The complex dynamics of the chips taking game
or
What can we get out of a simple combinatorial game?

István Miklós[1]

[1]Part of the presentation is a joint work with former BSM students
Mariam Abu-Adas and Logan Post

BSM seminar, November 10, 2022

What happens if...

your combinatorics professor gives the following exercises:

(a) An independent set of a graph $G = (V, E)$ is a subset of vertices $I \subseteq V$ such that for all $v_1, v_2 \in I$, $(v_1, v_2) \notin E$. An independent set I is maximal if there is no independent set I' such that $I \subset I'$. Give a recurrence that counts the maximal independent subsets in C_n, the cycle of n vertices, and solve it.

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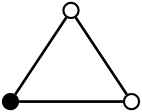

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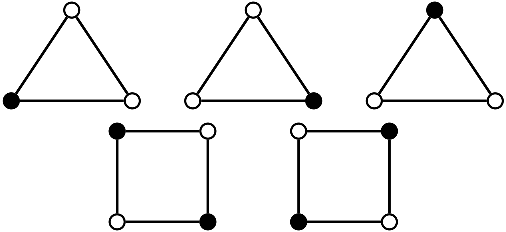

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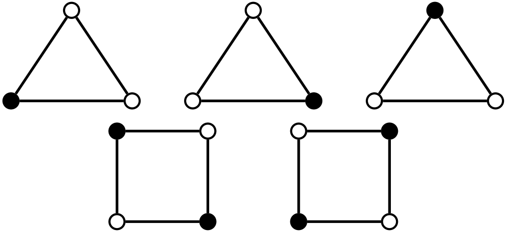

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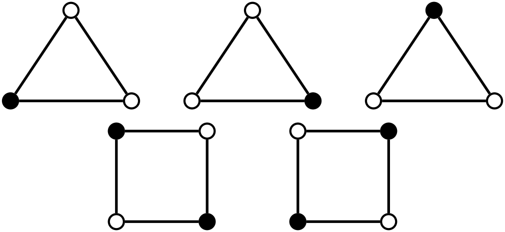

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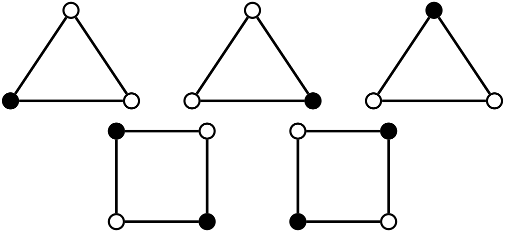

5 maximal independent sets in a pentagon,
5 maximal independent sets in a hexagon,
7 independent sets in a heptagon, etc.

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your combinatorics professor gives the following exercises:

(c) Let $[x]$ denote the closest integer to x. Prove that starting with $p = 7$, for all prime numbers p,

$$\left[\left(\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}\right)^p\right]$$

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For example,

- $\left(\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}\right)^7 \approx 7.1592$
- $\left(\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}\right)^{11} \approx 22.0474$
- $\left(\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}\right)^{13} \approx 38.6905$
- $\left(\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}\right)^{17} \approx 119.1511, \quad 119 = 7 \times 17$

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(d) Find a relationship between exercises (a), (b), (c) and the number 271441.

The story started at RES 2022 Spring...

The chips taking game

Let $A = \{a_1, a_2, \ldots a_k\}$ be a set of positive integers, let $n > \max\{A\}$ be a positive integer, and let g be a function mapping from $\{1, 2, \ldots, \max\{A\}\}$ to $\{0, 1\}$.

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Starting with n chips, Alice and Bob take any $a_i \in A$ chips from the pile of chips. The number of the chips in the pile will be eventually some $x \leq \max\{A\}$. The winner of the game is the current player if $g(x) = 1$, and the opposite player if $g(x) = 0$.

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The discrete mathematical problem is to compute who has the winning strategy.

Chips taking game example

$A = \{1, 2, 3, 4\}$, $g(1) = g(2) = g(3) = g(4) = 1$, $n = 20$
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Claim: Bob has the winnig strategy. The strategy is that if Alice takes a chips, Bob takes $5 - a$. Then after each pair of steps, the number of chips in the pile will be divisable by 5.

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Who has the winning strategy if the number of chips at the beggining is $n = 21$?

