1. The subject of this note is the following problem, proposed orally by G. Grünwald and D. Lázár. Let \( p_1, p_2, \ldots, p_k \) be any prime numbers. We may say that \( N \) is composed of the primes \( p_1, p_2, \ldots, p_k \) when every prime factor of \( N \) is one of these primes. Can we find an infinite set of different positive integers \( a_1, a_2, \ldots \) so that every sum \( a_i + a_j (i \neq j) \) is composed of \( p_1, p_2, \ldots, p_k \)? The answer that no such set exists was given by the proposers. Their proof depends on a theorem of Mr. Pólya asserting that if we denote \( q_1 < q_2 < \cdots < q_n \) the numbers composed of the primes \( p_1, p_2, \ldots, p_k \) then \( q_n + 1 \) \( q_n \) tends to infinity. But the proof of Pólya's theorem is not elementary; it seems therefore desirable to show the above result in an elementary way. On the other hand Pólya's theorem does not allow any further deductions in the following direction. Let \( a_1, a_2, \ldots, a_n \) be a finite set of positive integers such that the sums \( a_i + a_j \) contain no prime factors other than \( p_1, p_2, \ldots, p_k \); can we find an upper bound for the number \( n \) of such integers, depending on \( p_1, p_2, \ldots, p_k \) or on \( k \) only? (Plainly we can suppose that \( p_1 = 2 \), because if the \( p_1, p_2, \ldots, p_k \)
are all odd, we find \( n \leq 2 \). Indeed, otherwise at least one of \( a_1 + a_2 \), \( a_1 + a_3 \), \( a_2 + a_3 \) would be even.)

We present an answer to the last question containing also the original problem. We show in an elementary way that \( 3 \cdot 2^{k-1} - 1 \) is an upper bound for \( n \), i.e.

**Theorem I.** The two-term sums formed of \( 3 \cdot 2^{k-1} \) positive integers cannot all be composed of \( k \) given prime numbers.

From this we deduce as a corollary

**Theorem II.**

\[
\pi(n) > \log_2 \left( \frac{n}{3} \right)
\]

where \( \pi(n) \) denotes the number of primes \( < n \).

The bound given in theorem I is probably not exact. The order of the maximum \( n(k) \) of \( n \) belonging to a given number \( k \) of primes is probably

\[ n(k) = O(k^{1+\epsilon}) \text{ for any } \epsilon > 0 \]

but actually we cannot prove this relation.

In the same way we may treat the analogous problem:

Is it possible to find two infinite sets of positive integers

\[
a_1 < a_2 < \cdots \\
b_1 < b_2 < \cdots
\]

so that every sum \( a_i + b_j \) shall be composed of the given primes \( p_1, p_2, \cdots, p_k \)?

The answer is negative. The proof will show even more. We shall prove

**Theorem III.** The sums \((a_i + b_j)\) formed of the two sets

\[
a_1 < a_2 < \cdots < a_{k+1} \\
b_1 < b_2 < \cdots < b_r
\]

cannot be composed of only \( k \) primes if one of the \( b \)'s is greater than \( a_k + 1 \). (This surely occurs if \( r > a_k + 1 \).)

1. Before proving theorem I we shall prove the following

**Lemma:** Let \( a_1 < a_2 < \cdots < a_n \) be a set of positive integers and \( p > 2 \) a prime number. It is always possible to select out of this set at least \( \left\lfloor \frac{n}{2} \right\rfloor = N \) integers \( a_{i_1}, a_{i_2}, \cdots, a_{i_N} \) with the following property: if \( a_{i_\nu} \) is divisible exactly by \( p^{r_\nu} \), \( a_{i_\mu} \) by \( p^{r_\mu} \) and \( a_{i_\nu} + a_{i_\mu} \) by \( p^{r_\nu + r_\mu} \), then

\[ f(x) = O(g(x)) \text{ means that there exists a } B \text{ and an } A \text{ such that for all } x \geq B \text{ it is true that } |f(x)| < A g(x); \text{ see Landau, *Primzahlen*, vol. 1, p. 31.} \]

\[ \{x\} \text{ denotes the smallest integer } \geq x. \]
\[ \beta_{n'} = \min (\alpha_n, \alpha_s), \]

where \( \min (\alpha_n, \alpha_s) \) means the smaller of \( \alpha_n \) and \( \alpha_s \).

We divide every member of the set \( a_1, a_2, \cdots, a_n \) by the highest possible power of \( p \); thus we obtain the integers \( a'_1, a'_2, \cdots, a'_n \) (some of them being possibly equal). No member of this new set is divisible by \( p \). We divide the members of this set into two classes according as their smallest positive residue, \( \text{mod} \, p \), is less than or greater than \( p/2 \). At least one of these two classes must contain \( N \) of the \( a'_i \). We retain only these; it is clear that the two-term sums formed of these are not divisible by \( p \). The integers \( a \) corresponding to these \( a'_i \) satisfy the requirement of our lemma. (The lemma is trivial except when some of the \( a \)'s are divisible by the same power of \( p \).)

