Geometry of Multi-dimensional Dispersing Billiards

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Abstract

Geometric properties of multi-dimensional dispersing billiards are studied in this paper. On the one hand, non-smooth behaviour in the singularity submanifolds of the system is discovered (this discovery applies to the more general class of semi-dispersing billiards as well). On the other hand, a self-contained geometric description for unstable manifolds is given, together with the proof of important regularity properties. All these issues are highly relevant to studying the ergodic and statistical behaviour of the dynamics.

1 Introduction

Let $\mathbb{Q}$ be an open connected domain in $\mathbb{R}^d$ or on the d-dimensional torus T^d. Assume that the boundary $\partial\mathbb{Q}$ consists of a finite number of C^k smooth ($k \geq 3$) compact hypersurfaces (possibly, with boundary). Now let a pointwise particle move freely (along a geodesic line with constant velocity) in $\mathbb{Q}$ and reflect elastically at the boundary $\partial\mathbb{Q}$ (by the classical rule "the angle of incidence is equal to the angle of reflection"). This is what is commonly refered to as a billiard dynamical system.

Billiards make an important class in the modern theory of dynamical systems. Many classical and quantum models in physics belong to this class, most notably, the Lorentz gas [Si] and hard ball gases studied as early as the XIX century by L. Boltzmann [Bo].

The periodic Lorentz process is obtained by fixing a finite number of disjoint convex bodies $B_1, \ldots, B_s \subset \mathrm{T}^d$ with smooth boundary and putting the moving particle in the exterior domain $\mathbb{Q} = \mathrm{T}^d \setminus (\cup B_i)$. This system models the motion of an electron among a periodic array of molecules in a metal, as it was introduced by H. Lorentz in 1905.

Mathematical studies of billiards have begun long ago. Ya. Sinai in his seminal paper of 1970 [Si] described the first large class of billiards with truly chaotic behavior – with nonzero Lyapunov exponents, positive entropy, enjoying ergodicity, mixing, and (as was later discovered by G. Gallavotti and D. Ornstein [GO]) the Bernoulli property. Sinai billiards are defined in two dimensions ($d = 2$), i.e. for $\mathbb{Q} \subset \mathbb{R}^2$ or $\mathbb{Q} \subset \mathrm{T}^2$, and the boundary of $\mathbb{Q}$ must be concave (i.e., convex inward $\mathbb{Q}$), similarly to the Lorentz process (where the bodies B_i are convex). Due to the geometric concavity, the boundary $\partial\mathbb{Q}$ scatters or disperses bundles of geodesic lines falling upon it, see Fig. 1. For this reason, Sinai billiards are said to be *dispersing*.

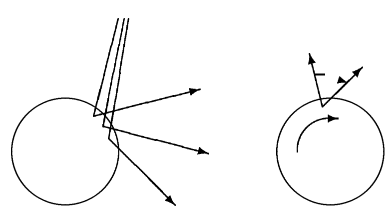

Figure 1: Scattering effect

Lorentz processes in two dimension have been studied very thoroughly since 1970. Many fine ergodic and statistical properties have been established by various researchers, including P. Bleher, L. Bunimovich, N. Chernov, J. Conze, C. Dettmann, G. Gallavotti, A. Krámli, J. Lebowitz, D. Ornstein, K. Schmidt, N. Simányi, Ya. Sinai, D. Szász, and others (see the references). The latest major result for this model (the exponential decay of correlations) was obtained by L.-S. Young [Y1]. The success in these studies had significant impact on modern statistical mechanics. The methods and ideas originally developed for the planar Lorentz process were applied to many other classes of physical models – see recent reviews by Cohen, Gallavotti, Ruelle and Young [GC, Ru, Y2].

On the other hand, the progress in the study of the multidimensional Lorentz process (where $d > 2$) has been much slower and somewhat controversial. Relatively few papers were published covering specifically the case $d > 2$, especially in contrast to the big number of works on the 2-D case. Furthermore, the arguments in the published articles were usually rather sketchy, as in Chernov's paper [Ch1]. It was commonly assumed that the geometric properties of the multidimensional Lorentz process were essentially similar to those of the 2-D system, and so the basic methods of study should be extended from 2-D to any dimension at little cost. Thus, the authors rarely elaborated on details.

Recent discoveries proved that spatial dispersing billiards are *very much* different from planar ones. Bunimovich and Reháček studies of *astigmatism* [BR], in the somewhat different context of focusing billiards, emphasized the known fact that the billiard trajectories may focus very rapidly in one plane and very slowly in the orthogonal planes. Astigmatism is unique to 3-D (and higher dimensional) billiards, it cannot occur on a plane. It plays an improtant role in higher dimensional focusing billiards as investigated in [BR].

In this paper we consider multidimensional *dispersing* billiards. We show that multidimensionality has great effect on the dynamics in the dispersing case as well – the system requires much more elaborated study than the 2D process. What is worse (cf. section 3), the singularity manifolds in the phase space of a spatial Lorentz process have pathologies – points exist where the sectional curvature is unbounded (blows up). Actually, singularity manifolds are in these pathologies – which form a two-codimensional submanifolds of them – not even differentiable. Indeed, as it will be shown in section 3, the unit normal vector to the singularity manifold has different directional limits at the pathological points – the geometry is pretty much like the classical Whitney umbrella $x^2 z = y^2$ in $\mathbb{R}^3$. This phenomenon is again unique to billiards in dimension $d \geq 3$. All these facts call for a revision of some earlier arguments and results on the multidimensional Lorentz process. This is much the more important since the studies of physically relevant multiparticle systems will require the same methods as those used for the high-dimensional Lorentz process.

Throughout the paper we conduct a systematic study of the geometry of the Lorentz process in any dimension $d > 2$, aiming at the future investigation of its ergodic and statistical properties (in particular, the decay of correlations). First we describe our recent discovery – pathological behavior of singularity manifolds – and show exactly where it occurs (in order to "localize the pathology"). Then we develop tools for the study of basic geometric properties of the dynamics – operator techniques in the Poincaré section of the phase space. By applying these geometric tools we provide rigorous proofs of important properties for unstable manifolds: we show absolute continuity, distorsion bounds, curvature bounds and alignment. All these facts are absolutely important for the studies of ergodic and statistical properties of the Lorentz gas, but strangely enough, their proofs (in the case of dimension $d > 2$) have never been published before. Lastly, we show how our results can be used in the study of the decay of correlations, which will be done in a separate paper.

2 Preliminaries

There are two ways of considering billiard dynamics, the motion of a point particle in a connected, compact domain $\mathbb{Q} \subset \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, $d \geq 2$ with a piecewise C^3-smooth boundary. The phase space of the **flow** can be identified with the unit tangent bundle over $\mathbb{Q}$ – the configuration space is $\mathbb{Q}$ while the phase space is $\mathcal{M} := \mathbb{Q} \times \mathbf{S}^{d-1}$ ($\mathbf{S}^{d-1}$ is the surface of the unit d-ball). In other words, every phase point x is of the form (q, v) where $q \in \mathbb{Q}$ and $v \in \mathbf{S}^{d-1}$. We denote the flow by $S^t : -\infty < t < \infty$.

On the other hand there is a naturally defined cross-section for this flow. The phase space of the Poincaré section **map** (or simply, of the billiard map) is $M := \partial\mathbb{Q} \times \mathbf{S}_+^{d-1}$, where $+$ means that we only take into account the hemisphere of the outgoing velocities (for a more precise definition of the phase space, see subsection 4.1). For any $x \in \mathcal{M}$ we set $t^+(x) := \inf\{t > 0 |\ S^t x \in M\}$, and $T^+x := S^{t^+(x)}x$ (of course, $T^+ : \mathcal{M} \to M$). Then the Poincaré section map $T : M \to M$ is defined as follows: $Tx := T^+x$ for $x \in M$.

We require the following properties from the system to be studied:

- Our billiard is **dispersing** (a Sinai-billiard): each $\partial\mathbb{Q}_i$ is strictly convex (had we required convexity only, our billiard would be *semi-dispersing*).
- The scatterers B_i are disjoint. This ensures the C^3-smoothness of the boundary $\partial\mathbb{Q}$, i.e. that there are **no corner points**.
- The condition that **the horizon is finite** says exactly that $t^+(x) < \infty$ for any $x \in M$.

Finally, some more notation. Let $n(q)$ be the unit normal vector of the boundary component $\partial\mathbb{Q}_i$ at $q \in \partial\mathbb{Q}_i$ directed inwards $\mathbb{Q}$. Then the invariant Liouville-measure of the discretized map is

$$d\mu(q, v) := \text{const.}\langle n(q), v\rangle dq dv \tag{2.1}$$

where dq is the induced Riemannian measure on $\partial\mathbb{Q}$ whereas dv is the Lebesgue-measure on $\mathbf{S}_+^{d-1}$.

Throughout the paper, unless otherwise emphasized, we are considering this discretized dynamics.

2.1 Fronts

In billiard theory, several basic constructions and concepts are based on the notion of a local orthogonal manifold, which - for simplicity - we will call front. A front $\mathcal{W}$ is defined in the whole phase space rather than in the Poincaré section. Take a smooth 1-codim submanifold E of the whole configuration space, and add the unit normal vector $v(r)$ of this submanifold at every point r as a velocity, continuously. Consequently, at every point the velocity points to the same side of the submanifold E. Then

$$\mathcal{W} = \{(r, v(r)) | r \in E\} \subset \mathcal{M},$$

where $v : E \to \mathbf{S}^{d-1}$ is continuous (smooth) and $v \perp E$ at every point of E. The derivative of this function v, called B plays a crucial role: $dv = Bdr$ for tangent vectors (dr, dv) *of the front.* B acts on the tangent plane $\mathcal{T}_r E$ of E, and takes its values from the tangent plane $\mathcal{J} = \mathcal{T}_{v(r)}\mathbf{S}^{d-1}$ of the velocity sphere. These are both naturally embedded in the configuration space $\mathbb{Q}$, and can be identified through this embedding. So we just write $B : \mathcal{J} \to \mathcal{J}$. B is nothing else than the curvature operator of the submanifold E. Yet we will prefer to call it second fundamental form (s.f.f.), in order to avoid confusion with other curvatures that are coming up. Obviously, B is symmetric.

Notice that fronts remain fronts during time evolution - at least locally, and apart from some singularity lines.

When we talk about a front, we sometimes think of it as the part of the (whole) phase space just described (for example, when we talk about time evolution under the flow), but sometimes just as the submanifold E (for example, when we talk about the tangent space or the curvature of the front). This should cause no confusion.

2.2 Evolution of fronts

The evolution of a front during free propagation (that is, from one collision to the other) is described by the formula

$$B_1^- = ((B^+)^{-1} + \tau Id)^{-1} \tag{2.2}$$

where τ is the length of the free run between the two collisions, B^+ is the s.f.f. of the front just after the first collision, and B_1^- is the s.f.f. just before the next one.

For this formula – and the next one – to make sense, we need to identify the tangent planes of the front at different moments of time. Let $\mathcal{T} = \mathcal{T}_r\partial\mathbb{Q}$ be the tangent plane of the scatterer at a collision point r. Just like $\mathcal{J}$, $\mathcal{T}$ is viewed together with its natural embedding into $\mathbb{Q}$. The identification of different $\mathcal{J}$'s is done in the usual way (cf. [SCh], [KSSz]):

- by translation parallel to v from one collision to the other,
- by reflection with respect to $\mathcal{T}$ (or, equivalently, by projection parallel to n) from pre-collision to post-collision moments.

Notation for the unitary operator that executes this identification is U, however, for brevity, we will often omit U if it causes no confusion.

