FORMALIZING SET THEORY IN WEAK FRAGMENTS OF ALGEBRAIC LOGIC (UPDATED IN JUNE 2011)

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Announcement: Theorem 7 way below solves some long-standing open problems from the literature discussed following the statement of Problem 1. What comes below is the old 2009 version of a note, updated with the new result Thm.7. (The results annunced in the 2009 note become corollaries of the new Thm.7.)

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In this note we recall some results in the subject mentioned in the title and we state open problems. We use the notation in [6], [8], [9].

Definition 1. *The class of α-dimensional substitution-cylindric algebras* SC_α *is defined as* $\mathbf{S}\{\langle A,+,-,\mathsf{c}_i,\mathsf{s}^i_j\rangle_{i,j<\alpha} : \langle A,+,-,\mathsf{c}_i,\mathsf{d}_{ij}\rangle_{i,j<\alpha} \in \mathsf{CA}_\alpha\}$.

See [3, sec.3, p.184], and for an equational basis see [1, Def. 4.14]. SC_α's are also mentioned in [6, p.267, 5.6.18(13)] as Pinter's algebras.[1] We denote the corresponding logic by $\mathcal{L}^{s\neq}_\alpha$, this is in the spirit of [3, p.229, sec.II.7, Ex.7].

Next we recall some results from [9], cf. also [10]. For stronger results see [9], [10], [8], [5], [2]. Fm^k_n is the set of formulas with k free variables of $\mathcal{L}_n$, the n-variable fragment of first-order logic (FOL).

Theorem 1. *Set theory can be built up in the equational theory of* CA_3*, and equivalently in first-order logic* $\mathcal{L}_3$ *with three variables. In more detail:*
There is a computable translation function $\kappa : Fm^2_\omega \to Fm^1_3$ *for which the following are true for all* $\varphi \in Fm^0_\omega$

(i) $ZF \models \varphi \quad \Leftrightarrow \quad \kappa^*(ZF) \vdash_3 \kappa\varphi$,

(ii) $ZF \models \varphi \leftrightarrow \kappa\varphi$.

The above theorem follows from [10, Thm.12,Thm.17(vi)] and from the fact that $ZF \models \pi$, for the formula π introduced in [8] as well as in [9].

Theorem 2. *Free* CA_3*'s are not atomic (except for the 0-generated one).*

Theorem 3. *The logic* $\mathcal{L}_3$ *has Gödel's incompleteness properties.*

The proofs of Thm.s 1-3 can be found in [8], [9].

Tarski proved that set theory can be built up in the equational theory of relation algebras [13], and hence in 4-variable logic $\mathcal{L}_4$. Thus Thm.1 above is an improvement of Tarski's result (solving open problems from [13]).

While CA_α is the algebraic counterpart of first-order logic with equality, the class SC_α is the natural algebraic counterpart of logic without equality $\mathcal{L}_\alpha^{s\neq}$. In the next theorems we generalize the above three results to SC_3 from CA_3.

Theorem 4. *Set theory can be built up in the equational theory of* SC_3*, we mean this in the same sense as in Thm.1 (hence in the same sense as in* [13]*,* [8]*,* [9]*). In more detail: there is a computable* $\kappa : Fm^2_\omega \to \mathcal{L}_3^{s,\neq}$ *satisfying (i),(ii) of Thm.1.*

Theorem 5. *Free* SC_3*'s are not atomic (except for the 0-generated one).*

Theorem 6. *The logic* $\mathcal{L}_3^{s\neq}$ *has Gödel's incompleteness properties.*

On the proof of Thm.s 4-6: The proofs in [8] and in [9] can be pushed through for logic without equality (if we have substitutions), hence for SC_3 in place of CA_3. One of the ideas is that in set theory we can define the equality relation by the extensionality axiom of set theory. Indeed, we add the formulas

$$(*)\ \{\forall xy(\forall z(z \in x \leftrightarrow z \in y) \to \forall z(x \in z \leftrightarrow y \in z)) : \{x, y, z\} = \{v_0, v_1, v_2\}\}$$

to the formula π in the proof of Thm.1 in [9], [8] (as conjuncts). We define the new formula π^+ as $(\pi \wedge (*))$.

