Hardy and Ramanujan* proved that \( v(m) \) is almost always \( \log \log n \), i.e. that for any positive \( \epsilon \) there are only \( o(n) \) integers \( m \leq n \) for which either \( v(m) > (1+\epsilon) \log \log n \) or \( v(m) < (1-\epsilon) \log \log n \).

We use the following notation:

1. \( T \) denotes the closed interval \([\log n, n^{\log \log n}]\),
2. \( v'(m) \) the number of different prime factors of \( m \) in \( T \),
3. \( q_1, q_2, \ldots, q_v \) symbols for the \( v \) primes \( q \) of \( T \),
4. \( a_1, a_2, \ldots \) the integers composed of \( q_i \),
5. \( a_1^{(k)}, a_2^{(k)}, \ldots \) the integers whose factors are powers of \( k \) different \( q_i \) \( (k < 2 \log \log n) \),
6. \( A(m) \) the greatest \( a_i \) contained in \( m \),
7. \( U_k \) the number of integers \( m \leq n \) for which \( A(m) \) is an \( a^{(k)} \),
8. \( c_1, c_2, \ldots \) absolute constants,
9. \( x = \sum_{q} \frac{1}{q} \); from the formula \( \sum_{p < y} \frac{1}{p} = \log \log y + c_1 + o(1) \), it immediately follows that \( x = \log \log n - 4 \log \log \log n - \log 6 + o(1) \).

We require four lemmas.

**Lemma 1.** The number of integers \( m \leq n \) for which

\[
v(m) - v'(m) > (\log \log n)^2
\]

is \( o(n) \).

We evidently have

\[
\sum_{m=1}^{n} \left( v(m) - v'(m) \right) = \sum_{p < n} \left[ \frac{n}{p} \right] - \sum_{q} \left[ \frac{n}{q} \right]
\]

\[
= \sum_{p < \log n} \left[ \frac{n}{p} \right] + \sum_{n^{\log \log n} < p < n} \left[ \frac{n}{p} \right]
\]

\[
= O(n \log \log \log n),
\]

which implies Lemma 1.

**Lemma 2.**

\[
\frac{x^k}{k!} - o\left(\frac{1}{(\log n)^2}\right) < \sum_{i} \frac{1}{a_i^{(k)}} < \frac{x^k}{k!},
\]

where the dash in the summation means that the summation is extended over the square-free $d^{(k)}$'s only.

We have

$$\sum' \frac{1}{a_i^{(k)}} < \left( \sum \frac{1}{q} \right)^k \frac{k!}{k!} = \frac{x^k}{k!}.$$  

By expanding $\left( \sum \frac{1}{q} \right)^k / k!$ by the multinomial theorem we see that the coefficient of the terms whose denominator is a square-free $d^{(k)}$ is 1, but the other terms contain in their denominator the square of a $q$, i.e. a square greater than $(\log n)^{12}$ and have coefficients less than 1. Finally, the denominators are all less than $n^{2/(\log \log n)^2}$, since $k < 2 \log \log n$. Thus

$$\sum' \frac{1}{a_i^{(k)}} > \left( \sum \frac{1}{q} \right)^k \frac{k!}{k!} - O \left( \frac{1}{(\log n)^{12}} \right),$$  

and hence

$$\sum' \frac{1}{a_i^{(k)}} > \frac{x^k}{k!} - o \left( \frac{1}{(\log n)^2} \right),$$

which establishes Lemma 2.

**Lemma 3.**

$$U_k = n e^{-x} \frac{x^k}{k!} + o \left( \frac{n}{(\log n)^2} \right).$$

First we evaluate the number of integers $m \leq n$ for which $A(m) = a_i^{(k)}$. The number of the $m \leq n$ divisible by the square of a $q$ is less than

$$\sum_{i=1}^{n} q^2 = O \left( \frac{n}{(\log n)^8} \right).$$  

If $m$ is not divisible by the square of a $q$, $A(m)$ is square-free, and the number of the $m$ for which $A(m) = a_i^{(k)}$ is equal to the number $z$ of integers

$$m \leq \frac{n}{a_i^{(k)}},$$

no one of which is divisible by a $q$. We calculate $z$ by Brun's method. We have

$$z = \left[ \frac{n}{a_i^{(k)}} \right] - \sum_{q} \left[ \frac{n}{q a_i^{(k)}} \right] + \sum_{q_1 < q_2} \left[ \frac{n}{q_1 q_2 a_i^{(k)}} \right] - \ldots$$

$$+ (-1)^r \sum_{q_1 < q_2 < \ldots < q_r} \left[ \frac{n}{q_1 q_2 \ldots q_r a_i^{(k)}} \right] + \ldots \quad (1)$$
We write
\[ s_r = \sum_{q_1 < q_2 < \ldots < q_r} \left( \frac{n}{q_1 q_2 \ldots q_r a_r^{(k)}} \right) \]
and
\[ s_r' = \sum_{q_1 < q_2 < \ldots < q_r} \frac{n}{q_1 q_2 \ldots q_r a_r^{(k)}} \]
so that we have
\[ z = \sum_{r=0}^{n} (-1)^r s_r. \]  
(1’)

