$$
$$= N_1(\Phi(S_m) - \Phi(\hat{S}_m)) = 0\,.$$

Thus S^*_{m+1} is really an extremal graph.

4. Proofs of Theorems 1.2

Let us notice that Theorem 1 is already proved: using the notations of the proof above, we have

$$S^{h+mdN_1} \in \mathsf{D}^m(S^h) \subseteq \mathsf{G}(h+mdN_1, h, d)\,.$$

Here h depends also on N_2, but since it is enough to consider a finite number of possible choices for N_2, there exist an r majorant for all of them. Now $\mathsf{G}(n, r, d)$ contains at least one extremal graph for every n large enough.

To prove Theorem 2 let us consider the graphs U^h and V^h and let us fix the systems $T_{p,j} \subset U^h$ and $Z_{p,j} \subset V^h$, so that D^{*m} could be applied to them. First we prove that if for infinitely many m,

$$\mathsf{D}^{*m}(U^h) \simeq \mathsf{D}^{*m}(V^h)\,, \tag{25}$$

then $U^h \simeq V^h$. Trivially, (25) implies that by an appropriate indexing, one may achieve that $T_{p,j} \simeq Z_{p,k}$. Now we choose a maximal system of symmetric subgraphs $T^*_{p,j}$ of $\mathsf{D}^{*m}(U^h)$ such that

$$\mathop{\times}_{p} \sum_{j} T^*_{p,j} \subseteq \mathsf{D}^{*m}(U^h)$$

(where the symmetry is required for every fixed p and the maximality means the maximality of $\Sigma\,\Sigma\, v(T^*_{p,j})$). Clearly, if m is large enough, then this system is just the system of subgraphs symmetric to one of $T_{p,1}$, $p = 1, \ldots, d$. Omitting for every p exactly $mN_1/v(T_{p,1})$ symmetric subgraphs from these ones, we obtain always the same graph up to an isomorphism and consequently, just U^h. Therefore we can describe how to reconstruct from $\mathsf{D}^{*m}(U^h)$ the original graph U^h in terms, invariant of isomorphisms: maximal set of symmetric subgraphs, omitting a given number of these subgraphs. Therefore the isomorphism of graphs $\mathsf{D}^{*m}(U^h)$ and $\mathsf{D}^{*m}(V^h)$ implies $U^h \simeq V^h$.

Now we can prove Theorem 2 very easily:
Let U^h and V^h be given extremal graphs, h be large enough and let us consider $\mathsf{D}^{*m}(U^h)$ and $\mathsf{D}^{*m}(V^h)$. They are extremal graphs and either they are not isomorphic for all sufficiently large m, then there is nothing to prove ($r_0 = h$), or they are, and then U^h and V^h are also isomorphic graphs. (Notice that $\mathsf{D}^{*m}(U^h), \mathsf{D}^{*m}(V^h) \in \mathsf{G}(n, h, d)$!)

Appendix

(A) *The outline of the proof of Lemma 3.1.1*

Let G^n be a graph satisfying the conditions of Lemma 3.1.1. Theorem B is applicable to G^n for every $\epsilon > 0$, thus G^n can be obtained from a product $\mathsf{X}\, G^{n_p}$ by omitting $o(n^2)$ edges. Here

$$\sum e(G^{n_p}) = o(n^2), \qquad |n_p - n/d| = o(n)\,.$$

Let

$$\begin{aligned} A_p &= \{x : x \in G^{n_p},\ \alpha(x) < \epsilon^2 n,\ \beta(x) < \epsilon n\}\,, \\ B_p &= \{x : x \in G^{n_p},\ \alpha(x) \geq \epsilon^2 n\}\,, \\ C_p &= \{x : x \in G^{n_p},\ \alpha(x) < \epsilon^2 n,\ \beta(x) \geq \epsilon n\}\,. \end{aligned}$$

Since

$$o(n^2) = e(G^{n_p}) = \tfrac{1}{2} \sum_{x \in G^{n_p}} \alpha(x) \geq \tfrac{1}{2} \sum_{x \in B_p} \alpha(x) \geq |B_p|\, \epsilon^2 n\,,$$

thus $|B_p| = o(n)$. Similarly, $\beta(G^{n_p}) = o(n^2)$ implies that $|C_p| = o(n)$. Therefore the proof of $|B_p| = O_\epsilon(1)$ (see [12, p.316]) remains valid without any essential change. Thus $|B_p| = O_\epsilon(1)$. Now, if $\epsilon < \frac{1}{2}$, then C_p contains only vertices of valence $< (1-(1/d)-\frac{1}{2}\epsilon)n$. *This will imply* $|C_p| = O_\epsilon(1)$. Indeed, if $x_1, \ldots, x_\nu$ are in C_p, let $G^{n-\nu} = G^n - \{x_1, \ldots, x_\nu\}$.

