On sequences of positive integers.

By

H. Davenport (Cambridge) and P. Erdős (Manchester).

I. Let \( a_1, a_2, \ldots \) be any sequence of (different) positive integers, and let \( b_1, b_2, \ldots \) be the sequence consisting of all positive integers which are divisible by at least one \( a \). We define

\[
A_1 = \frac{1}{a_1},
\]

\[
A_2 = \frac{1}{a_2} - \frac{1}{[a_1, a_2]},
\]

\[
\vdots
\]

\[
A_n = \frac{1}{a_n} - \sum_{i=1}^{n-1} \frac{1}{[a_i, a_n]} + \sum_{i=1}^{n-1} \frac{1}{[a_i, a_n, a_i]} - \ldots,
\]

where \([a, b, c, \ldots]\) denotes the least common multiple of \( a, b, c, \ldots \). Then \( A_n \) is easily seen to be the density of those integers which are divisible by \( a \), but not by any one of \( a_1, \ldots, a_{n-1} \). Hence \( A_n \geq 0 \), and \( \sum_{i=1}^{m} A_i \), being the density of those integers which are divisible by at least one of \( a_1, \ldots, a_m \), is less than 1. If we define

\[
A = \sum_{i=1}^{\infty} A_i,
\]

then \( 0 < A \leq 1 \), and it is reasonable to expect that \( A \) is the density
in some sense of the sequence \( \{b_i\} \). It was proved by Besicovitch\(^1\) that the sequence \( \{b_i\} \) may have different upper and lower densities. We shall prove (§2) that the “logarithmic density” of \( \{b_i\} \) exists and has the value \( A \), and also that the lower density of \( \{b_i\} \) has the value \( A \).

In § 3 we use the former of these results to prove that if a sequence \( a_1, a_2, \ldots \) of positive integers has the property

\[
\lim_{x \to \infty} \frac{1}{(\log x)^{s-1}} \sum_{a_m \leq x} a_m^{-1} > 0,
\]

then it has a subsequence \( a_{i_1}, a_{i_2}, \ldots \) in which \( a_{i_k} \mid a_{i_{k+1}} (k = 1, 2, \ldots) \). Naturally every sequence of positive lower density satisfies the condition.

2. Let \( \theta(n) \) be 1 if \( n \) is a \( b_i \) (i.e. if there is an \( a_j \mid n \)) and 0 otherwise. Let

\[
F(s) = \sum_{n=1}^{\infty} \theta(n) n^{-s} \quad (s > 1).
\]

Let

\[
A_n(s) = \frac{1}{a_s} - \sum_{p < q} \frac{1}{[a_p, a_q]^{s}} + \sum_{i < p < q} \frac{1}{[a_i, a_p, a_q]^{s}} - \ldots,
\]

so that \( A_n(1) = A_n \), and

\[
A(s) = \sum_{n=1}^{\infty} A_n(s).
\]

Then it is easily seen that

\[
F(s) = \zeta(s) A(s)
\]

for \( s > 1 \).

**Lemma 1:** If \( 1 < s_1 < s_2 \), then for any \( m \),

\[
\sum_{i=1}^{m} A_{i}(s_2) \leq \sum_{i=1}^{m} A_{i}(s_1).
\]

**Proof:** Let \( \theta_m(n) \) be 1 if \( n \) is divisible by any one of \( a_1, \ldots, a_m \) and 0 otherwise, and let \( F_m(s) = \sum_{n=1}^{\infty} \theta_m(n) n^{-s} \). As before

\[
F_m(s) = \zeta(s) \sum_{n=1}^{m} A_{i}(s).
\]

We have the inequality

\(^1\) Math. Annalen 110 (1934), 336 — 341.
On sequences of positive integers.