At a moment of collision the curvature of the front changes non-continuously (the front is "scattered"):

$$B^+ = B^- + 2\Theta = B^- + 2\langle n, v\rangle V^*KV \tag{2.3}$$

where[1]

[1] This convention on the collision term ($\Theta = \langle n, v\rangle V^*K$) will be useful in the geometric description of the phase space, see section 4.

- $B^- : \mathcal{J} \to \mathcal{J}$ is the s.f.f. just before collision,
- $B^+ : \mathcal{J} \to \mathcal{J}$ is the s.f.f. just after collision,
- $V : \mathcal{J} \to \mathcal{T}$ is the projection parallel to v: $Vdv = dv - \frac{\langle dv,n\rangle}{\langle v,n\rangle} v \in \mathcal{T}$ for $dv \in \mathcal{J}$,
- $V^* : \mathcal{T} \to \mathcal{J}$ (the adjoint of V) is the projection parallel to n: $V^*dq = dq - \frac{\langle dr,v\rangle}{\langle n,v\rangle} n \in \mathcal{J}$ for $dq \in \mathcal{T}$,
- $K : \mathcal{T} \to \mathcal{T}$ is the s.f.f. of the scatterer at the collision point,
- $\langle n, v\rangle = \cos\phi$, where $\phi \in [0, \frac{\pi}{2}]$ is the so-called collision angle,
- and the operator $\Theta : \mathcal{J} \to \mathcal{J}$: $\Theta = \langle n, v\rangle V^* K V$ is the so-called collision term.

2.3 Singularities

As it can be easily seen the billiard map T is discontinuous at pre-images of tangential reflections. Indeed, consider the set of tangential reflections:

$$\mathcal{S}_0 := \partial M = \{(q, v) \mid \langle v, n(q)\rangle = 0\}$$

(which is nothing else than the boundary of the phase space). Its pre-images are:

$$\mathcal{S}_k = T^{-k}\mathcal{S}_0 \quad (k > 0).$$

(From section 4 on it will be useful to introduce the notation $\mathcal{S}^{(k)}$ for the set of all singularities up to k, i.e. $\mathcal{S}^{(k)} = \cup_{i=1}^{k} \mathcal{S}_i$.) The map T is discontinuous precisely at the points of $\mathcal{S}_1 (= \mathcal{S}^{(1)})$. Furthermore – related to the smallness of the term $\langle n, v\rangle$ – the derivative DT is unbounded near $\mathcal{S}_1$. As a consequence, to get a well-behaved dynamics, the phase space is partitioned into homogeneity layers by introducing secondary singularities (for a detailed discussion see [BSC2] or subsection 4.1).

To consider higher iterates of the dynamics – the maps T^k $(k \geq 1)$ – the sets $\mathcal{S}_k$ are to be investigated. We view all these sets as (finite unioins of) topologically embedded one codimensional compact submanifolds with boundary. They have smooth manifold structure in the interior, however, in the multi-dimensional case (as it is demonstrated in subsection 3.1) the behaviour at the boundary is irregular (the curvature diverges). This behaviour is related to the fact that in the multi-dimensional case, in addition to unbounded derivatives, the dynamics is highly non-isotropic near singularities.

3 Geometry of singularities

In several papers that appeared, singularities were assumed – either explicitely or implicitely – to consist of smooth 1-codim submanifolds of the phase space. Often, even a uniform bound on the curvature was assumed, independent of the order of the singularity. This is true in 2-dimensional billiards. However, it is not true in higher dimensions. In this section we present a counter-example in a 3-dimensional dispersing billiard. In correspondence with the notations introduced in subsection 2.3, we will use the notation $\mathcal{S}_1$ and $\mathcal{S}_2$ for the set of those phase points the trajectories of which have tangential first and second collisions, respectively. We will demonstrate that already the curvature of $\mathcal{S}_2$ has no upper bound, i.e. the curvature blows up near a point where the singularity manifold is not even differentiable.

To avoid confusion let us make one further remark. As already mentioned, billiard dynamics has singularities: points where the billiard map is not continuous. These singularities occur on one codimensional submanifolds of the phase space. The development of the theory is based on considering connected and essentially smooth components of the singularity manifolds. The recently discovered phenomenon described below shows that these components are, indeed, only essentially smooth. On certain two-codimensional submanifolds of them pathologies occur: singularities in the sense of algebraic singularity theory. To avoid confusion we will refer to these singular two-codimensional submanifolds as *pathologies* (in contrast to the *singularities*, the singularity manifolds of the dynamics themselves).

3.1 Counter-example for bounded curvature

In this section we prove that even in a $3D$ dispersing billiard, already the two-step singularities have no bounded curvature. The proof is rather implicit. We start with the indirect assumption that the curvature is bounded, and find that the two-step singularity intersects the one-step singularity tangentially at every point of their intersection, except for a one-codimensional degeneracy, where the intersection is not tangent. However – as a consequence of bounded curvature – our indirect asumption implies that the unit normal vector of $\mathcal{S}_2$ is a continuously differentiable function of its base point. Thus the set of those points where the two singularity manifolds intersect non-tangentially is open in $\mathcal{S}_1 \cap \mathcal{S}_2$. This way we get a contradiction.

Consider the situation demonstrated on Figure 2. To perform as transparent an argument as possible

- the parameters on the figure and in the calculations below are different,
- the first scatterer, the surface where the trajectories start out is a plane – thus it is not strictly convex.

Nevertheless the reader can easily see that these modifications have no real significance. We are in 3 dimensions, so take a standard 3D Cartesian coordinate system. Let the first 'scatterer' be the $\{z = 0\}$ plane. Let the second scatterer be the sphere with centre $O_1 = (0, -1, 1)$ and radius $R = 1$. Let the third scatterer be the sphere with centre $O_2 = (1, 0, 2)$ and radius $R = 1$. We look at the component of the phase space corresponding to the first scatterer, near the phase point $(x_0 = 0, y_0 = 0, v_{x0} = 0, v_{y0} = 0)$. Of course, $v_{z0} = 1$, and the trajectory is the z axis. We are interested in the singularity manifold belonging to a tangent *second* collision. To describe this, let $D \in \mathbb{R}^4$ be the set of those points (x, y, v_x, v_y) the trajectories of which hit the first sphere. Let $r : D \to \mathbb{R}$ be the distance of the trajectory and O_2. That is, the singularity manifold we are looking at is the set $\mathcal{S}_2 = \{(x, y, v_x, v_y) \in D | r(x, y, v_x, v_y) = 1\}$. So, if we want to construct the normal vector of the singularity manifold, we just need to calculate the gradient of r. We will directly calculate the partial derivatives. Since $(x_0, y_0, v_{x0}, v_{y0}) = (0, 0, 0, 0)$ is on the boundary of D, we can only hope to find one-side partial derivatives. What is even worse: $(x, y, v_x, v_y) = (x, 0, 0, 0) \in D$ only if $x = 0$, so we cannot differentiate with respect to x. The same is true for v_x. What we can do is take these partial derivatives at the points $(0, y, 0, v_y)$ and than the limits

$$\lim_{y\to 0} \lim_{v_y \to 0} \frac{\partial}{\partial x} r(x, y, v_x, v_y) mid_{x=v_x=0}.$$

(we will see that it is important to fix $x = v_x = 0$).

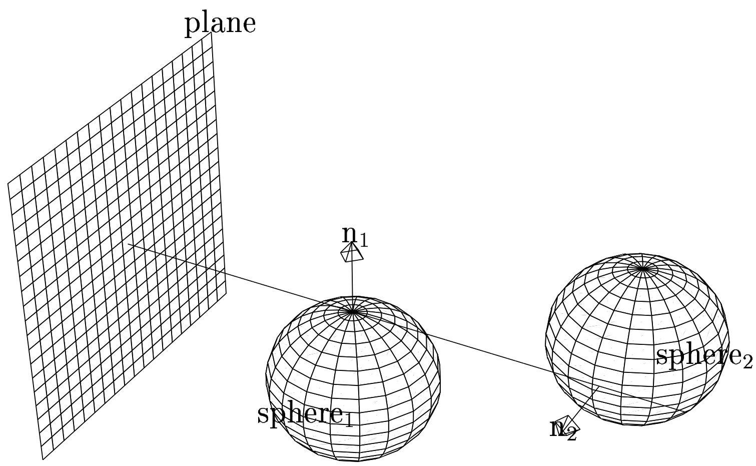

Figure 2: The studied billiard configuration

We start with the indirect assumption that $\mathcal{S}_2$ has bounded curvature. This implies that the unit normal vector of $\mathcal{S}_2$ is a continuously differentiable function of its base point with bounded derivative. In this way it makes sense to define the normal vector of $\mathcal{S}_2$ on the boundary points of $\mathcal{S}_2$ as the limit of (unit) normal vectors on the interior. For us the indirect assumption will mean that the limit

$$\mathrm{grad} r(0, 0, 0, 0) := \lim_{(x,y,v_x,v_y)\to(0,0,0,0)} \mathrm{grad} r(x, y, v_x, v_y)$$

exists.

The closer a reflection is to tangential, the less effect it has on the "neutral" direction. In our case, the reflection on the first sphere causes "no scattering" in the x direction.

That is, let (v_x', v_y', v_z') be the velocity after the first collision. The "x" direction being the "neutral" direction means that

$$\lim_{y\to 0} \frac{\partial}{\partial v_x} v_x'(0, y, 0, 0) = 1$$

which implies that

$$\lim_{y\to 0} \frac{\partial}{\partial v_x} r(0, y, 0, 0) = -2$$

Similarly,

$$\lim_{y\to 0} \frac{\partial}{\partial x} v_x'(0, y, 0, 0) = 0$$

which implies that

$$\lim_{y\to 0} \frac{\partial}{\partial v_x} r(0, y, 0, 0) = -1.$$

According to our indirect assumption, this means that

$$\frac{\partial}{\partial x} r(0, 0, 0, 0) = -1$$

and

$$\frac{\partial}{\partial v_x} r(0, 0, 0, 0) = -2.$$

For the other two components, fix $x = v_x = 0$. So the trajectory is in the $\{x = 0\}$ plane, the scattering is just a 2D problem. We will calculate the one-side partial derivatives $\frac{\partial}{\partial y} r(0,0,0,0)$ and $\frac{\partial}{\partial v_y} r(0,0,0,0)$.

To find out about v_y', let ϕ be the angle of the first sphere's radius at the first collision point and the $(0,1,0)$ vector. If $v_y = 0$, then $1 - \cos\phi = -y$ ($y < 0$, of course), which, in leading order, gives $\phi = \sqrt{-2y}$. It can be seen that after the reflection $v_y' = \sin 2\phi$. That is, the trajectory is far from being a line. However, it is diverted in the very direction which - in the first order - does not affect its distance from O_2. Instead, in leading terms, $r^2 = 1 + (v_y')^2$.

Putting these together, we get $r = \sqrt{1 - 8y}$, that is,

$$\frac{\partial}{\partial y} r(0, 0, 0, 0) = -4.$$

If we fix $y = 0$, the exact same consideration gives $r = \sqrt{1 - 8v_y}$, that is,

$$\frac{\partial}{\partial v_y} r(0, 0, 0, 0) = -4$$

as well. All together, we get

$$\mathrm{grad}\, r(0, 0, 0, 0) = (-1, -4, -2, -4).$$

This is (the limit of) the normal vector of the singularity at the point $(x = 0, y = 0, v_x = 0, v_y = 0)$.

It is easy to see that the singularity corresponding to a tangent reflection on the first sphere has the normal vector

$$\text{grad} r_0(x, y, v_x, v_y) = (0, -1, 0, -1).$$

That is, the two singularities are not tangent at this point.