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Who has the winning strategy if the number of chips at the beggining is $n = 21$?
It is now Alice. She takes 1 chips, so now $n = 20$, and Bob is the first player, Alice is the second one, and she has the winning strategy, see above.

Dynamic programming recursion

Claim

We define $f(n) = 1$, *if Alice has the winning strategy starting by n chips, and* $f(n) = 0$ *if Bob has the winning strategy. Then* $f(n) = g(n)$ *for all* $n \leq \max\{A\}$, *and for* $n > \max\{A\}$

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For example, if $A = \{1, 2, 3, 4\}$, and $g(1) = g(2) = g(3) = g(4) = 1$, then

n	1	2	3	4	5	6	7	8	9	10	11
f(n)	1	1	1	1	0	1	1	1	1	0	1

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From this proof, it is clear that the largest possible period is $2^{\max\{A\}}$. It is an open question if there exists a period larger than a suprapolynomial function of $\max\{A\}$. The typical period length is a linear function of $\max\{A\}$.

Periods of $A = \{n, m\}$, $g.c.d(n, m) = 1$

Theorem

Let $A = \{n, m\}$ and let n and m be relatively primes. Then for all $d|(n+m)$, $d \neq 1, 4, 6$, there exists a g such that the period length of f is d. No other period length is possible.

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The corollary is that any superperiod is tiled by patterns 01 and 011. Then any period is a divisor of k, and the period length cannot be 1 (not tilable), 4 and 6 (these are also superperiods).

How the heck will we prove in 10 minutes from now that for any prime number p at least 7, p divides

$$\left[\left(\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}\right)^p\right] ?!?!?!$$

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Corollary: any period length $d|k$ $d \neq 1, 4, 6$ is possible as periods with such lengths have a tiling with 01 and 011 patterns that are not superperiods.

Logan's trick

Going from $A = \{1, k-1\}$ to $A = \{n, m\}$

If n and m are relatively primes, then so n and $n + m$.

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It can be shown (non-trivial!) that this bijection preserves the period lengths, too.

Number of periods

Proposition

Let $s(k,d)$ denote the number of g functions such that $f_{\{1,k-1\},g}$ has superperiod d, and let $n(k,d)$ denote the number of g functions factorized by cyclic permutations such that $f_{\{1,k-1\},g}$ has period d. Then

$$n(k,d) = \frac{s(k,d) - \sum_{d'|d}(d' \times n(k,d'))}{d}$$

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Observation

Observe that for any $d|k$, $s(k,d) = s(d,d)$ and $n(k,d) = n(d,d)$. Therefore, if we denote $s(k,k)$ by $s(k)$ and $n(d,d)$ by $n(d)$, we get that

$$n(k) = \frac{s(k) - \sum_{d|k}(d \times n(d))}{k}$$

Number of superperiods and maximal independent sets of C_k

Claim

There is a 1-1 correspondence between superperiods of length k built from 01 *and* 011 *and the maximal independent sets of C_k.*

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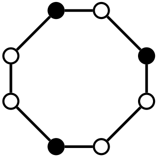
Example:



01011011

Recurrence for the number of maximal independent sets

Claim

Let $S(k)$ denote the number of maximal independent sets in C_k. Then

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Furthermore,

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$$\rho_1 = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}}.$$

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Because both ρ_2 and ρ_3 are smaller than 1 in absolute value!

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Definition

A Pisot-Vijayaragavhan number or simply a PV number or Pisot number is a positive real algebraic integer larger than 1 *such that all of its Galois conjugates have absolute value less than* 1*.*

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That is, a PV number is a positive, greater than 1 root of a polynomial with integer coefficients and leading coefficient 1, such that all other roots of that polynomial are smaller than 1 in absolute value.
The powers of the PV numbers modulo 1 have a very biased distribution. That is for any PV number ρ, it holds that

$$\lim_{n\to\infty} |\rho^n - [\rho^n]| = 0.$$

Funny properties of the Fibonacci numbers

The golden ratio is also a PV number!