Now, using π^+ in place of π we can push through the proof of Thm.1 in [9], [8] to proving Thm.s 4-6. In particular, we can define a translation mapping $\kappa^+ : Fm^2_\omega \to \mathcal{L}_3^{s,\neq}$ analogously to κ in Thm.1 (of course, by using the new π^+). $\square$

For an independent, different kind of proof for Thm.s 5,6 we refer to Gyenis [5].

The above leads up to the following problem which has been an open conjecture ever since 1987.

$\mathcal{L}_3^{\neq}$ denotes three-variable FOL without equality and without substitutions. (Hence, $\mathcal{L}_3^{\neq}$ is restricted FOL in the sense of the cylindric algebra monograph [6, sec.4.3].) This is the logic corresponding to Df_3 (i.e., Boolean algebras with three commuting complemented closure operators). The logic corresponding to Df_3 can be regarded as a multimodal propositional logic with 3 commuting S5-modalities. The multimodal propositional logics $[S5, S5, S5]$ and $S5 \times S5 \times S5$ are equivalent with the logical counterparts of Df_3 and RDf_3 respectively. In particular,

$[S5, S5, S5]$ is equivalent with $\mathcal{L}_3^{\neq}$. Cf. Gabbay et al [4, p.379, lines 15-20].

Problem 1. *(Solved) Do theorems 4-6 generalize from* SC_3 *to the class* Df_3 *of diagonal-free* CA_3*'s? More concretely:*
Problem 1.1: *Can set theory be formalized in* $\mathcal{L}_3^{\neq}$ *similarly to Thm.1?*
Problem 1.2: *Are finitely generated free* Df_3*'s not atomic?*
Problem 1.3: *Does the logic* $\mathcal{L}_3^{\neq}$ *corresponding to* Df_3 *enjoy the Gödel incompleteness properties in a sense analogous to that of Thm.s 3,6?*

For the statement of this problem see also [11, p.12, Open Question 1], [12, p.476, Open Question 1]. The same problems are raised for the class BSR of Boolean semigroups in [3, pp.152,153]. Since the equational theories of $\mathsf{Crs}_3, \mathsf{WA}, \mathsf{NA}$ are decidable, set theory cannot be formalized in these. Problem 1 was highlighted in the problem sessions of the international conferences Logic in Hungary 2005 (Budapest, 2005) and Logic, Algebra, Relativity - 2002 (Budapest, 2002).

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With H. Andréka we proved in 2011 that the answer to the above Problem 1 is affirmative, e.g., the 1-generated free Df_3 is not atomic, see Thm.7 below.

Theorem 7. *The answers to Problem 1 above are in the affirmative. More concretely:*
(7.1) *Set theory can be formalized in* $\mathcal{L}_3^{\neq}$ *in complete analogy with Thm.s 1,4 and their proofideas above.*
(7.2) *Finitely generated free* Df_3*'s are not atomic (except for the 0-generated one). This is a corollary of* (7.1).
(7.3) *The logics* $\mathcal{L}_3^{\neq}$*,* $[S5, S5, S5]$ *and* $S5 \times S5 \times S5$ *have Gödel's incompleteness property in analogy with Thm.s 3,6.*
In particular, Thm.s 1-3 remain true if we replace CA_3 *and* $\mathcal{L}_3$ *in them with* Df_3 *and* $\mathcal{L}_3^{\neq}$ *(or equivalently* $[S5, S5, S5]$*) everywhere.*

For more detail on Thm.7 the reader is referred to Andréka-Németi [2]. Details are available from the authors via e-mail.

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Notes

[1]Warning: there is a typo in [6, p.267]: the reference there should be Pinter [73'], [73c'].

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