NOTE ON SOME ADDITIVE PROPERTIES
OF INTEGERS

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1. It is well known that for suitable n's both the equations \( n = x^2 - y^2 \)
and \( n = x^2 + y^2 \) have more than \( n^{\log \log n} \) solutions. I show that for suitable
n's the number of solutions of the equations \( n = p^2 - q^2 \) resp. \( n = p^2 + q^2 \)
\((p, q \text{ primes})\) is greater than \( n^{\log \log n} \).

I sketch the proof for \( n = p^2 - q^2 \).

Let \( A = 2 \cdot 3 \cdots p_r \), the product of consecutive primes, be sufficiently
large. By elementary method we prove that the number of solutions of the
congruence \( p^2 - q^2 \equiv 0 \pmod A \) with \( 0 < q < p < A \) is greater than
\( A^{1 + 1/\log \log A} \). But the integers of the form \( p^2 - q^2 \) with \( 0 < q < p < A \) lie
all between 0 and \( A^2 \), hence there exists a multiple of A say \( n(< A^2) \)
such that the number of solutions of the equation \( n = p^2 - q^2 \) is greater than
\( A^{1/\log \log A} > n^{1/\log \log n} \).

The proof for \( n = p^2 + q^2 \) is much more complicated but also elementary.
It requires Brun's method.

2. Schnirelmann proved that there exists a constant \( c_8 \) such that
every integer is the sum of \( c_8 \) or less primes. Some time ago Heilbronn-
Landau-Scherk proved that \( c_8 \leq 71 \). By Brun's method I proved that there
exists a constant \( c_4 \) such that any integer is the sum of \( c_4 \) or less positive
and negative squares of primes. The same result holds for any powers
of primes. It can be proved also that the density of integers of the form
\( p^2 + q^2 - r^2 - s^2 \) is positive.

3. Now I sketch some new results of N. P. Romanoff (Tomsk).

Let us denote by \( f(x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_l) \) the number of
integers not exceeding \( n \) belonging to the sequence \( x_i, y_j \) and \( x_i + y_i \).
It is an old and most important problem of the additive theory of numbers
to determine the value of \( f \) for given \( x_i \) and \( y_j \). But this can be solved
only for special sequences. Romanoff deduced 4 formulas for the mean
value of \( f \) for general sequences of integers.
First mean-value-theorem:

\[ \sum_{1 \leq x_1 < x_2 < \ldots < x_{k_1} \leq n} f(x_1, x_2, \ldots, x_{k_1}; y_1, y_2, \ldots, y_{k_2} | n) = n C_{k_1}^n - \sum_{z=1}^{n-k_2} C_{k_2}^{n-1-y_z+z} \]

where \( C_k^n = \binom{n}{k} \) and \( y_z \) denotes the complementary sequence of \( y_z \).

Second mean-value-theorem:

\[ \sum_{x_1=1}^{n} \sum_{x_{k_1}=1}^{n} \cdots \sum_{x_{k_1}=1}^{n} f(x_1, x_2, \ldots, x_{k_1}; y_1, y_2, \ldots, y_{k_2} | n) = n^{k+1} - \sum_{z=1}^{n-k_2} (n - 1 - y_z + z)^{k_1}. \]

Third mean-value-theorem:

\[ \sum_{1 \leq x_1 < x_2 < \ldots < x_{k_1} \leq n} f(x_1, x_2, \ldots, x_{k_1}; y_1, y_2, \ldots, y_{k_2} | n) = n C_{k_1}^n C_{k_2}^{n} - C_{k_1+1}^n C_{k_2}^{n} + C_{k_1+1}^n C_{k_2}^{n-1} + C_{k_1}^n C_{k_2}^{n-k_1+1} \]

Fourth mean-value-theorem:

\[ \sum_{1 \leq x_1 < x_2 < \ldots < x_{k_1} \leq n} f(x_1, x_2, \ldots, x_{k_1}; y_1, y_2, \ldots, y_{k_2} | n) = n C_k^n - 2 C_{k+2}^n C_{k+1}^n + 2^{k+3} C_{k+2}^\left[\frac{n}{2}\right] + 2^{k+1}(1 + 2 \varepsilon_n) C_{k+1}^\left[\frac{n}{2}\right] \]

with \( \varepsilon_n = 0 \) if \( n \) even, and \( \varepsilon_n = 1 \) if \( n \) odd.

The proof depends on elementary combinatoric methods.