It is known that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$$

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$\sqrt{5}\times 1 \approx 2.236$	$\sqrt{5}\times 1 \approx 2.236$
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$\sqrt{5}\times 5 \approx 11.180$	$\sqrt{5}\times 8 \approx 17.888$
$\sqrt{5}\times 13 \approx 29.068$	$\sqrt{5}\times 21 \approx 46.957$
$\sqrt{5}\times 34 \approx 76.026$	$\sqrt{5}\times 55 \approx 122.984$
$\sqrt{5}\times 89 \approx 199.010$	$\sqrt{5}\times 144 \approx 321.994$
$\sqrt{5}\times 233 \approx 521.004$	$\sqrt{5}\times 377 \approx 842.998$

Pisot-Vijayaragavhan numbers

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Pisot-Vijayaragavhan numbers

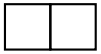
Salem proved that the set of PV numbers is closed, therefore there is a smallest PV number (as they are greater than 1). The smallest PV number is

$$\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}},$$

called the plastic number.

The geometry of the plastic number

Golden ratio spiral



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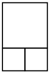

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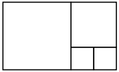

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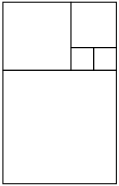

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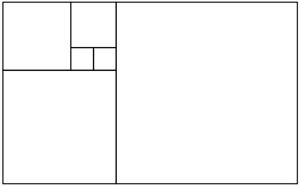

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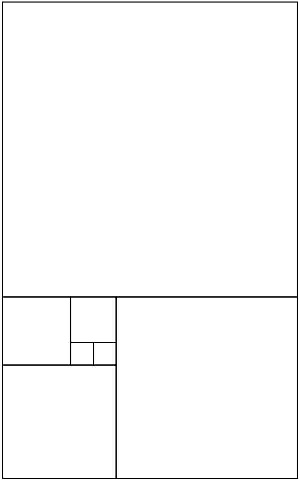

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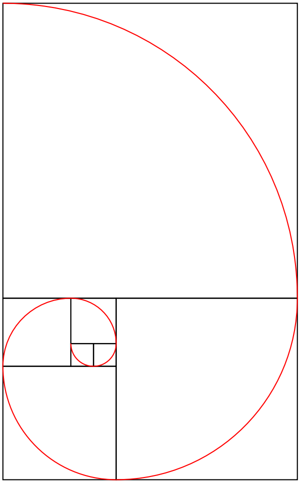

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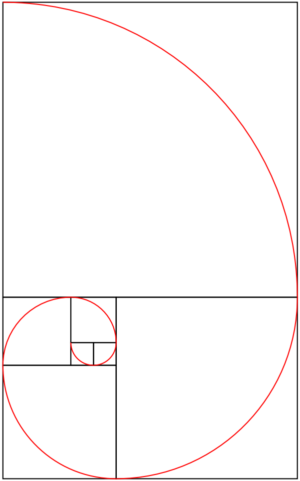

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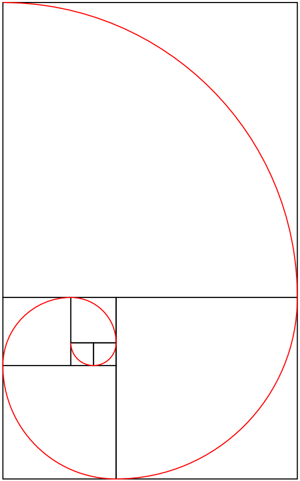

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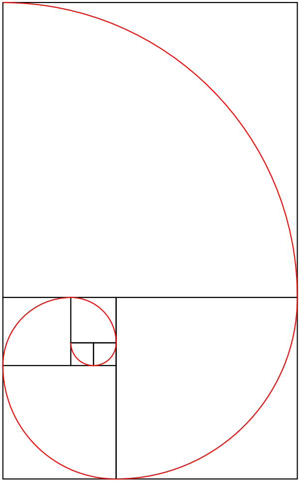

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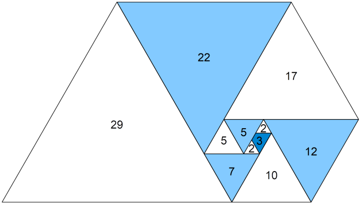

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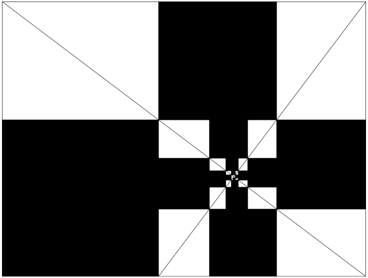

