EXTREMAL GRAPH PROBLEMS

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Notations. \( \nu(G), \sigma(G), \chi(G) \) denote the number of vertices, edges and the chromatic number of the graph \( G \). Here the graphs have no directed, multiple or loop edges. \( \times \sum_{i=1}^{d} G_i \) denotes the product of graphs \( G_i \), i.e. the graph, obtained by joining vertices of \( G_i \) to the vertices of the other \( G_i \)-s.

Generalizing a well-known theorem of Turán [1] Erdős and I have proved independently [3], [4] that for any given graph \( M_1, \ldots, M_k \) and fixed \( n \) if \( K_n \) has maximum number of edges among graphs of \( n \) vertices, not containing any \( M_i \) as a subgraph, then

**Theorem A.** There exist graphs \( N_1, \ldots, N_d \), \((d+1 = \min \chi(M_i))\) such that \( K_n \) can be obtained from \( \times \sum_{i=1}^{d} N_i \), omitting \( O(n \frac{2}{d-1}) \) edges from it. Here is an integer depending only on \( M_1, \ldots, M_k \) and

\[
\nu(N_i) = \frac{n}{d} + O(n \frac{1}{d-1}), \quad \sigma(N_i) = O(n \frac{2}{d-1})
\]

and

\[
\nu(N_i) = \frac{n}{d} + O(n \frac{1}{d-1})\]

(2) any vertex of \( N_i \) has valence \( \geq \frac{n}{d} (d-1) + O(n \frac{1}{d-1}) \)

(3) the number of vertices of \( N_i \) joined to at least one vertex of the same \( N_i \) is 0 (1).

The graph \( K_n \) is called the extremal graph for \( M_1, \ldots, M_k \). Theorem A shows that the extremal graphs for \( M_1, \ldots, M_k \) are fairly well determined by \( \min \chi(M_i) \), they depend loosely on the structure of \( M_i \)-s.

How the structure of \( M_i \)-s influence the structure of the extremal graphs? Erdős and I have proved [5] that the extremal graphs for \( K(3, r_1, \ldots, r_d) \) are products: \( K^n = \times \sum_{i=1}^{d} N_i \) where \( 3 \leq r_1 \leq r_d \) and

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(1) \( \nu(N_t^d) = \frac{n}{d} + o(n^{2/3}) \)

(2) \( N_t^d \) is an extremal graph for \( K(3,r) \).

(3) \( N_1, \ldots, N_d \) are extremal graphs for \( K(n, r) \).

Here 3 can be replaced by 2 or 1 as well.

I have found the following generalization of this latest theorem:

**Notation.**

(1) \( f(n, M_1, \ldots, M_d) \) denotes the number of edges of the extremal graphs for \( M_1, \ldots, M_d \).

(2) Let \( \chi(M) = 2 \) and colour both \( M \) and \( K(n, n) \) by two colours: red and blue. We consider subgraphs \( G^{2n} \) of \( K(n, n) \) such that if \( M \) is the subgraph of \( G^{2n} \), then the class of blue vertices of \( M \) is not contained by the class of blue vertices of \( K(n, n) \). The maximum of \( \varepsilon(G^{2n}) \) will be denoted by \( h(n, G^{2n}) \).

**Definition.** \( x \in M \) is a weak point for \( M_1, \ldots, M_d \) if \( \chi(M_1) = 2 \) and \( h(n; M_1 - x) = o(f(n; M_1, \ldots, M_d)) \).

**Remark.** If there exists an automorphism of \( M_1 - x \) changing the colours, then our condition with \( f(n; M_1 - x) = o(f(n; M_1, \ldots, M_d)) \).

**Examples.**

(1) \( K(r_0, \ldots, r_d) \) has weak points if either \( r_0 \not\in \mathbb{Z} \), or if \( n_0^2 - 3r_0 + 3 > r_1 \). [5] Probably it always has.

(2) If \( M \) is not a tree, but \( M - x \) is, \( \chi(M) = 2 \) then \( x \in M \) is a weak point of it.

(3) Let \( C(2l) \) be a circuit of \( 2l \) vertices, \( x \in C(2l) \) and let \( z \) be joined to 5 or more vertices of \( C(2l) \) so that the obtained graph \( M \) is two-chromatic. Then \( x \in M \) is a weak point of it.

(4) Let \( M \) be a graph, obtained from two \( C(2l) \) or from two \( K(r, r) \) by joining them by a path of length 2. Then \( M \) has no weak point.

**Theorem.** Let \( M \) be a \( d+1 \) chromatic graph and let us colour it by \( 1, 2, \ldots, d+1 \). \( L_{i,j} \) denotes the subgraph of \( M \) spanned by the vertices of the \( i \)-th and \( j \)-th colours. If \( x \in L_{i,j} \) is a weak point of \( (L_{i,j}) \) and \( K^d \) is an extremal graph for \( M \), then \( K^d \) can be obtained from a suitable product \( \prod_{i=1}^{d} N_i \) omitting \( o(n) \) edges from it. Here
(1) \( \nu(N_i) = \frac{n^2}{d} + o(n) \)

(2) \( N_i \) is almost an extremal graph for \( (L_{d,j}^i) \) it has
\( f(n; \ldots, L_{d,j}^i, \ldots) + o(n) \) edges, but it does not contain any \( L_{d,j}^i \).

(3) The vertices of \( N_i \) (i=2,\ldots,d) are joined to less than \( s \) other vertices of \( N_i \), if \( s \) is joined to \( e \) vertices of the 3rd colour.

**Theorem 2.** If in Theorem 1, \( r \leq 3 \), then \( o(n) \) can be replaced by \( o(1) \).

If \( r \leq 2 \), then there exists an extremal graph \( K^3 \) such that
\[
K^3 = \times_{i=1}^d N_i \text{ whenever } n \text{ is large enough.}
\]

**Remarks.**

(1) Similar theorems hold if \( M \) is replaced by \( M_1^1, \ldots, M_\mu^\mu \). The only change is that \( L_{d,j}^i \)-s must be replaced by those subgraphs of \( N_i^i, \ldots, M_\mu^\mu \), for which \( \chi(N_d^i - L_e) = \min \chi(N_d) - 2 \) if \( L_e \subseteq N_d^i \).

(2) Theorem 1 has "assymptotic" character, but it has many corollaries of "exact" character. One of them is the theorem of Erdös and mine about the extremal graphs for \( K(3, P_1, \ldots, P_d) \).

Another one is

**Theorem 3.** Let \( \Gamma(3k) \) be the graph, having the vertices \( x_1, \ldots, x_k \);
\( y_1, \ldots, y_k \); \( z_1, \ldots, z_k \) and defined by

(1) \( x_i \rightarrow y_i \rightarrow z_i \rightarrow x_i \) is an automorphism of \( \Gamma(3k) \).

(11) \( x_1, \ldots, x_k, y_1, \ldots, y_k \) determine a \( C(2l) \).

Then for \( n > n_0 \) any extremal graph \( K^n \) for \( \Gamma(3k) \) is a product:
\[
K^n = k_1 \times k_2 \text{ where } v(k_1) = \frac{n}{2}, e(k_2) = 0 \text{ and } K_1 \text{ is an extremal graph for } \{ \ldots, C(2l), \ldots \} \frac{k_1}{2} \leq l \leq k.
\]

**References**

2. Turán, P., Matematikai Lapok, 48 (1941), 436-452. (in Hungarian).

