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Note

A Simple Proof of the Erdös-Chao Ko-Rado Theorem

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A simple proof is given of the theorem of Erdös, Chao Ko, and Rado.

THEOREM (Erdös, Chao Ko, Rado [1]). *Let* $X = \{1, 2, \ldots, n\}$ *be a finite set and* $A_1, A_2, \ldots, A_m$ *be different subsets of* X *such that*

$$|A_i| = k \qquad (1 \leqslant i \leqslant m,\ k \leqslant n/2 \text{ fixed})$$

and

$$A_i \cap A_j \neq \varnothing \qquad (1 \leqslant i \leqslant j \leqslant m).$$

Then

$$m \leqslant \binom{n-1}{k-1}.$$

Proof. For $1 \leqslant i \leqslant n$ let B_i be the set of those numbers among $1, 2, \ldots, n$ which are congruent, mod n, to one of

$$(i-1)k + 1, \quad (i-1)k + 2, \ldots, ik.$$

B_i's are not necessarily different.

1. If $1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant n$ and $B_{i_1}, \ldots, B_{i_d}$ are pairwise non-disjoint, then $d \leqslant k$. To prove this, let us fix B_{i_1}. By the symmetricity we may assume that $i_1 = 1$. B_j is non-disjoint with B_1 if and only if either

$$jk = q_1 n + r_1, \tag{1}$$

where $0 \leqslant q_1 < k$, $1 \leqslant r_1 \leqslant k$, or

$$(j-1)k + 1 = q_2 n + r_2, \tag{2}$$

where $0 \leqslant q_2 < k$, $1 \leqslant r_2 \leqslant k$.

For fixed q_1 there is at most one (j, r_1) satisfying (1), and for fixed q_2 there is at most one (j, r_2) satisfying (2). If (1) holds for some j with $r_1 = k$, then $(j-1)k + 1 = q_1 n + 1$, and (2) can hold only for the same j. If (1) holds for some $j = j_0$ and $1 \leqslant r_1 < k$, then (2) can hold, with $q_2 = q_1$, only for $j = j_0 + 1$ since $j_0 k + 1 = q_1 n + r_1 + 1$. However $B_{j_0} \cap B_{j_0+1} \neq \varnothing$ since $k \leqslant n/2$. Thus, for every q, there is at most one j with $B_1 \cap B_j \neq \varnothing$ so that either (1) or (2) holds, and if $q = 0$ then $j = 1$.

Our first statement is proved.

Denote by $F_1, \ldots, F_{n!}$ the sequences obtained from $F_1 = (B_1, \ldots, B_n)$ by permutation of the elements of X.

2. Count in two different ways the pairs (F_i, A_j), where A_j is in the sequence F_i:

$$n! \cdot k \geqslant m \cdot n \cdot k!(n-k)!. \tag{3}$$

Here, on the left-hand side, we used the result of point 1, that F_i can contain at most k A_j's. On the right-hand side, $n \cdot k!(n-k)!$ is the number of F_i's containing a fixed A_j, because we find $k!(n-k)!$ permutations transforming a fixed B_r onto A_j. (3) is equivalent to the statement of the theorem.

This proof can be considered as a further development of Lubell's method [2].

References

1. P. Erdös, Chao Ko, and R. Rado, Intersection theorem for system of finite sets, *Quart. J. Math. Oxford Ser.* **12** (1961), 313–318.
2. D. Lubell, A short proof of Sperner's lemma, *J. Combinatorial Theory* **1** (1966), 299.

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