The previous consideration for grad$r$ also shows that this behaviour is exceptional. It is the result of the fact that in the first order r was unaffected by v'_y. If the radii at the reflection points $(x, y, z) = (0, 0, 1)$ and $(x, y, z) = (0, 0, 2)$ had not been orthogonal, the result would have been

$$\frac{\partial r}{\partial y} = \infty, \frac{\partial r}{\partial v_y} = \infty,$$

corresponding to a normal vector $(0, 1, 0, 1)$, meaning that the two singularities are tangent. Non-tangentiallity of the two singularities is a one-codimensional degeneracy.

As we have pointed out at the beginning of the subsection, this contradicts our indirect assumption on the boundedness of the curvature. In this way we have only proven that the assumption was false. However, we believe that the picture of the singularity suggested above is correct, the singularities *are* tangent almost everywhere, and their curvature only blows up near the pathological points described.

3.2 Discussion

For a rigorous proof of some finer properties (such as correlation decay) of multi-dimensional dispersing billiards it seems essential to characterize singularities in a systematic way. Such a characterization should be subject to future research (some possible ideas related to this question are discussed in [BChSzT]). In this subsection we do not plan to give rigorous proofs; we would like to point out some analogies to and emphasize some interesting features of the irregularities demonstrated above.

The Whitney-umbrella. Consider the one-codimensional set in $\mathbb{R}^3$ defined by the polynomial equation:

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 z = y^2\},$$

the Whitney-umbrella (for more details see [AGV]). 'One half' of this set (its intersection with the quadrants $xy \geq 0$) is shown on Figure 3. For simplicity we use the notations: W_2 for this 'half-umbrella' and W_1 for the $\{y = 0\}$ plane. Clearly

- W_2 terminates on W_1 (in the points of the x-axis), thus $W_1 \cap W_2 = \partial W_2$.
- at every point of the x-axis where $x \neq 0$ the intersection of W_2 and W_1 is tangential.
- W_2 has smooth manifold structure in its interior; nevertheless, near the origin its curvature is unbounded as the normal vector changes rapidly (actually, the normal vector does not even have a well-defined limit at the origin).

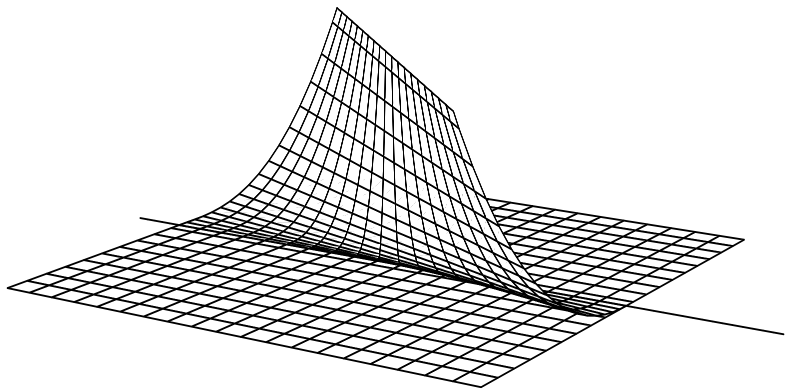

Figure 3: The Whitney Umbrella

By these properties the geometry of singularities described in subsection 3.1 is analogous to Figure 3. [2] W_1 corresponds to $\mathcal{S}_1$, W_2 corresponds to $\mathcal{S}_2$ while the origin corresponds to the set of those doubly tangential reflections where the two radii are orthogonal (this set is one-codimensional in $\mathcal{S}_1 \cap \mathcal{S}_2$).

Generalization I. First let us consider the first-step singularity $\mathcal{S}_1$. By the notations of the previous subsection we may characterize the points of (x, y, v_x, v_y) belonging to $\mathcal{S}_1$ easily. These are precisely those for which $d(x, y, v_x, v_y) = 1$, where $d(.,.,.,.)$ is the distance of the point $O_1 = (0, -1, 1)$ from the line that passes through the point $(x, y, 0)$ and has direction specified by the velocity components v_x, v_y. As d is a smooth function of its variables there is no curvature blow-up for $\mathcal{S}_1$ – and, for first-step singularities in general. Thus $\mathcal{S}_2$ is a *pre-image of a smooth one-codimensional compact submanifold*, however, *the map* under which the pre-image is taken *has unbounded derivatives and is highly an-isotropic.* Curvature blow-up occurs only at those points of $\mathcal{S}_2$ (near its intersection with $\mathcal{S}_1$) where the map behaves irregularly.

In correspondence with the above observation we conjecture that curvature blow-up is not a peculiar feature of $\mathcal{S}_2$, it is present in the pre-images of one-codimensional smooth submanifolds in general. Consider for example two-step *secondary* singularities Γ_2 – those phase points for which at the second iterate instead of tangentiality the collision angle $(\langle n, v \rangle)$ is a given constant (see section 4 for more detail). In the specific example of subsection 3.1 such secondary singular trajectories are precisely those that touch tangentially a sphere of radius R' ($R' < 1$) at the second iterate. It is clear that the geometry of Γ_2 is completely analogous to $\mathcal{S}_2$.

Generalization II. Our calculations in subsection 3.1 do not use any speciality of the explicitly given billiard configuration. Doubly tangential reflections for which the normal

[2] To be precise, the situation on Figure 3. has one dimension less – in contrast to W_2 the singularities are 3-dimensional manifolds – but this has little significance to the analogy.

vectors of the scatterers at the consecutive collisions are orthogonal can be found in *any multi-dimensional semi-dispersing billiard*. Near such trajectories a similar calculation can be performed.

Generalization III. All in all, the discovered pathology is general. In addition, *the higher step singularities* $\mathcal{S}_k$; ($k \geq 3$) may show even wilder behaviour near their intersections. Nevertheless, we strongly conjecture that a nice geometric characterization – suggested by the analogy with the Whitney-umbrella in the case of $\mathcal{S}^{(2)}$ – can be performed. This question is subject to future research.

4 Geometric properties of u-manifolds

Throughout sections 4 and 5 we investigate u-manifolds (their counterparts, s-manifolds can be treated similarly). u-manifolds are $d-1$-dimensional submanifolds of the phase space with tangent planes in the (appropriately defined) unstable cone. Possibly the most important tools in studying ergodic and statistical properties, local unstable manifolds (or LUMs for short) are suitable limits of u-manifolds (for details see [Y1, Ch2, Ch3]). In contrast to the $2d-3$-dimensional (one-codimensional) singularity manifolds, u-manifolds behave in a uniformly regular way. In section 4 we introduce a natural geometrical description that turns out to be very useful for studying multi-dimensional dispersing billiards. Proofs for some basic properties of u-manifolds are also included. More involved technicalities – that play a crucial role in investigating the statistical behaviour of a billiard system (cf. [Y1, Ch2, Ch3]) – are discussed in section 5.

4.1 The phase space

We shall work with the discrete time (collision to collision) dynamical system, thus our phase space – which we denote by M – is the Poincaré phase space, the collection of possible collision points supplied with outgoing velocities. Mathematically this space is a bundle over the scatterers $\partial\mathbb{Q}$, the fibers of which consist of the possible outgoing velocities. At every base point q the fiber is the $(d-1)$-dimensional hemisphere with boundary which we shall denote by $\mathbf{S}_+^{d-1}$. Note that this bundle can be viewed as a subbundle (of vectors of unit length) in the direct sum of the tangent and normal bundles over the scatterers. Thus, by the Riemannian structure of $\partial\mathbb{Q}$, there is a naturally defined parallel translation on our bundle (see the description of the tangent plane below). Local coordinates on our phase space will be denoted $x = (q, v)$. Additionally we shall use all the notations for local quantities introduced in the previous section(s) (eg. $n(q)$, ϕ).

Some conventions. Throughout the paper the superscripts '+' and '−' denote post- and precollisional values, respectively, for certain functions, operators, hyperplanes etc. (e.g. v^+ and v^-). The dynamics and its derivative are denoted by T and DT, respectively. In correspondence with $x_1 = Tx$ ($\delta x_1 = DT\delta x$), the subscript '1' means the value of a certain quantity at the first iterate. We shall usually prime the points, trajectories, operators etc. infinitesimally close to a reference point or trajectory.

The tangent plane. At any point $x = (q, v)$ the tangent plane has a natural splitting $\mathcal{T}_x M = \mathcal{T}_q \partial\mathbb{Q} + \mathcal{T}_v \mathbf{S}_+^{d-1} = \mathcal{T} + \mathcal{J}$. The two planes $\mathcal{J}$ and $\mathcal{T}$ are related by the projection operator $V : \mathcal{J} \to \mathcal{T}$ and its adjoint V^* (for their description see the section 2).

For two points $x = (q, v)$ and $x' = (q', v')$ infinitesimally close, the tangent vector pointing from x to x' is

$$\delta x = (\delta q, \delta v) \qquad \delta q = q - q'; \; \delta v = Q_0^{-1} v' - v$$

where Q_0 is the rotator that takes $\mathcal{T}$ to $\mathcal{T}'$. Up to first order:

$$Q_0 u = u - \langle u, dn \rangle n + \langle u, n \rangle dn \quad \text{for} \;\; u \in \mathbb{R}^d; \tag{4.1}$$

$$Q_0^{-1} u = u + \langle u, dn \rangle n - \langle u, n \rangle dn \quad \text{for} \;\; u \in \mathbb{R}^d \tag{4.2}$$

and thus:

$$\delta v = dv - \langle v, n \rangle V^* dn$$

Here $dv = v' - v$ and $dn = n' - n$. These formulas execute (up to first order) the parallel translation of the bundle M.

4.2 Important submanifolds

Singularity manifolds. The dynamics T is discontinuous, the singularity manifold is $\mathcal{S}^{(1)} = \mathcal{S}_1 = T^{-1}\mathcal{S}_0$ where $\mathcal{S}_0 = \partial M = \{(q, v) | \langle v, n \rangle = 0\}$ is just the boundary of the phase space. However, as already mentioned, to get a well-behaved dynamics we should partition the original phase space into homogeneity layers:

$$\begin{aligned} I_k &= \{(q, v) \in M | \, (k+1)^{-2} < \langle v, n(q) \rangle < k^{-2}\} \quad \text{and} \\ I_0 &= \{(q, v) \in M | \, \langle v, n(q) \rangle > k_0^{-2}\} \end{aligned} \tag{4.3}$$

Here the integer constant k_0 is arbitrary. The boundary of this partitioned phase space, $\bar{M}$ is

$$\Gamma_0 = \partial \bar{M} = \cup_{k=k_0}^{\infty} \{(q, v) | \langle v, n \rangle = k^{-2}\}$$

Correspondingly, the countably many manifolds in the set $\Gamma^{(1)} = T^{-1}\Gamma_0$ are the so called *secondary singularities*. For a higher iterate of the dynamics, T^n, the primary and secondary singularities are, respectively:

$$\mathcal{S}^{(n)} = \mathcal{S}^{(1)} \cup T^{-1}\mathcal{S}^{(1)} \cup \cdots T^{-n+1}\mathcal{S}^{(1)}; \quad \Gamma^{(n)} = \Gamma^{(1)} \cup T^{-1}\Gamma^{(1)} \cup \cdots T^{-n+1}\Gamma^{(1)}.$$

Fronts. As introduced in section 2, $(d-1)$-dimensional submanifolds in $\mathbb{Q}$, the configurational space of the flow, everywhere orthogonal to the flow direction will be referred to as fronts. When supplied with their normal vectors v (the velocities), fronts can be viewed as submanifolds of the flow phase space $\mathcal{M}$. Vectors (in the tangent bundle over $\mathcal{M}$) tangent to fronts are denoted by $(dr, dv) = (dr, Bdr)$ where B is the second fundamental form (s.f.f.) of our submanifold in $\mathbb{Q}$ (here, of course, $dr \perp v$).