3. We can now prove theorem I. Let \( n = 3 \cdot 2^{k-1} \) and \( a_1, a_2, \ldots, a_n \) be any positive integers. Suppose that all two-term sums of these are composed of \( k \) primes \( p_1 = 2, p_2, \ldots, p_k \); we shall prove that this supposition leads to a contradiction.

We apply our lemma with \( p = p_k \); we obtain then \( 3 \cdot 2^{k-2} \) integers \( a_r \) with the property in the lemma. Repeat the same process with \( p = p_{k-1} \) upon this system of \( 3 \cdot 2^{k-2} \) integers and so on. Finally we obtain three numbers \( a_1, a_2, a_3 \) of the same property with respect to the primes \( p_2, p_3, \ldots, p_k \). Let

\[
\begin{align*}
(1) \quad a_1 + a_2 &= 2^{\beta_1} p_2 \cdots p_k \\
(2) \quad a_1 + a_3 &= 2^{\beta_1} p_2 \cdots p_k \\
(3) \quad a_2 + a_3 &= 2^{\beta_2} p_2 \cdots p_k
\end{align*}
\]

then \( a_1 \) and \( a_2 \) are divisible by \( p_2^e, \ldots, p_k^e \); therefore \( a_1 \) and \( a_2 \) cannot be divided by \( 2^{a_1} \). Hence by (1) \( a_1 \) and \( a_2 \) must contain the same power of 2. This evidently holds for \( a_1 \) and \( a_3 \) also. Let us denote this common exponent by \( \gamma \). Then dividing (1), (2) and (3) by \( 2^\gamma \) and denoting \( a_i/2^\gamma \) by \( b_i \) we have

\[
\begin{align*}
(4) \quad b_1 + b_2 &= 2^{\beta_1} p_2 \cdots p_k \\
(5) \quad b_1 + b_3 &= 2^{\beta_1} p_2 \cdots p_k \\
(6) \quad b_2 + b_3 &= 2^{\beta_2} p_2 \cdots p_k
\end{align*}
\]

Here \( b_1, b_2 \) and \( b_3 \) are odd and each member of the left side of (4), (5) and (6) is divisible by the odd prime-powers on the respective right side. Dividing (4) by \( p_1^{\delta_1}, \ldots, p_k^{\delta_k} \) we get a number >2, for the members on the left side are different odd numbers. By this \( \delta \geq 2 \) and by analogous reasoning \( \epsilon \geq 2 \) and \( \theta \geq 2 \). Thus from (4), (5) and (6) it follows that the two-term sums formed of three different odd numbers are all divisible by 4, which is impossible.

4. In order to obtain the inequality of theorem II, let \( a_n = v \) for \( v = 1, 2, \ldots, \lfloor n/2 \rfloor \). Then the prime divisors of the sums \( a_i + a_j \) are the primes \( \leq n \). Hence by theorem I, \( n/2 < 3 \cdot 2^{v(n-1)} \), from which we immediately obtain the inequality stated in the introduction.
5. Finally we will prove our theorem III. Let
\[a_1 < a_2 < \cdots < a_{k+1},\]
\[b_1 < b_2 < \cdots < b_k,\]
be given integers, \(b_l > a_{k+1}\) and suppose that the sums \(a_i + b_l\) are all composed of \(k\) prime factors \(p_1, p_2, \ldots, p_k\). Let us consider the sums
\[a_1 + b_l, a_2 + b_l, \ldots, a_{k+1} + b_l.\]
We next show that one of these \(a_l + b_l\) contains a power of one of the given primes, say \(p^{\alpha_l}_{l_1}\), so that
\[p^{\alpha_l}_{l_1} > a_{k+1} \quad (l = 1, 2, \ldots, k + 1).\]
This we deduce from the fact that \(a_l + b_l > a_{k+1}\) and that \((a_l + b_l)\) can have only \(k\) different prime factors. We call this prime \(p_{l_1}\) (or if there are several, any one of them) "the prime belonging to \(a_l\)." We assert that the primes belonging to different \(a_l\) are different. For if the same \(p\) should belong to \(a_{l_1}\) and \(a_{l_2}\), then \((a_{l_1} - a_{l_2})\) would be divisible by \(p^m\), where \(m\) is the smaller of \(\alpha_{l_1}\) and \(\alpha_{l_2}\); but according to what has been said before, \(p^m > a_{k+1}\), whereas both of the numbers \(a_{l_1}\) and \(a_{l_2}\) are positive and <\(a_{k+1}\). Since the same prime can not belong to two integers, it is impossible that \(k\) primes shall belong to \((k + 1)\) integers. Hence the supposition that all the sums \(a_l + b_l\) are composed of the \(k\) primes must be false.