149

for all \( n \). For if \( \theta_m(n) = 0 \) then \( \theta_m(d) = 0 \) for all \( d \mid n \), and if \( \theta_m(n) = 1 \) then

\[
\log n = \sum_{d \mid n} \Lambda \left( \frac{n}{d} \right) = \sum_{d \mid n} \theta_m(d) \Lambda \left( \frac{n}{d} \right).
\]

From (1):

\[
\sum_{n=1}^{\infty} \theta_m(n) \log n n^{-s} \geq \left( \sum_{n=1}^{\infty} \theta_m(n) n^{-s} \right) \left( \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \right)
\]

for \( s > 1 \), i.e.

\[
-F_m'(s) \geq F_m(s) \left( -\frac{\zeta'(s)}{\zeta(s)} \right),
\]

hence

\[
\frac{d}{ds} \left( \sum_{i=1}^{m} A_i(s) \right) \leq 0
\]

for \( s > 1 \), which proves the Lemma.

**Lemma 2:** \( A(s) \to A \) as \( s \to 1 \) \( (s > 1) \).

**Proof:** By Lemma 1, we have for \( s > 1 \) and any \( m \),

\[
\sum_{i=1}^{m} A_i(s) \leq \lim_{s \to 1} \sum_{i=1}^{m} A_i(s) = \sum_{i=1}^{m} A_i \leq A,
\]

hence \( A(s) \leq A \). But

\[
\lim_{s \to 1} A(s) \geq \lim_{s \to 1} \sum_{i=1}^{m} A_i(s) = \sum_{i=1}^{m} A_i,
\]

and so

\[
\lim_{s \to 1} A(s) \geq A,
\]

which proves the Lemma.

**Theorem 1:** (a) \( \lim_{x=\infty} \sum_{n=1}^{x} \theta(n) n^{-1} \) exists and has the value \( A \).

(b) \( \lim_{x=\infty} x^{-1} \sum_{n=1}^{x} \theta(n) = A \).

**Proof:** By lemma 2,

\[
F(s) = \sum_{i=1}^{\infty} \theta(n) n^{-s} \sim \frac{A}{s - 1}
\]
as $s \to 1$ ($s > 1$). Part (a) of the Theorem follows from this by a Tauberian theorem due to Hardy and Littlewood. 

As regards (b), it is obvious from the meaning of $\sum_{i} A_{i}$ as a density that the lower limit in (b) is $\geq A$, and if equality did not hold we should have

$$s_{n} = \sum_{i=1}^{n} \delta (l) > (A + \delta) n$$

for some $\delta > 0$ and all $n \geq N$, and so

$$F (s) = \sum_{n=1}^{\infty} s_{n} \left( n^{-s} - (n+1)^{-s} \right) > (A + \delta) \sum_{N=1}^{\infty} n^{-s},$$

which on making $s \to 1$ contradicts (2).

3. Theorem 2: If $a_{1}, a_{2}, \ldots$ is a sequence of (different) positive integers, and

$$x = \lim_{x \to \infty} (\log x)^{-1} \sum_{a_{n} \leq x} a_{n}^{-1} > 0,$$

then there exists a subsequence $a_{i_{1}}, a_{i_{2}}, \ldots$ such that $a_{i_{k}} | a_{i_{k+1}}$ ($k = 1, 2, \ldots$).

Proof: It suffices to prove that there exists an $a_{i}$ such that

$$\lim_{x \to \infty} (\log x)^{-1} \sum_{a_{n} \leq x} a_{n}^{-1} > 0.$$  

We take $r$ so large that

$$\sum_{i \leq r} A_{i} < x,$$

and we shall prove that there exists an $a_{i}$ with $i \leq r$ satisfying (3). If the left side of (3) were zero for $i \leq r$, we should have

$$x = \lim_{x \to \infty} (\log x)^{-1} \sum_{a_{n} \leq x} a_{n}^{-1} \quad \text{if} \quad a_{i} + a_{n}, \ldots, a_{r} + a_{n}$$

On sequences of positive integers. 151

\[ \lim_{x \to \infty} (\log x)^{-1} \sum_{n=1}^{x} \theta(n) n^{-1}. \]

By Theorem 1 (a) the last expression has the value

\[ A - \sum_{r=1}^{r} A_{r}. \]

From (4) we have a contradiction.

The condition in Theorem 2 is easily seen to be best possible of its kind, i.e. one can construct sequences \( \{a_{i}\} \) for which

\[ (\log x)^{-1} \sum_{a_{n} \leq x} a_{n}^{-1} \]

tends to zero arbitrarily slowly, but in which no subsequence with the desired property exists.

(Received 10 January, 1936.)