Let us consider a front directly after (before) collision. It leaves a trace of velocities on the scatterer which can be viewed either as a (unit) vector field over $\partial\mathbb{Q}$ or as a $(d-1)$-dimensional submanifold in the Poincaré phase space. Direct calculations show that for a vector $(dr, dv) = (dr, B^+dr)$, tangent to the post-collisional front, the corresponding vector in the Poincaré phase space is $\delta x = (\delta q, \delta v)$ where:

$$\begin{aligned} \delta q &= V dr; \\ \delta v &= dv - \langle v, n\rangle V^* dn = dv - \langle v, n\rangle V^* K \delta q = \\ &= (B^+ V^{-1} - \langle v, n\rangle V^* K)\delta q = F\delta q. \end{aligned} \tag{4.4}$$

The operator $F : \mathcal{T} \to \mathcal{J}$ plays an important role, it describes the tangent plane of our $(d-1)$-dimensional manifold in the Poincaré phase space.

A front will be called *convex/diverging* whenever B^+ is positive semi-definite ($B^+ \geq 0$). Convex fronts remain convex under time evolution. The *convex cone* consists of those tangent vectors δx that are tangent to some convex front.

Lemma 4.1. *There are constants $m_0 \in I\!N$ and $\phi_0 < \frac{\pi}{2}$ that depend only on the billiard domain itself such that out of m_0 consecutive reflections at least for one of them for the collison angle ϕ we have: $\phi < \phi_0$.*

Proof. Let us assume the contrary: there is a sequence x_n of phase points which have trajectories with n consecutive collisions, all with collision angle $\phi > \frac{\pi}{2} - \frac{1}{n}$. By compactness there is a limit phase point with infinitely many consecutive tangential reflections. This, however, contradicts the finite horizon assumption. □.

u-manifolds and homogeneous u-manifolds. We shall consider C_x^u, the m_0-image of the convex cone as our *unstable cone*. A manifold is a *u-manifold* if it has all tangent vectors in C_x^u. u-manifolds remain u-manifolds as C_x^u is invariant under the positive powers of T.

A u-manifold is said to be *homogeneous* if it is contained in one homogeneity layer.

There will be two metrics used on u-manifolds. Before their introduction we mention that for any vector dz in $\mathcal{T}$ or in $\mathcal{J}$ $\|dz\|$ is the notation for the Euclidean length and for oprators O acting on these spaces $\|O\|$ denotes the naturally induced norm.

The *p-metric*

$$\|\delta x\|_p = \|dr\|$$

measures distances on the corresponding front while the *Euclidean metric*

$$\|\delta x\|_e = \sqrt{\delta q^2 + \delta v^2}$$

in the Poincaré phase space. A priori the p-metric seems to be degenerate but as we shall see it is a good metric on the cone C^u_x. Time evolution in the p-metric is given by:

$$\|\delta x_1\|_p = \|dr_1\| = \|dr + \tau dv\| = \|(I + \tau B^+)dr\| \qquad (4.5)$$

Some further notation. For any u-manifold W; the quantities $J^p_W(x)$ and $J^e_W(x)$ are the Jacobians of the dynamics in the p- and e-metrics, respectively.
Remark. All the above introduced concepts have their natural counterparts (with the corresponding nice properties) for the reversed dynamics: concave/convergent fronts, s-manifolds etc.

4.3 Properties of F and equivalence of metrics

Some conventions. Constants that depend only on the billiard table itself (like τ_{min}, ϕ_0...) will be called *global constants.*
For an invertible operator O the meaning of the relations $c \prec O \prec C$ is that there are two positive global constants C_1 and C_2 that bound the norms of the operator and its inverse:

$$\|O\| < C_1; \qquad \|O^{-1}\| < C_2.$$

Note that the operator O is not necessarily symmetric, even more, it need not be an automorphism. The values of the constants C_1 and C_2 are usually irrelevant.
Two quantities f and g defined on the unstable cones will be called *equivalent* ($f \sim g$) if there are some global constants C_1 and C_2 such that $C_1 f \leq g \leq C_2 f$.

Throughout this subsection we restrict our considerations on the vectors of the unstable cone.

Sublemma 4.2. *Let us consider any u-front with incoming and outgoing s.f.f.-s B^- and B^+, respectively. Then $c \prec B^+$ and $c \prec B^- \prec C$.*

Proof. By the collision equations the operator $B^+ - B^-$ is always positive semi-definite, thus it is enough to prove $c \prec B^- \prec C$ as it implies $c \prec B^+$. The upper bound is trivial by (2.2) and the lack of corner points (there is a lower bound on the free path: $\tau \geq \tau_{min}$). Thus it remains to prove $c \prec B^-$, what is an easy consequence of Lemma 4.1. Indeed, our submanifold is an m_0-iterate of a convex front . By the lemma out of these m_0 reflections there is definitely at least one with collision angle smaller than ϕ_0. We shall denote the collision term that corresponds to this particular reflection by Θ_0. Of course, $c \prec \Theta_0$ as the spectrum of Θ_0 is bounded below by $k_{min} \cos\phi_0$ (here k_{min} is the lower bound on the spectrum of K – the curvature operator of the scatterers $\partial\mathbb{Q}$). Now let us consider any $dr \in \mathcal{J}$. By the evolution equations (2.2) and (2.3):

$$\langle dr, B^- dr\rangle \geq \langle dr, ((\Theta_0)^{-1} + m_0\tau_{max} I)^{-1} dr\rangle \geq ((k_{min}\cos\phi_0)^{-1} + m_0\tau_{max})^{-1}\langle dr, dr\rangle.$$

Thus we have the desired lower bound. $\square$.

Now we can formulate our most important technical lemma.

Lemma 4.3. *Assume $K' : \mathcal{T} \to \mathcal{T}$ and $B' : \mathcal{J} \to \mathcal{J}$ are both symmetric, positive definite and $c \prec B', K' \prec C$. Then:*

$$c \prec B'V^{-1} + \langle v, n\rangle V^* K' \prec C.$$

Proof. The upper bound is obvious since $\|V^{-1}\| = 1$ and $\langle v, n\rangle \|V^*\| = 1$.
By the definition of V, we have

$$Vu = u - \frac{\langle u, n\rangle}{\langle v, n\rangle} v \qquad \text{for} \quad u \in \mathcal{J}$$

and

$$V^{-1}u = u - \langle u, v\rangle v \qquad \text{for} \quad u \in \mathcal{T}$$

Similarly,

$$V^*u = u - \frac{\langle u, v\rangle}{\langle v, n\rangle} n \qquad \text{for} \quad u \in \mathcal{T} \tag{4.6}$$

and

$$(V^*)^{-1}u = u - \langle u, n\rangle n \qquad \text{for} \quad u \in \mathcal{J}$$

It is then easy to arrive at

$$(V^*)^{-1}V^{-1}u = u - \langle u, v\rangle v + \langle u, v\rangle \langle v, n\rangle n$$

and

$$\langle v, n\rangle^2 V V^* u = \langle v, n\rangle^2 u + \langle u, v\rangle v - \langle u, v\rangle \langle v, n\rangle n$$

Adding the two equations above yields

$$(V^*)^{-1}V^{-1} + \langle v, n\rangle^2 V V^* = (1 + \langle v, n\rangle^2) I \tag{4.7}$$

where I is the identity operator in $\mathcal{T}$.

Another useful observation: Since $\|(B')^{-1}\| \leq C$ and $\|(K')^{-1}\| \leq C$ for a global constant $C > 0$, all the eigenvalues of B' and K' are bounded below by $c' = 1/C$. Hence

$$\langle B'u, u\rangle > c'\|u\|^2 \qquad \text{for} \quad u \in \mathcal{J} \tag{4.8}$$

and

$$\langle K'u, u\rangle > c'\|u\|^2 \qquad \text{for} \quad u \in \mathcal{T} \tag{4.9}$$

Now, let $u \in \mathcal{T}$, $\|u\| = 1$. Then $\|V^{-1}u\| \leq 1$, and

$$\langle B'V^{-1}u + \langle v, n\rangle V^* K' u, V^{-1}u\rangle = \langle B'V^{-1}u, V^{-1}u\rangle + \langle v, n\rangle \langle K'u, u\rangle$$

Here all three scalar products are positive, hence

$$\|B'V^{-1}u + \langle v,n\rangle V^*K'u\| \geq c'\|V^{-1}u\| \tag{4.10}$$

due to (4.8). Next, we have $\langle v,n\rangle\|V^*u\| \leq 1$, and

$$\langle B'V^{-1}u + \langle v,n\rangle V^*K'u, \langle v,n\rangle V^*u\rangle = \langle B'V^{-1}u, \langle v,n\rangle V^*u\rangle + \langle K'u, \langle v,n\rangle^2 VV^*u\rangle$$

Substitution of (4.7) and using (4.9) gives

$$\|B'V^{-1}u + \langle v,n\rangle V^*K'u\| \geq c'\|u\|^2 - c''\|V^{-1}u\| = c' - c''\|V^{-1}u\|$$

for some global constant $c'' > 0$. Combining this with (4.10) yields

$$\|B'V^{-1}u + \langle v,n\rangle V^*K'u\| \geq c$$

with $c = c'/(1 + c''/c')$. The lower bound is proved. □

Corollary 4.4. *There are global constants c and C such that for any u-front $c \prec F \prec C$. As a consequence, for all vectors of the unstable cone, $\delta x \in C^u_x$ the norm $\|\delta x\|_e$ is uniformly equivalent to both $\|\delta q\|$ and $\|\delta v\|$. Furthermore, the p-metric is non-degenerate on the cone C^u_x (nonzero vectors in C^u_x have nonzero p-length).*

Proof. This is an easy application of Lemma 4.3 with $B' = B^-$ and $K' = K$ (see also formula (4.4)). □.

Corollary 4.5. *The p-metric and the e-metric are equivalent in a 'dynamical' sense: for any $\delta x \in C^u_x$: $\|DT\delta x\|_p \sim \|\delta x\|_e$.*

Proof. Indeed, by the evolution equation (4.5):

$$\|DT\delta x\|_p = \|(I + \tau B^+)dr\| = \|(I + \tau B^+)V^{-1}\delta q\|.$$

Now we may apply Lemma 4.3 with $K' = 2K$ and $B' = I + \tau B^-$ (remember that the free path τ is uniformly bounded from below and above). Together with Corollary 4.4 we get:

$$\|(I + \tau B^+)V^{-1}\delta q\| \sim \|\delta q\| \sim \|\delta x\|_e.$$

The two equations together give Corollary 4.5. □.

Before going into further details we would like to make an important remark.

Remark 4.6. *From the next section on we turn to a closer investigation of u-manifolds. We will see that – as long as the properties discussed in the rest of the paper are concerned – u-manifolds are no less regular in multi-dimensional billiards than in the planar ones. This can be easily checked if our results are compared to those proved in the literature for the two-dimensional case, see especially [Ch2], Section 6 and the references cited there.*

Nevertheless, there are important differences from planar billiards in the way how u-manifolds are actually described. Anisotropy of the geometry is reflected in the use of linear operators. It is of course much more difficult to handle operators than numbers, thus the proof of the very same regularity properties becomes more technical as one switches from dimension two to three.

4.4 Geometry and hyperbolicity of u-manifolds

Now we would like to turn to the hyperbolic and geometric properties of the unstable cone. Unless otherwise stated, any vector δx mentioned is an element of the u-cone C_x^u.