Now

$$\text{(A1)} \qquad e(G^{n-\nu}) \geq e(G^n) - n(1-(1/d)) - \tfrac{1}{2}\epsilon\, \nu n + \binom{\nu}{2}.$$

If we knew

$$\text{(A2)} \qquad f(n; L_1, ..., L_\lambda) - f(n-\nu; L_1, ..., L_\lambda) \geq \nu n(1-(1/d) + o(1))$$

for $\nu < n^{1/4}$, then we should have

$$\text{(A3)} \qquad \begin{aligned} e(G^{n-\nu}) &\geq f(n; L_1, ..., L_\lambda) - Kn - n\,\nu(1-(1/d) - \tfrac{1}{2}\epsilon) + \binom{\nu}{2} \\ &\geq f(n-\nu; L_1, ..., L_\lambda) - o(n) + (\tfrac{1}{2}\epsilon\nu - K)n\,. \end{aligned}$$

On the other hand ,

$$\text{(A4)} \qquad e(G^{n-\nu}) \leq f(n-\nu; L_1, ..., L_\lambda)\,.$$

Thus $(\frac{1}{2}\epsilon\nu - K)n = O(n)$, i.e. $\nu = O(1)$. This proves that

$$|C_p| = O(1)\,.$$

Let $U \subset G(A_p)$ be a subgraph of $\leq \tau$ vertices. We can determine recursively τ vertices in each $G(A_j)$ $(j \neq p)$, determining a

$$U \times K_{d-1}(\tau, ..., \tau) \subset G^n.$$

Therefore $G(A_p)$ does not contain T. One can easily prove that $f(n; T) = O(n)$. Therefore

$$e(G^{n_p}) \leq e(G(A_p)) + O(n) = O(n)\,.$$

This means that (supposing (A2)) we have proved already (iv), (ii) and (v). In order to prove (i) we choose a K^* such that $e(G(A_p)) < K^* n$ (for every p) and suppose that at least $2Kn + K^* dn$ edges were omitted from the product $\times\, G^{n_p}$. Then

$$\begin{aligned} f(n; L_1, ..., L_\lambda) - Kn \leq e(G^n) &\leq e(K_d(n_1, ..., n_d)) + K^* dn - K^* dn - 2Kn \\ &= f(n; L_1, ..., L_\lambda) - 2Kn\,. \end{aligned}$$

This contradiction proves (i). The proof of (iii) is the same as in [12], (iii) is an immediate consequence of [14, Lemma 1]. Therefore the only thing we have to prove is (A2).

Let now G^n be an extremal graph for $L_1, ..., L_\lambda$. There exist in A_1, $y_1, ..., y_\tau$ such that

$$\sum \alpha(y_i) + \beta(y_i) = \mathrm{o}(n).$$

We "quasisymmetrize" ν new vertices, i.e. $z_1, ..., z_\nu \notin G^n$ are joined just to the vertices of $\bigcap_{j \leq \nu} \mathrm{st}\, y_i$. The proof of Lemma 3.4.1 now gives that the obtained $G^{n+\nu}$ does not contain any L_i. Thus

$$\begin{aligned} f(n+\nu; L_1, ..., L_\lambda) &\geq e(G^{n+\nu}) \geq e(G^n) + \nu n(1-(1/d)\,\mathrm{o}(1)) + \nu \mathrm{O}(1) \\ &= f(n; L_1, ..., L_\lambda) + \nu n(1-(1/d) + \mathrm{o}(1)). \end{aligned}$$

This proves (A2).

(B) *On the chromatic conditions*

First of all, why are the chromatic conditions called so? Because of the first example after Definition 1.5.

Now we prove that the first and second examples are really chromatic conditions according to our definition. It is enough to consider only the second case. (i) is trivial. In order to prove (ii), let us consider a graph of chromatic number $t + u$ each circuit of which is longer than ω. (According to [10] there exist such graphs.) This proves (ii). Now, if G is an arbitrary graph and $T_1, ..., T_\rho$ are symmetric subgraphs in it and omitting $x_1, ..., x_u$ from it we obtain a $< t$-chromatic graph, then we colour this last graph by $t-1$ colours. If $\rho > u$, at least one T_i does not contain any x_j. Let us colour all the vertices of the other T_i's in the same way, i.e. let the corresponding vertices have the same colour. This is a good colouring of the graph $G - \{x_1, ..., x_u\}$ by $t-1$ colours. This proves (iii).

(C) Let S^h be a given graph and let the parameters of D be fixed. *When are the graphs* $\mathsf{D}^{*m}(S^h)$ *extremal graphs for some sample graphs* $L_1, ..., L_\lambda$?

Definition A.1. If for every k omitting any $k\,d\,N_1$ vertices of $\mathsf{D}^{*m}(S)$, we obtain a graph having at most as many edges as (strictly less edges than) $\mathsf{D}^{*m-k}(S)$ has, then $\mathsf{D}^{*m}(S)$ will be called a (strictly) balanced regular sequence of graphs.

Theorem A.2. *The following assertions are equivalent*:

(i) $\mathsf{D}^{*m}(S)$ *is* (*strictly*) *balanced*;

(ii) *there exist sample graphs* $L_1, ..., L_\lambda$ *such that if m is large enough*, $\mathsf{D}^{*m}(S)$ *is an* (*is the only*) *extremal graph for* $L_1, ..., L_\lambda$ *for the considered numbers of vertices.*

This theorem shows that our result, formulated in Theorem 1 is the best possible. The proof will be published elsewhere. (ii) → (i) is trivial.

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