Now, evidently,
\[ \sum_{r < 10 \log \log n} (-1)^r s_r - \sum_{r > 10 \log \log n} \leq z \leq \sum_{r < 10 \log \log n} (-1)^r s_r + \sum_{r > 10 \log \log n} s_r, \]  
(2)
but
\[
\sum_{r > 10 \log \log n} s_r \leq \sum_{r > 10 \log \log n} s_r' < \sum_{r > 10 \log \log n} \frac{\sum_{q=1}^{r} \frac{1}{q}}{r!} < \frac{n}{a_r^{(k)}} \sum_{r > 10 \log \log n} (\log \log n)^r \left( \frac{\log \log n + 1}{r!} \right) < \frac{2n}{a_r^{(k)}} 10^{10 \log \log n} \]
\[
< \frac{2n e^{10 \log \log n} (10 \log \log n + 1)}{a_r^{(k)}} \]
\[
< \frac{2n e^{10 \log \log n}}{a_r^{(k)}} \]
\[
< \frac{2n}{a_r^{(k)}} 2^{10 \log \log n} \]
(3)
since
\[ y! > \frac{y^y}{e^y}. \]

Hence, from (1’), on noting the right-hand inequalities in (2) and (3) and omitting the square brackets, we obtain
\[ z = \sum_{r < 10 \log \log n} (-1)^r s_r' + O \left( \left( 1 + \frac{1}{v} \right)^{10 \log \log n} \right) + O \left( \frac{n}{a_r^{(k)}} 2^{10 \log \log n} \right), \]
(4)
the \( v \) term arising from a possible error \( 1 + \frac{1}{v} + \left( \frac{n}{2} \right) + \ldots \) up to \( 10 \log \log n \) terms.

From (3), (4), and \( 1 + \frac{1}{v} < n^{(\log \log n)^{-1}} \), we obtain
\[ z = \sum_{r} (-1)^r s_r' + O \left( n^{10/(\log \log n)^3} \right) + O \left( \frac{n}{a_r^{(k)}} 2^{10 \log \log n} \right). \]
(5)

Now we have
\[ \sum_{r} (-1)^r s_r' = \frac{n}{a_r^{(k)}} \prod_{r} \left( 1 - \frac{1}{q} \right) = \frac{n}{a_r^{(k)}} e^{-\left( \frac{1}{q} \right) + O(1/q^2)} \]
\[ = \frac{n}{a_r^{(k)}} e^{-\frac{1}{q} + O(1/q^2)} \]
\[ = \frac{n}{a_r^{(k)}} e^{O(1/(\log n)^3)} \]
\[ = \frac{n}{a_r^{(k)}} e^{x} \left( 1 + O \left( \frac{1}{(\log n)^{y}} \right) \right). \]
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Thus
\[ z = \frac{n}{a_1^{(k)}} n^{-x} \left( 1 + O\left( \frac{1}{(\log n)^x} \right) \right) + O(n^{10/(\log \log n)^3}) + O\left( \frac{n}{a_1^{(k)}} \frac{1}{210 \log \log n} \right). \]  (6)

From (6) we easily obtain
\[ U_k = n e^{-x} \left( 1 + O\left( \frac{1}{(\log n)^x} \right) \right) \sum_{i} \frac{1}{a_i^{(k)}} + O(n^{20/(\log \log n)^3}) \]
\[ + O\left( \frac{n}{\log \log n} \sum_{i} \frac{1}{a_i^{(k)}} \right) + O\left( \frac{n}{(\log n)^4} \right), \]  (7)

since the number of the square-free \( a_i^{(k)} \leq n \) is less than
\[ (1+\epsilon)^k < n^{10/(\log \log n)^3}, \]

and finally, from Lemma 2 and from
\[ \sum_{i} \frac{1}{a_i^{(k)}} < \sum_{i<n} \frac{1}{i} = O(\log n), \]

we have
\[ U_k = n e^{-x} \frac{x^k}{k!} + o\left( \frac{n}{(\log n)^2} \right). \]  (8)

Thus Lemma 3 is proved.

**Lemma 4.** The number of integers \( m \leq n \) for which \( v'(m) \geq \log \log n \) is
\[ \frac{1}{2} n + o(n). \]

Evidently \( v'(m) = v[A(m)] \); thus we have only to consider the integers for which \( v[A(m)] > \log \log n \).