Uniform **hyperbolicity in the p-metric** is guaranteed by the uniform bound $\tau > \tau_{min}$ and Sublemma 4.2. Indeed:

$$\|DT\delta x\|_p = \|(Id + \tau B^+)dr\| > \Lambda \|\delta x\|_p.$$

Here $\Lambda > 1$ is a global constant. On the other hand, by Sublemma 4.2 again (together with the evolution equations) for the $(d-1)$ eigenvalues of the symmetric operator B^+:

$$\lambda_1 \sim (\cos\phi)^{-1}; \qquad \lambda_i \sim 1, \quad i = 2 \dots d-1.$$

As a consequence, for an arbitrary u-manifold W the Jacobian in the p-metric behaves as

$$J_W^p(x) \sim (\cos(\phi))^{-1}.$$

In the **e-metric** we have by Corollary 4.5:

$$\|DT^n\delta x\|_e \geq \|DT^n\delta x\|_p > \Lambda^{n-1}\|DT\delta x\|_p > C\Lambda^n\|\delta x\|_e. \tag{4.11}$$

This implies that for a sufficiently high fixed power of the dynamics, $T_1 = T^{m_1}$:

$$\|DT_1\delta x\|_e > \Lambda_1\|\delta x\|_e \qquad \text{with } \Lambda_1 > 1 \text{ global.} \tag{4.12}$$

To calculate $J_W^e(x)$ for any u-manifold W consider the operator $G : \mathcal{T} \to \mathcal{T}_x W$ that acts by the rule $\delta q \mapsto (\delta q, F(\delta q)) = \delta x$. Then one can easily check that in our notation

$$DT\mid_W(x) = G_1 \circ V_1 \circ U_1 \circ (I + \tau B^+) \circ V^{-1} \circ G^{-1}$$

in correspondence with equation (4.5) that describes evolution in the p-metric. Now we may get a formula for the Jacobian in the e-metric:

$$J_W^e(x) = \det G_1 \, \det V_1 \, J_W^p(x) \, (\det V)^{-1} \, (\det G)^{-1}. \tag{4.13}$$

We observe that

$$(\det G)^2 = \det(I + F^* F) \tag{4.14}$$

Indeed, there is an orthonormal basis in $\mathcal{T}$ and an orthonormal basis in $\mathcal{J}$ such that $F : \mathcal{T} \to \mathcal{J}$ is represented, in those bases, by a diagonal matrix (this follows from the singular value decomposition theorem in linear algebra). For a diagonal matrix F, the relation (4.14) is easily verified by direct inspection.

Now it is easy to see that there are global constants c and C such that: $c < \det G < C$ for the operator G at any u-manifold. Direct calculation gives:

$$J^e_W(x) \ \sim \ \det(\, V_1\,) \ \sim \ (\cos(\phi_1))^{-1}. \tag{4.15}$$

Let us consider a further restriction of DT onto a subspace $R \subset \mathcal{T}_x W$ of the tangent plane. Applying the above argument for the restriction $DT\,|_R$ we get:

$$\det(\, DT\,|_R) \ \sim \ \det(V_1\,|_{R'}) \tag{4.16}$$

where $R' = (V_1^{-1} \circ G_1^{-1} \circ DT)(R)$.

Now we turn to some geometric properties of our submanifolds. **Transversality** – the property that the stable and unstable cones are uniformly transversal – is justified by the following theorem:

Theorem 4.7. *The u-manifolds and s-manifolds in M are uniformly transversal. Precisely, there is a global constant $c_0 > 0$ such that for any u-manifold W_u and any s-manifold W_s at any point of intersection $x \in W_u \cap W_s$ the angle between W_u and W_s is greater than c_0.*

Proof. We use the subscripts u and s to denote various quantities and operators related to the submanifolds W_u and W_s, respectively. According to (4.4),

$$F_u = UB_u^- U^{-1} V^{-1} + \langle v, n \rangle V^* K$$

and

$$F_s = B_s^+ V^{-1} - \langle v, n \rangle V^* K$$

Note that the operator $-B_s^+$ is symmetric, positive definite and satisfies $c \prec -B_s^+ \prec C$ (this is the counterpart of the previously established property $c \prec B_u^- \prec C$). Hence, the operator $B' := UB_u^- U^{-1} - B_s^+$ is symmetric, positive definite and satisfies $c \prec B' \prec C$. Now Lemma 4.3 implies

$$c \prec F_u - F_s \prec C \tag{4.17}$$

Next assume that Theorem 4.7 is false. Then, by using Corollary 4.4, one can easily conclude that for any $\varepsilon > 0$ there are a u-manifold W_u, an s-manifold W_s intersecting W_u at some point $x = (q, v)$, and a nonzero vector $\delta q \in \mathcal{T}$ such that

$$\|F_u(\delta q) - F_s(\delta q)\| \le \varepsilon\,\|\delta q\|$$

This clearly contradicts (4.17). Theorem 4.7 is proved. □

Remark: Observe that the above proof goes through even if instead of the s-manifold W_s we have just an arbitrary convergent front W_0. Indeed, for the crucial equation (4.17) it is enough to have the upper bound $-B_0^+ \prec C$ (which trivially holds for any convergent front W_0), the lower bound $c \prec -B_s^+$ – which is only true for s-manifolds – is, however, not essential.
As a consequence we are able to prove the so-called **alignment** property.

Corollary 4.8. *The u-manifolds are uniformly transversal to all the singularity manifolds $S \subset \mathcal{S}^{(n)}$ and $S \subset \Gamma^{(n)}$, $n \geq 1$. Precisely, there is a global constant $c_0 > 0$ such that for any u-manifold W_u intersecting any manifold $S \subset \mathcal{S}^{(n)}$ or $S \subset \Gamma^{(n)}$ at a point x there is a $(d-1)$-dimensional submanifold $S' \subset S$ through x such that the angle between W_u and S' is greater than c_0.*

Proof. We have $S = T^{-k}S_0$ for some $1 \leq k \leq n$ and a domain $S_0 \subset \mathcal{S}_0$ (or $S_0 \subset \Gamma_0$). Let $x_0 = (q_0, v_0) = T^k x \in S_0$. Define a small $(d-1)$-dimensional submanifold $S_0' \subset S_0$ through x_0 by $S_0' = \{y = (r, v) \in M | v = Q_0 v_0\}$, where Q_0 is the rotator of $\mathbb{R}^d$ taking $n(q_0)$ to $n(q)$, as defined by (4.1).

First let us discuss the primary singularities (i.e. the case $S_0 \subset \mathcal{S}_0$). We claim that $S' = T^{-k}S_0'$ is a limit, in C^0 metric, of a sequence of convergent fronts. Indeed, we first approximate S_0' by a sequence of $(d-1)$-dimensional manifolds $S_0^{(i)}$ defined as follows. Pick a sequence of vectors $v_0^{(i)} \in S^{d-1}$ such that $v_0^{(i)} \to v_0$ as $i \to \infty$ and $\langle v_0^{(i)}, n(q_0)\rangle > 0$ for all i. Then we put $S_0^{(i)} = \{y = (q, v) \in M : v = Q_0 v_0^{(i)}\}$. For each submanifold $S_0^{(i)}$, the tangent plane at every point $(q, v) \in S_0^{(i)}$ is characterized by $\delta v = 0$, hence $F = 0$ in our notation. According to (4.4), we now have $UB^-U^{-1} = -\langle v, n\rangle V^* K V^{-1}$, which is a negative definite operator. So, the trajectories of $S_0^{(i)}$, as they flow backward in time, make a convergent front. Therefore, $T^{-k}S_0^{(i)}$ is a convergent front for every i. As $i \to \infty$, these fronts converge to $S' = T^{-k}S_0'$, as we claimed. Now, Theorem 4.7 (in view of the remark above) completes the proof for the case of primary singularities.

In the secondary case (i.e. $S \subset \Gamma^{(n)}$) the $(d-1)$-dimensional manifold $S' = T^{-k}S_0'$ is a convergent front itself. Thus we may refer to the theorem and the remark directly. □

Remark. Recall that singularity manifolds are $2d-3$-dimensional. The above Corollary roughly states that there is a $d-1$-dimensional subbundle in their tangent bundle that lies in the stable cone field. However, the tangent space may behave wildly in the further $d-2$ directions, in correspondence with the curvature blow-up discussed in section 3.

5 Technical bounds on u-manifolds

After introducing the basic structures and tools now we would like to turn to the discussion of some more complicated technical properties. Unless otherwise stated, all calculations refer to the unstable cone (field) C_x^u and we use all other conventions from the previous section as well (e.g. quantities corresponding to a trajectory infinitesimally close to a reference one are primed).

Our main reference will be Lemma 4.3. Before discussing the important specific properties in the subsections, we record a few immediate consequences of this Lemma. For every u-manifold W, at every reflection we have

$$c \prec B^+V^{-1} \prec C. \tag{5.1}$$

This bound has its adjoint version

$$c \prec (V^*)^{-1} B^+ \prec C. \tag{5.2}$$

Let τ be the time between the current and the next reflections (or, more generally, any number satisfying $\tau_{\min}/10 < \tau \le \tau_{\max}$). Then

$$c \prec (I + \tau B^+) V^{-1} \prec C \tag{5.3}$$

and we also have an adjoint version of (5.3)

$$c \prec (V^*)^{-1} (I + \tau B^+) \prec C. \tag{5.4}$$

Note that if $c \prec A \prec C$ for any operator A, then also $c \prec A^{-1} \prec C$. Hence, all the above inequalities remain true for the inverse operators as well. For example, we have

$$(I + \tau B^+)^{-1} V^* \prec C \qquad \text{and} \qquad V(I + \tau B^+)^{-1} \prec C. \tag{5.5}$$

5.1 Curvature bounds on u-manifolds

In this subsection we would like to prove that there is a uniform bound on the curvature of u-manifolds. More precisely we prove that the tangent plane of a u-manifold is a Lipschitz function of the base point, with a uniform (global) Lipschitz constant. The tangent plane is described by the operator F, thus we should prove that F depends smoothly enough on the base point.

First we will get the relevant *curvature bounds in the phase space of the flow*; in other words, we investigate the smoothness of the dependence for s.f.f.-s B that describe any front corresponding to some u-manifold (which we refer to as u-fronts for short). Let $\mathcal{W}$ be any such u-front and $x = (r, v) \in \mathcal{W}$. Let $x' = (r', v') \in \mathcal{W}$ be infinitesimally close to x, and $dr = r' - r$, $dv = v' - v$ the infinitesimal displacement vectors in $\mathbb{Q}$ and $\mathbf{S}^{d-1}$, respectively. Clearly, $dr, dv \in \mathcal{J}$ and $dv = [B_{\mathcal{W}}(x)](dr)$. Consider the evolution of the displacement vector $(dr_t, dv_t) = S^t(dr, dv)$. If no collisions occur on an interval $(t, t+\Delta t)$, then $dv_{t+\Delta t} = dv_t$ and

$$dr_{t+\Delta t} = dr_t + \Delta t\, dv_t = [I + \Delta t\, B_t](dr_t) \tag{5.6}$$

where $B_t = B_{\mathcal{W}_t}(x_t)$. By Sublemma 4.2 we know that $\langle B_t u, u\rangle \ge b_{\min} \|u\|^2$ for all $u \in \mathcal{J}$. Therefore

$$\|dr_{t+\Delta t}\| \ge (1 + \Delta t\, b_{\min}) \|dr_t\| \tag{5.7}$$

hence

$$\|(I + \Delta t\, B_t)^{-1}\| \le (1 + \Delta t\, b_{\min})^{-1}. \tag{5.8}$$

Now consider a moment of reflection. The tangent vector $dx_t = (dr_t, dv_t)$ changes discontinuously, in correspondence with (2.3): $dr = dr^+ = Udr^-$ and $dv = dv^+ = U(dv^-) + \Theta(dr^+)$. The two trajectories reflect at the points $q, q' \in \partial Q$ in the time moments t, t' , respectively. For the infinitesimal differences we use the notations $dt \in \mathbb{R}$, $\delta q \in \mathcal{T}$ and $dn = n(q') - n(q) = K\delta q \in \mathcal{T}$. As to their relations:

$$\|dr^+\| \le \|\delta q\|; \qquad |dt| \le 2\|\delta q\|; \qquad \|dn\| \le C\|\delta q\| \quad \text{and} \quad \|dv\| \le C\|\delta q\|. \tag{5.9}$$

Indeed, these bounds are straight consequences of the formulas (2.3) and (4.4), the boundedness of K, the triangle inequality $|dt| \le \|dq\| + \|dr^+\|$ and our crucial Lemma 4.3.