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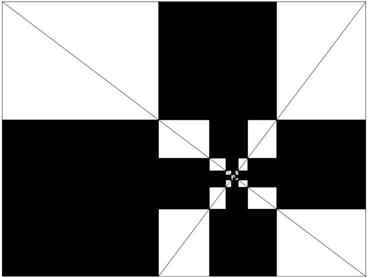

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The 1967 St. Benedictusberg Abbey church by Hans van der Laan has plastic-number proportions,
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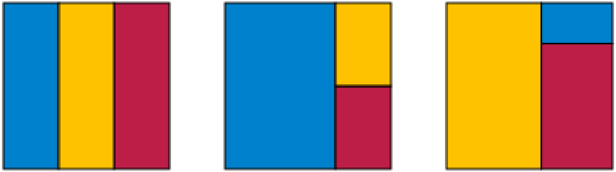
The geometry of the plastic number

Solving subexercise (b)



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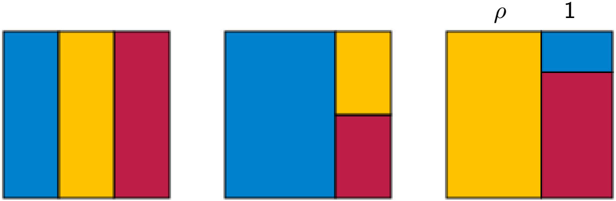

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The Perrin pseudoprimes

Solving subexercise (d)

The sequence

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, \ldots$$

is also called the Perrin sequence.

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Geometric/combinatorial proof of Euler's theorem

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$$a^p \equiv a \mod p$$

More general divisibility rule for PV numbers

(courtesy of prof. Péter Maga)

Claim

Let ρ be a PV number such that it is a root of $f(x) = \sum_{k=0}^{n} a_k x^k$ with $a_{n-1} = 0$. Then there exists a p_0 such that for all prime numbers $p \geq p_0$, $p | [\rho^p]$.

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Applying the multinomial expansion theorem:

$$0 = \sum_{k=0}^{n} \rho_k^p + \sum_{s_1+s_2+\ldots+s_n=p,\ s_k \neq p} \binom{p}{s_1, s_2, \ldots, s_n} \prod_{k=0}^{n} \rho_k^{s_k}.$$

□

Proof (cont'd)

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But $\sum_{k=0}^{n} \rho_k^p$ is an integer (for example, due to Newton's sums), thus the algebraic integer in question is rational.

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That is,

$$0 = \sum_{k=0}^{n} \rho_k^p + p \times [\text{algebraic integer}].$$

But $\sum_{k=0}^{n} \rho_k^p$ is an integer (for example, due to Newton's sums), thus the algebraic integer in question is rational.However, any rational algebraic integer is an integer, thus we get that

$$\sum_{k=0}^{n} \rho_k^p \equiv 0 \mod p.$$

Due to the PV property, we get that for large p,

$$p | \left[\rho^p\right].$$

□

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Claim

Starting with $p = 5$, for all prime numbers p,

$$p| \left[\sqrt{5}F_{p-1} - 1\right].$$

Generalizing the dynamic programming recursion

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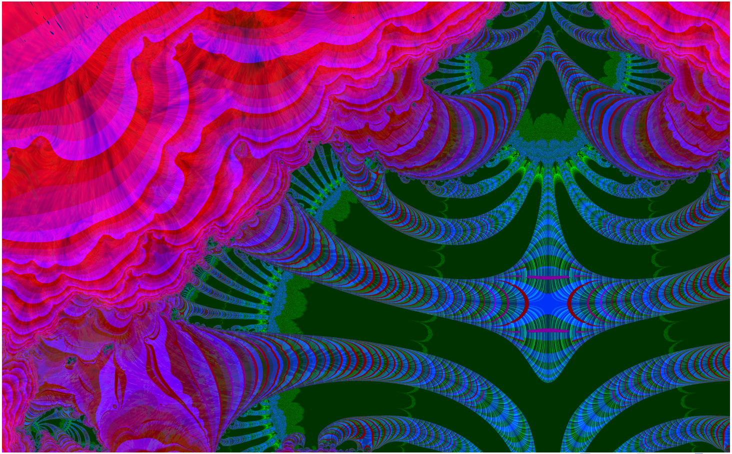

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This function is not continuous at the positive real axis, but we can live with that.

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Life is complicated :)

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- Combinatorial and algebraic methods will be widespread.
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Thank you!