First we prove that the number of integers for which \( v[A(m)] > x \) is
\[ \frac{1}{2} n + o(n), \]

\[ \sum_{k \geq x} U_k = \frac{1}{2} n + o(n). \]

Since \( \sum_{r=1}^{n} d(r) = O(n \log n) \), the number of integers \( m \leq n \) for which \( v(m) > 2 \log \log n \) is
\[ O(n \log n / 2^{\log \log n}) = o(n), \]

so that we have to prove
\[ \sum_{k \geq x} U_k = \frac{1}{2} n + o(n), \]

i.e., by Lemma 3,
\[ n e^{-x} \sum_{k \geq x} \frac{x^k}{k!} = \frac{1}{2} n + o(n). \]  (9)

But it is known that*
\[ \sum_{k \geq x} \frac{x^k}{k!} = \frac{1}{2} e^x + o(e^x) \]  (10)

* Srinivasa Ramanujan, *Collected papers*, 323, Question 294.
and
\[ \sum_{k > \frac{2 \log \log n}{2 \log \log \log n}} \frac{x^k}{k!} < \frac{2^{2 \log \log n} e^{2 \log \log n + 1}}{2^{2 \log \log n} (\log \log n)^{2 \log \log n}} = o(e^x), \tag{11} \]
and (9) is an immediate consequence of (10) and (11). We now have to prove that there are only \( o(n) \) integers \( m \leq n \) for which
\[ x \leq \nu'(m) \leq \log \log n. \]

From Lemma 3 we see that, since \( x^k/k! \) assumes its maximum value for \( k = \lceil x \rceil \), the number of integers \( m \leq n \) for which \( \nu'(m) = k \), is, by Stirling's formula, at the utmost
\[ n e^{-x} \frac{x^\lceil x \rceil}{\lceil x \rceil!} + o \left( \frac{n}{\log^2 n} \right) < c_n x. \tag{12} \]
Hence the number of integers \( m \leq n \) for which \( x < \nu'(m) \leq \log \log n \) is
\[ O \left( \frac{n}{\log \log n} \right) = O \left( \frac{n \log \log \log n}{(\log \log n)^4} \right) = o(n), \]
which completes the proof of Lemma 4.

We now proceed to prove our main theorem.

By Lemma 4, we have only to prove that the number of integers \( m \leq n \) for which \( \nu'(m) \leq \log \log n \) but \( \nu(m) > \log \log n \) is \( o(n) \).

We divide these integers into two classes.
In the first class are the integers for which
\[ \nu'(m) < \log \log n - (\log \log \log n)^2. \]
For these, \( \nu(m) - \nu'(m) > (\log \log n)^2 \), and so, from Lemma 1, the number of them is \( o(n) \).
For the integers of the second class
\[ \log \log n - (\log \log n)^2 \leq \nu'(m) \leq \log \log n. \]
From (12), it follows that the number of them is less than
\[ \frac{c_n n}{\sqrt{x}} (\log \log n^2 + 1) = O \left( \frac{n (\log \log n)^2}{(\log \log n)^4} \right) = o(n). \]
Thus our theorem is established.

In consequence of the exceedingly slow increase of \( \log \log n \) we can easily deduce from our theorem that the number of integers \( m \leq n \) for which \( \nu(m) > \log \log m \) is also \( \frac{1}{2} n + o(n) \).
Let $f(m)$ be the number of prime factors of $m$, multiple factors being counted multiply. We easily deduce that for every $\epsilon$ there exists a $c_\epsilon$ such that the number of integers $m \leq n$ for which $f(m) - \nu(m) > c_\epsilon$ is less than $\epsilon n$, and from this it is clear that the number of integers $m \leq n$ for which

$$f(m) > \log \log n$$

is $\frac{1}{2}n + o(n)$.

By similar methods we can prove the following theorems:

**Theorem 1.** Let $\nu_1(m)$ and $\nu_2(m)$ denote the numbers of prime factors of $m$ of the forms $4k-1$ and $4k+3$ respectively. The number of integers $m \leq n$ for which $\nu_1(m) > \nu_2(m)$ is $\frac{1}{2}n + o(n)$. The same holds for $\nu_1(m) < \nu_2(m)$ and hence the number of integers $m \leq n$ for which $\nu_1(m) = \nu_2(m)$ is $o(n)$.

**Theorem 2.** Let $A_1(m)$ and $A_2(m)$ denote the product of all prime factors of $m$ of the forms $4k-1$ and $4k+3$ respectively, multiple factors being counted multiply. The number of integers $m \leq n$, for which $A_1(m) > A_2(m)$ is $\frac{1}{2}n + o(n)$.

**Theorem 3.** The number of integers $m \leq n$, the greatest prime factor of which is a prime of the form $4k+1$, is $\frac{1}{2}n + o(n)$.

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