We need to **compare the operators Θ and Θ' taken at the points (q, v) and** (q', v'), respectively. They act in the hyperplanes $\mathcal{J}$ and $\mathcal{J}'$ orthogonal to v and v', respectively. Consider the operators V^*, K, V entering (2.3) at the reference point (q, v) and their counterparts $(V')^*, K', V'$ at the nearby point (q', v'). Let $Q = Q_{v,v'}$ be the rotation in $\mathbb{R}^d$ taking v to v' and leaving invariant all the vectors perpendicular to v and v'. Then Q takes $\mathcal{J}$ to $\mathcal{J}'$. More specifically, Q acts by the rule

$$Qu = u - \langle u, dv\rangle v \qquad \text{for} \quad u \in \mathcal{J} \tag{5.10}$$

and its inverse acts by

$$Q^{-1}u = u + \langle u, dv\rangle v \qquad \text{for} \quad u \in \mathcal{J}' \tag{5.11}$$

where the terms of the second order in dv are dropped. Furthermore we shall use another rotator, Q_0, that takes $\mathcal{T}$ to $\mathcal{T}'$: this later one we have already introduced at the description of the parallel translation of the tangent bundle (see (4.1), (4.2)).

Instead of V and V^*, it is now more convenient to work with more "tame" operators $\tilde{V} = \langle v, n\rangle V$ and $\tilde{V}^* = \langle v, n\rangle V^*$. They act by the rules

$$\tilde{V}u = \langle v, n\rangle u - \langle u, n\rangle v \qquad \text{for} \quad u \in \mathcal{J} \tag{5.12}$$

and

$$\tilde{V}^*u = \langle v, n\rangle u - \langle u, v\rangle n \qquad \text{for} \quad u \in \mathcal{T} \tag{5.13}$$

Similar formulas hold for $\tilde{V}'$ and $(\tilde{V}')^*$, where v', n' are substituted for v, n.
Put $\Delta\tilde{V} = Q_0^{-1}\tilde{V}'Q - \tilde{V}$, $\Delta\tilde{V}^* = Q^{-1}(\tilde{V}')^*Q_0 - \tilde{V}^*$ and $\Delta K = Q_0^{-1}K'Q_0 - K$. Direct calculations based on (5.12), (5.10) and (4.2) yield

$$[\Delta\tilde{V}](u) = (\langle dv, n\rangle + \langle v, dn\rangle)u + (\langle v, n\rangle\langle u, dn\rangle - \langle u, n\rangle\langle v, dn\rangle)n - \langle u, dn\rangle v - \langle u, n\rangle dv$$

hence

$$\|\Delta\tilde{V}\| \le 2\,\|dv\| + 4\,\|dn\| \tag{5.14}$$

Note that $\Delta\tilde{V}^*$ is the adjoint of $\Delta\tilde{V}$, hence

$$\|\Delta\tilde{V}^*\| = \|\Delta\tilde{V}\| \le 2\,\|dv\| + 4\,\|dn\| \tag{5.15}$$

Now, because $\partial\mathbb{Q}$ is C^3 smooth we have

$$\|\Delta K\| \le C\|\delta q\| \tag{5.16}$$

for some global constant $C > 0$.

Sublemma 5.1. *There is a global constant $C > 0$ such that for any $\tau \in (\tau_{\min}/10, \tau_{\max})$*

$$\|(I + \tau B^+)^{-1}(Q^{-1}\Theta' Q - \Theta)(I + \tau B^+)^{-1}\| \le C\,\|\delta q\|$$

Proof. Recall that

$$\Theta = 2\langle v, n\rangle V^* K V = 2\langle v, n\rangle^{-1}\tilde{V}^* K \tilde{V}$$

and similar formulas hold for Θ'. We have, to the first order of $\|\delta q\|$,

$$\begin{aligned} Q^{-1}\Theta' Q - \Theta \;\; &= \;\; 2(\langle v', n'\rangle - \langle v, n\rangle)V^* K V \\ &\quad + 2\langle v, n\rangle^{-1}(\Delta\tilde{V}^* K\tilde{V} + \tilde{V}^*\Delta K\tilde{V} + \tilde{V}^* K\Delta\tilde{V}) \end{aligned} \tag{5.17}$$

Note that $\langle v', n'\rangle - \langle v, n\rangle = (\langle dv, n\rangle + \langle v, dn\rangle)$, to the first order in $\|\delta q\|$. Thus we can rewrite (5.17) in this way:

$$Q^{-1}\Theta' Q - \Theta = 2(\langle dv, n\rangle + \langle v, dn\rangle)V^* K V + 2(\Delta\tilde{V}^* K V + V^*\Delta K\tilde{V} + V^* K\Delta\tilde{V})$$

Now we apply (5.5) and then (5.14)-(5.16) with (5.9). This completes the proof of the sublemma. □

After so much preparation we are ready to **discuss curvature bounds for the flow**, i.e. for u-fronts $\mathcal{W}$.
We need to estimate the 'derivative' of the second fundamental form $B_{\mathcal{W}}(x)$ with respect to $x \in \mathcal{W}$. The operator $B_{\mathcal{W}}(x)$ acts in the hyperplane $\mathcal{J}$ that also depends on x. For points $x' = (r', v') \in \mathcal{W}$ infinitesimally close to x, let $Q = Q_{v,v'}$ be the rotator in $\mathbb{R}^d$ that takes $\mathcal{J}$ to $\mathcal{J}'$ as defined by (5.10). Then the 'increment' of B is defined by $Q^{-1}B'Q - B$, where $B = B_{\mathcal{W}}(x)$ and $B' = B_{\mathcal{W}}(x')$. Now consider

$$D_{\mathcal{W}}(x) := \max_{dr \neq 0} \|Q^{-1}B'Q - B\|/\|dr\|$$

where the maximum is taken over all nonzero infinitesimal displacement vectors $dr = r' - r$.

Lemma 5.2 (Curvature bounds - I). *There is a constant $D_{\max}$ such that for any divergent wave front $\mathcal{W}$ and $x \in \mathcal{W}$ there is a $t_0 = t_0(\mathcal{W}, x)$ such that for all $t > t_0$ we have the following: if no collisions occur in the interval $(t - \tau_{\min}/2, t)$, then $D_{\mathcal{W}_t}(x_t) \le D_{\max}$.*

Proof. For short, we put $D_t = D_{\mathcal{W}_t}(x_t)$. First we show that D_t decreases during free runs between collisions.

Sublemma 5.3. *If there are no collisions in a time interval $(t, t+\Delta t)$, then*

$$D_{t+\Delta t} \leq (1 + \Delta t\, b_{\min})^{-3} D_t$$

Proof. For short, we put $B = B_{\mathcal{W}_t}(x_t)$ and $B_1 = B_{\mathcal{W}_{t+\Delta t}}(x_{t+\Delta t})$. Similarly, we define B' and B_1' at the points x_t' and $x_{t+\Delta t}'$. Now, if A_1 and A_2 are two invertible linear operators acting in the same space, then obviously

$$A_1 - A_2 = -A_1(A_1^{-1} - A_2^{-1})A_2 \tag{5.18}$$

Applying this trick twice and using (2.2) yields

$$Q^{-1}B_1'Q - B_1 = Q^{-1}(I + \Delta t\, B')^{-1}Q\,[Q^{-1}B'Q - B]\,(I + \Delta t\, B)^{-1}$$

Now the sublemma easily follows, with the help of (5.7) and (5.8). □

Sublemma 5.4. *If there is a collision in a time interval $(t, t+\tau_{\min}/4)$, then*

$$D_{t+\tau_{\min}/2} \leq D_t + \bar{D}$$

where $\bar{D} > 0$ is a global constant.

Proof. Let $s = t + \tau_{\min}/2$. Note that there are no collisions in the interval $(t+\tau_{\min}/4, s)$. For short, we put $B = B_{\mathcal{W}_s}(x_s)$ and $B' = B_{\mathcal{W}_s}(x_s')$. Denote by t_1 and t_1' the moments of reflection of the trajectories of the points x_t and x_t', respectively, that occur in the interval $(t, t+\tau_{\min}/4)$. Put $dt = t_1' - t_1$, $\tau = s - t_1$ and $\tau' = s - t_1'$. Note that $\tau > \tau_{\min}/4$ and $\tau' > \tau_{\min}/4$. Put $B^+ = B_{\mathcal{W}_{t_1+0}}(x_{t_1+0})$ and $B'^+ = B_{\mathcal{W}_{t_1'+0}}(x_{t_1'+0})$. Let Q be the rotation of $\mathbb{R}^d$ that takes $v = v_s$ to $v' = v_s'$. It acts on $\mathcal{J} = \mathcal{J}_{x_s}$ by the rule (5.10). Applying the trick (5.18) twice yields

$$\begin{aligned} Q^{-1}B'Q - B \;&=\; -Q^{-1}B'Q(dt\, I)B \\ &\quad + Q^{-1}(I + \tau' B'^+)^{-1}Q\,[Q^{-1}B'^+Q - B^+]\,(I + \tau B^+)^{-1} \end{aligned} \tag{5.19}$$

Note that $\|B\| \leq 1/\tau \leq 4/\tau_{\min}$, and likewise $\|B'\| \leq 4/\tau_{\min}$. Hence,

$$\| -Q^{-1}B'Q(dt\, I)B\| \leq C|dt|$$

for a global constant $C > 0$. Next, we have $B^+ = UB^-U^{-1} + \Theta$ by (2.3), and, similarly $B'^+ = U'B'^-U'^{-1} + \Theta'$. Then we can further decompose the last term in (5.19):

$$\begin{aligned} \|Q^{-1}B'Q - B\| \;&\leq\; C|dt| + \|Q^{-1}U'B'^-U'^{-1}Q - UB^-U^{-1}\| \\ &\quad + \|Q^{-1}(I + \tau' B'^+)^{-1}Q\,[Q^{-1}\Theta'Q - \Theta]\,(I + \tau B^+)^{-1}\| \end{aligned}$$

Using Sublemma 5.1 (and its notation) gives, up to the first order in $\|\delta q\|$,

$$\|Q^{-1}(I+\tau' B'^+)^{-1}Q\,[Q^{-1}\Theta' Q-\Theta]\,(I+\tau B^+)^{-1}\|$$

$$=\|(I+\tau B^+)^{-1}[Q^{-1}\Theta' Q-\Theta]\,(I+\tau B^+)^{-1}\|\le C\|\delta q\|$$

Note that

$$\|Q^{-1}U'B'^-U'^{-1}Q-UB^-U^{-1}\|=\|Q_1^{-1}B'^-Q_1-B^-\| \tag{5.20}$$

where $Q_1=U'^{-1}QU$ is the rotator that takes the hyperplane $\mathcal{J}^-=\mathcal{J}_{x_{t_1-0}}$ to $\mathcal{J}'^-=\mathcal{J}_{x'_{t'_1-0}}$. We apply the trick (5.18) twice and act as in (5.19) and easily obtain

$$\|Q_1^{-1}B'^-Q_1-B^-\|\le\|B'^-\|\,|dt|\,\|B^-\|+\|Q_1^{-1}B_1'Q_1-B_1\| \tag{5.21}$$

where $B_1=B_{\mathcal{W}_t}(x_t)$ and $B_1'=B_{\mathcal{W}_t}(x_t')$.

Combining the above estimates gives

$$\|Q^{-1}B'Q-B\|\le C|dt|+C\|\delta q\|+\|Q_1^{-1}B_1'Q_1-B_1\|$$

for some global constant $C>0$. Note that $dr_s=(I+\tau B^+)dr^+=(I+\tau B^+)V^{-1}\delta q$, and due to (5.3) we have $\|\delta q\|\le C\|dr_s\|$. Lastly, $|dt|\le 2\|\delta q\|$ by (5.9) and $\|dr_t\|<\|dr_s\|$, which easily follows from (5.7). Therefore,

$$\|Q^{-1}B'Q-B\|/\|dr_s\|\le\bar{D}+\|Q_1^{-1}B_1'Q_1-B_1\|/\|dr_t\|$$

where $\bar{D}$ is a global constant, which proves the sublemma. □

We now complete the proof of Lemma 5.2. Let $t>0$ satisfy the condition of the Lemma, and n be the number of collisions on the interval $(0,t)$. Then combining Sublemmas 5.3 and 5.4 gives

$$D_t\le\lambda^n D_0+(1+\lambda+\cdots+\lambda^n)\bar{D}$$

where $\lambda=(1+\tau_{\min}b_{\min}/4)^{-3}<1$. Since $\bar{D}$ is a global constant, the Lemma follows. □

In all that follows we will only consider u-fronts $\mathcal{W}$ for which $D_{\mathcal{W}}(x)\le D_{\max}$ for all $x\in\mathcal{W}$ provided the trajectory $S^t x$, $-\tau_{\min}/2<t<0$, does not collide with $\partial\mathbb{Q}$. As we are mainly interested in those u-manifolds that approximate LUM-s, this convention is justified by Lemma 5.2. Indeed, if the front $\mathcal{W}$ corresponds to a LUM, than $S^{-t}\mathcal{W}$ is a divergent front for any $t>0$.

Remark. A useful estimate (5.21) obtained in the proof of Sublemma 5.4 can now be restated. Recall that $|dt|\le 2\|\delta q\|$, $\|B'^-\|\cdot\|B^-\|\le 1/\tau_{\min}^2$ (a global bound) and

$$\|Q_1^{-1}B_1'Q_1-B_1\|\le D_{\max}\|dr_t\|$$

by the above convention. Also note that $\|dr_t\|\le\|dr^-\|=\|dr^+\|=\|V^{-1}\delta q\|\le\|\delta q\|$. Hence,

$$\|Q_1^{-1}B'^-Q_1-B^-\|\le C\|dr\| \tag{5.22}$$

with a global constant $C > 0$.

Finally we should prove the **curvature bounds on u-manifolds W in the Poincaré phase space**, in other words, that the 'derivative' of F along u-manifolds is uniformly bounded.

We will denote by $\text{dist}_W(x, y)$ the distance between $x, y \in W$ in the Euclidean metric on W. Let $x = (q, v)$ and $x' = (q', v')$ be two infinitesimally close points of a u-manifold W, and F and F' the corresponding operators at x and x'. Using our previous notation, we consider the increment of F defined by $Q^{-1}F'Q_0 - F$. Here again Q_0 is the rotator taking $n = n(q)$ to $n' = n(q')$ and Q is the rotator taking v to v'.

Theorem 5.5 (Curvature bounds - II). *There is a global constant $C > 0$ such that*

$$\|Q^{-1}F'Q_0 - F\| \leq C\,\|\delta q\|$$

Proof. Using the second formula in (4.4) and our earlier notation $\tilde{V}^* = \langle v, n\rangle V^*$ gives

$$\begin{aligned} \|Q^{-1}F'Q_0 - F\| \;&\leq\; \|Q^{-1}\tilde{V}'^*Q_0Q_0^{-1}K'Q_0 - \tilde{V}^*K\| \\ &\quad +\|Q^{-1}U'B'^-U'^{-1}QQ^{-1}V'^{-1}Q_0 - UB^-U^{-1}V^{-1}\| \end{aligned}$$

The first term is bounded by $C\,\|\delta q\|$ for some global constant $C > 0$, according to our earlier estimates (5.15) and (5.16). To bound the second term we need two more estimates. One is

$$\|Q^{-1}V'^{-1}Q_0 - V^{-1}\| \leq 4\,\|dv\| + 2\,\|dn\| \leq C\|\delta q\| \tag{5.23}$$

which is proved just like (5.14) and (5.15), we omit the details. The other is

$$\|Q^{-1}U'B'^-U'^{-1}Q - UB^-U^{-1}\| \leq C\|\delta q\| \tag{5.24}$$

for a global constant $C > 0$. In the proof of Sublemma 5.4 we introduced the rotator $Q_1 = U'^{-1}QU$ that takes the hyperplane $\mathcal{J}^-$ to $\mathcal{J}'^-$. With this, (5.24) is simply equivalent to our early estimate (5.22). Theorem 5.5 is now proved. □

5.2 Distorsion bounds

This subsection is devoted to the question, how smoothly the volume expansion rates vary at nearby points on the same u-manifold (distorsion bounds) and at different u-manifolds joint by holonomy maps along s-manifolds (absolute continuity). Actually, the reason for introducing homogeneity strips and secondary singularities (see (4.3)) is that we would like to control these distorsions. Let us consider the evolution under T^n of a u-manifold W. Due to (4.11) distances grow exponentially in n, and the same is true for the $(d-1)$-dimensional volume of T^nW. However, at almost grazing reflections, when

$\langle v, n\rangle \approx 0$, the expansion of u-manifolds is highly nonuniform, and so distortions are unbounded. Nevertheless, as we shall prove in Theorem 5.7, the situation is much better with homogeneous u-manifolds.

Throughout the subsection all metric quantities (norms, distances, volume elements, Jacobians) are understood in the e-metric, thus we often drop the sub- or superscripts e.

Sublemma 5.6. *If W is a homogeneous u-manifold, then for any two points $x = (q, v)$ and $\bar{x} = (\bar{q}, \bar{v})$ of W we have*

$$|\langle \bar{v}, \bar{n}\rangle - \langle v, n\rangle| \leq C \langle v, n\rangle \left[\mathrm{dist}_W(x, \bar{x})\right]^{1/3}$$

where $\bar{n} = n(\bar{q})$ and $C > 0$ is a global constant.

Proof. Let $W \cap I_k \neq \emptyset$ for some k. Then

$$|\langle \bar{v}, \bar{n}\rangle - \langle v, n\rangle| \leq C_1 (k+1)^{-3} \tag{5.25}$$

with a global constant C_1, according to our construction of I_k. Next, for any point $x' = (q', v')$ infinitesimally close to x, we have, up to the first order in $\|\delta x\| (= \|\delta x\|_e)$,

$$|\langle v', n'\rangle - \langle v, n\rangle| = |\langle dv, n\rangle + \langle v, dn\rangle| \leq C_2 \|\delta q\| \leq C_3 \|\delta x\| \tag{5.26}$$

with some global constants C_2, C_3, see (5.9) and Corollary 4.4. Integrating (5.26) from x to $\bar{x}$ yields

$$|\langle \bar{v}, \bar{n}\rangle - \langle v, n\rangle| \leq C_3 \,\mathrm{dist}(x, \bar{x}) \tag{5.27}$$

Now (5.25) and (5.27) give

$$|\langle \bar{v}, \bar{n}\rangle - \langle v, n\rangle|^3 \leq C_1^2 C_3 (k+1)^{-6} \,\mathrm{dist}(x, \bar{x})$$

Lastly, recall that $\langle v, n\rangle \geq (k+1)^{-2}$ if $k > 0$ and $\langle v, n\rangle \geq k_0^{-2}$ if $k = 0$, hence $\langle v, n\rangle \geq k_0^{-2}(k+1)^{-2}$ for any k. Therefore,

$$|\langle \bar{v}, \bar{n}\rangle - \langle v, n\rangle|^3 \leq C_1^2 C_3 k_0^6 \langle v, n\rangle^3 \,\mathrm{dist}(x, \bar{x})$$

This proves the sublemma. □

Let W be a u-manifold, $x \in W$ and T^n continuous at x. Denote by $J_{W,n}(x)$ the expansion factor of the $(d-1)$-dimensional volume of the manifold W under T^n at the point x, i.e. $J_{W,n}(x) := |\det DT^n|_W (x)|$.

Theorem 5.7 (Distorsion bounds). *Let W be a small u-manifold on which T^n is continuous. Assume that $W_i := T^i W$ is a homogeneous u-manifold for each $0 \leq i \leq n$. Then for all $x, \bar{x} \in W$*

$$|\ln J_{W,n}(\bar{x}) - \ln J_{W,n}(x)| \leq C \cdot \left[\mathrm{dist}_{W_n}(T^n x, T^n \bar{x})\right]^{1/3}$$

for a global constant $C > 0$.

Proof. Note that $J_{W,n}(x) = \prod_{i=0}^{n-1} J_{W_i,1}(T^i x)$. Hence, it is enough to prove the lemma for $n = 1$, because $\mathrm{dist}(T^i x, T^i \bar{x})$ grows uniformly exponentially in i due to (4.11). So we put $n = 1$.

Denote $x_1 = Tx$ and $\bar{x}_1 = T\bar{x}$. We will also use a variable point $x' \in W$ infinitesimally close to x, and put $x_1' = Tx'$. For convenience, we will use the subscript 1 to denote quantities (including operators, hyperplanes, etc.) related to the points $x_1, \bar{x}_1$ and x_1'. In a similar way, bars are used to denote quantities related to the points $\bar{x}$ and $\bar{x}_1$, and primes are used for quantities related to x' and x_1'. For example, we denote by B^+, $\bar{B}^+$ and B'^+ the second fundamental forms of the wave front (corresponding to the u-manifold W) at the points x, $\bar{x}$, and x', respectively. Similarly, F, $\bar{F}$, and F' denote the F operator (4.4) taken at x, $\bar{x}$ and x', respectively. In a similar way, F_1, $\bar{F}_1$, and F_1' are the F operators taken at x_1, $\bar{x}_1$ and x_1', respectively, etc.

Note that the basic quantity, $J_{W,1}(x)$ was already calculated as $J_W^e(x)$ in the previous section (formula (4.13)) where we also introduced the operator G. In view of this formula, to prove Theorem 5.7 with $n = 1$, it is now enough to prove three claims:

Claim 1. $|\ln\det\bar{V} - \ln\det V| \le C \cdot \Big[\mathrm{dist}_W(x,\bar{x})\Big]^{1/3}$.

Claim 2. $|\ln\det\bar{G} - \ln\det G| \le C \cdot \Big[\mathrm{dist}_W(x,\bar{x})\Big]$.

Claim 3. $|\ln\det(I + \bar{\tau}\bar{B}^+) - \ln\det(I + \tau B^+)| \le C \cdot \Big[\mathrm{dist}_{TW}(x_1,\bar{x}_1)\Big]^{1/3}$.

By C we denote some global constants. Indeed, the bounds in Claims 1 and 2 will also hold at the points x_1 and $\bar{x}_1$, because TW is a homogeneous u-manifold, and Theorem 5.7 will then easily follow.

Proof of Claim 1. Since $\det V = \langle v, n\rangle^{-1}$, the claim is a direct consequence of Sublemma 5.6.

Our proofs of Claims 2 and 3 use the following

Sublemma 5.8. *Let A be an invertible linear operator in an m-dimensional space, and ΔA an infinitesimal operator. Then, up to the first order of $\|\Delta A\|$,*

$$|\ln\det(A + \Delta A) - \ln\det A| = |\,\mathrm{tr}(A^{-1}\cdot\Delta A)| \le m\,\|A^{-1}\cdot\Delta A\|$$

Proof. We have $\ln\det(A + \Delta A) = \ln\det A + \ln\det(I + A^{-1}\cdot\Delta A)$, and the rest is straightforward. □

Proof of Claim 2. It is enough to prove

$$|\ln\det G' - \ln\det G| \le C\,\|\delta x\| \tag{5.28}$$

for infinitesimally close points $x, x' \in W$, then the integration from x to $\bar{x}$ will give the bound in Claim 2.

As to the value of $\det G$, we refer to formula (4.14). Now, by Sublemma 5.8, we have

$$\begin{aligned} |\ln\det G' - \ln\det G| &\le |\ln\det(I+F'^*F') - \det(I+F^*F)| \\ &= |\ln\det(I+Q_0^{-1}F'^*F'Q_0) - \det(I+F^*F)| \\ &\le (d-1)\,\|(I+F^*F)^{-1}(Q_0^{-1}F'^*F'Q_0 - F^*F)\| \end{aligned}$$

(the introduction of Q_0 defined by (4.1) was necessary to ensure that both operators act in the same space). It is obvious that $\|(I+F^*F)^{-1}\| \le 1$, and by Corollary 4.4 and Theorem 5.5 we have

$$\|Q_0^{-1}F'^*F'Q_0 - F^*F\| \le C\,\|dr\|$$

This proves (5.28), and so Claim 2 is proved.

Proof of Claim 3. To shorten some formulas, we put $R = I + \tau B^+$ (and, respectively, define $\bar{R}$ and R' at the points $\bar{x}$ and x'). It will be enough to prove that

$$|\ln\det R' - \ln\det R| \le C|\langle v',n'\rangle - \langle v,n\rangle|\,\langle v,n\rangle^{-1} + C\,\|\delta x\| + C\,\|\delta x_1\| \tag{5.29}$$

for infinitesimally close points $x, x' \in W$. Note that $\|\delta x\| \le C\|\delta x_1\|$ by (4.11). Then the integration of (5.29) from x to $\bar{x}$ (and, respectively, from x_1 to $\bar{x}_1$) will give

$$|\ln\det\bar{R} - \ln\det R| \le C|\langle\bar{v},\bar{n}\rangle - \langle v,n\rangle|\,\langle v,n\rangle^{-1} + C\left[\mathrm{dist}_{TW}(x_1,\bar{x}_1)\right]$$

After that Claim 3 will follow by Sublemma 5.6.

We now prove (5.29). By Sublemma 5.8 we have, to the first order in $\|\delta x\|$,

$$\begin{aligned} \ln\det R' - \ln\det R &= \ln\det Q^{-1}R'Q - \ln\det R \\ &= \mathrm{tr}\,[R^{-1}(\tau' Q^{-1}B'^+Q - \tau B^+)] \end{aligned} \tag{5.30}$$

(the introduction of Q defined by (5.10) was necessary to ensure that both operators act in the same space). Note that $\|R^{-1}\| \le C$ by (5.8). Next, we have, again to the first order in $\|\delta x\|$,

$$\begin{aligned} \tau' Q^{-1}B'^+Q - \tau B^+ &= d\tau\, B^+ + \tau(Q^{-1}U'B'^-U'^{-1}Q - UB^-U^{-1}) \\ &\quad + \tau(Q^{-1}\Theta' Q - \Theta) \end{aligned} \tag{5.31}$$

Observe that

$$\|VR^{-1}\| \le C \qquad \text{and} \qquad \|R^{-1}V^*\| \le C \tag{5.32}$$

according to (5.3) and (5.4). Using (2.3) now yields

$$\|R^{-1}B^+\| \le \|R^{-1}\|\,\|B^-\| + 2\,\|R^{-1}V^*K\tilde{V}\| \le C \tag{5.33}$$

Now recall that $|d\tau| \le 2\|\delta q\| + 2\|\delta q_1\|$ by (5.9). Hence we have, by (5.33),

$$|\,\mathrm{tr}\,(d\tau\, R^{-1}B^+)\,| \le (d-1)\,|d\tau|\,\|R^{-1}B^+\| \le C(\|\delta q\| + \|\delta q_1\|)$$

so the first term in the right hand side of (5.31) is properly taken care of.

Denote $\Delta B^- = Q^{-1}U'B'^-U'^{-1}Q - UB^-U^{-1}$. We then have, using (5.20) and (5.22),

$$\begin{aligned} |\,\mathrm{tr}\,(\tau\, R^{-1}\,\Delta B^-)\,| &\le (d-1)|\tau|\,\|R^{-1}\,\Delta B^-\| \\ &\le \tau_{\max}\|R^{-1}\|\,\|Q_1^{-1}B'^-Q_1 - B^-\| \\ &\le C\,\|\delta q\| \end{aligned}$$

which takes care of the second term in (5.31).

Lastly, we use (5.17) to handle the third term in (5.31):

$$\begin{aligned} |\,\mathrm{tr}\,(R^{-1}(Q^{-1}\Theta'Q - \Theta))\,| \le\ & 2\,|\langle v',n'\rangle - \langle v,n\rangle|\,|\,\mathrm{tr}\,(R^{-1}V^*KV)| \\ & +2\,|\,\mathrm{tr}\,(R^{-1}\Delta\tilde{V}^*KV)| + 2\,|\,\mathrm{tr}\,(R^{-1}V^*\Delta K\tilde{V})| \\ & +2\,|\,\mathrm{tr}\,(R^{-1}V^*K\Delta\tilde{V})| \end{aligned} \tag{5.34}$$

We note that

$$\mathrm{tr}\,(R^{-1}\Delta\tilde{V}^*KV) = \mathrm{tr}\,(\Delta\tilde{V}^*KVR^{-1}) = \mathrm{tr}\,(R^{-1}V^*K\Delta\tilde{V})$$

where the first equation follows from a general formula $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ in linear algebra, and the second is due to the fact that the operators $\Delta\tilde{V}^*KVR^{-1}$ and $R^{-1}V^*K\Delta\tilde{V}$ are adjoint to each other. Using this observation, we can rewrite (5.34) as

$$\begin{aligned} |\,\mathrm{tr}\,(R^{-1}(Q^{-1}\Theta'Q - \Theta))\,| \le\ & C\,|\langle v',n'\rangle - \langle v,n\rangle|\,\langle v,n\rangle^{-1}\,\|R^{-1}V^*K\tilde{V}\| \\ & +C\,\|\Delta\tilde{V}^*KVR^{-1}\| + C\,\|R^{-1}V^*\Delta K\tilde{V}\| \end{aligned}$$

We now apply (5.32) and (5.15)-(5.16) with (5.9) and obtain

$$|\,\mathrm{tr}\,(R^{-1}(Q^{-1}\Theta'Q - \Theta))\,| \le C|\langle v',n'\rangle - \langle v,n\rangle|\,\langle v,n\rangle^{-1} + C\,\|\delta x\|$$

This completes the proof of (5.29) and hence Claim 3. Theorem 5.7 is now proved. □

After proving that the expansion factors vary nicely between nearby points on the same u-manifold, we now investigate their behaviour at points of different u-manifolds that lie on the same s-manifold. This is the absolute continuity property. Just like it was with the distorsion bounds, it is important to consider homogeneous manifolds.

Theorem 5.9 (Absolute continuity). *Let W_s be a small s-manifold, $x, \bar{x} \in W_s$, and $W_u, \bar{W}_u$ two u-manifolds crossing W_s at x and $\bar{x}$, respectively. Assume that T^k is continuous on W_s and T^iW_s is a homogeneous s-manifold for each $0 \le i \le k$. Then*

$$|\ln J_{W_u,k}(x) - \ln J_{\bar{W}_u,k}(\bar{x})| \le C$$

where C is a global constant.

Proof. For any $z \in W_s$, let $J_{W_s,k}(z)$ be the volume expansion factor of W_s under T^k at the point z, i.e. $J_{W_s,k}(z) = |\det DT^k|_{W_s}(z)|$. By the analogue of Theorem 5.7 for homogeneous s-manifolds, we have

$$|\ln J_{W_s,k}(x) - \ln J_{W_s,k}(\bar{x})| \leq C' \tag{5.35}$$

for a global constant C'.

Let $|DT^k(x)|$ denote the Jacobian of T^k at a point $x = (q,v) \in M$ with respect to the Lebesgue measure $\delta q\, \delta v$ on M in our local coordinates (q,v). Note that the T-invariant measure is $d\nu = \langle v, n\rangle\, \delta q\, \delta v$. Hence, $|DT^k(x)| = \langle v,n\rangle / \langle v_k, n_k\rangle$ where $x_k = (q_k, v_k) = T^k x$ and $n_k = n(q_k)$. Similarly, $|DT^k(\bar{x})| = \langle \bar{v}, \bar{n}\rangle / \langle \bar{v}_k, \bar{n}_k\rangle$, where the notation is quite clear. Since both W_s and $T^k W_s$ are small homogeneous s-manifolds, Sublemma 5.6 implies that the quantity $\langle v, n\rangle$ does not vary much over either W_s or $T^k W_s$. In fact, $c < \langle v,n\rangle / \langle \bar{v}, \bar{n}\rangle < C$ and $c < \langle v_k, n_k\rangle / \langle \bar{v}_k, \bar{n}_k\rangle < C$ for global constants $C > c > 0$. Hence,

$$0 < c < |DT^k(x)| / |DT^k(\bar{x})| < C < \infty \tag{5.36}$$

for some global constants c and C. Now Theorem 5.9 follows easily from (5.35), (5.36), and Theorem 4.7. □

6 Outlook

The results of this paper can be summarized as follows. We have some bad news (non-smooth behaviour) related to the singularity submanifolds in multi-dimensional hyperbolic billiards. On the other hand, there are important good news related to the u-manifolds in the multi-dimensional dispersing case. It is proved that practically all important regularity properties (uniform hyperbolicity, alignment, curvature and distorsion bounds) are just as valid as they are in the multi-dimensional case (cf. Remark 4.6).

In billiard theory one is mainly interested in the ergodic and statistical properties of the dynamical system. We emphasize that the above results are highly relevant to these issues. As to the ergodic properties, a major breakthrough was achieved with the proof of the Fundamental (or Local Ergodicity) Theorem ([SCh, KSSz]). However, for some measure theoretic estimates, the original arguments in these papers implicitly assumed uniform curvature bounds on the singularities. Thus these proofs have to be checked. In a separate paper ([BChSzT]) we will show that – at least, for billiards with algebraic scatterers – the original proofs of local ergodicity remain valid if some suitable modifications are performed.

Much less is known about statistical properties. As to the multi-dimensional dispersing case, no optimal result (exponential decay of correlations) has been achieved so far. Nevertheless, we conjecture that the rate of mixing is, indeed, exponential. The recently developed method of Markov-returns ([Y1]) turned out to be especially powerful in the study of decay rates for planar billiards ([Ch2, Ch3]). It is the growth of unstable manifolds that is to be investigated for Young's method to work. Essentially all important features of unstable manifolds have been checked in sections 4 and 5 to control growth of LUMs, the only thing we do not know yet how to handle is the irregular behaviour of singularities. We conjecture that, given a systematic geometric characterization of singularities, exponential decay of correlations for multi-dimensional dispersing billiards could be proved.

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