NOTE ON THE TRANSFINITE DIAMETER

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I.

We begin with some definitions. A function \( h(X) \) is said to be a measuring function‡ if it has the following properties:—

(i) \( h(X) \) is defined and is continuous in some range \( 0 \leq X \leq h_1 \), and positive in \( 0 < X < h \);

(ii) \( h''(X) \) exists and is less than or equal to 0 in \( 0 < X \leq h \);

(iii) \( h(0) = 0 \).

Now let \( h \) be a measuring function and \( E \) a linear set. Given any positive number \( \rho \), we denote by \( I(\rho, E) \) any set of intervals \( \{I_k\} \) such that

(i) every point of \( E \) is interior to at least one of the \( I_k \)'s, and

(ii) the length \( d_k \) of \( I_k \) is less than or equal to \( \rho \) \( (k = 1, 2, \ldots) \).

Then we write \( m^{(n)}_h \) \( E \) for the lower bound of \( \sum_{k} (\frac{d_k}{h}) \) for all possible sets \( I(\rho, E) \).

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Clearly, as $\rho$ decreases $m^\rho_h(E)$ cannot decrease and so

$$\lim_{\rho \to 0} m^\rho_h E = m^*_h E$$

exists.

It is easily verified that $m^*_h$ satisfies all the axioms of a Carathéodory measure function and so measurable, measurable sets, etc., can be introduced in the usual way. In this note we are especially interested in the function $h(x) = 1/\log(1/|x|)$ and, for this $h(x)$, we write $m_h E = \lambda E$.

Now suppose that there is some origin of coordinates on the line on which the set $E$ lies. We denote by $x$ indifferently a point and its distance from the origin, provided that no ambiguity can arise. Now we define

$$d_n(E) = \text{upper bound} \left\{ \prod_{i=1}^{n} |x_i - x_j|^{1/(n-1)} \right\}_{i \neq j}$$

Then it is known that $\lim d_n$ exists; it is called the transfinite diameter of $E$. We denote it by $\tau(E)$.

It is clear that, if $\bar{E}$ is the closure of $E$, $\tau(\bar{E}) = \tau(E)$. The following are the most important of the known relations between $\tau(E)$ and $m_h E$ for closed sets:

(A) If $m_h E > 0$, where $\int_0^1 \frac{h(t)}{t} \, dt$ is finite, then $\tau(E) > 0$.

(B) If $\lambda E = 0$, then $\tau(E) = 0$.

(C) It has been conjectured that if $\int_0^1 \frac{h(t)}{t} \, dt$ diverges and $m_h E$ is finite, not necessarily zero, then $\tau E = 0$. This result has not yet been proved generally, but Nevanlinna has proved it for the special case where $E$ is a "Cantor-set". In this note we prove it for a general closed set but

\[ \text{Cf. Hausdorff, op. cit.} \]
\[ \| R. Nevanlinna, "Über die Kapazität der Cantorschen Punktmengen", Monatshefte für Math. und Phys., 43 (1936), 435-447.} \]
\[ \| R. Nevanlinna, loc. cit. \]
only for the special function

\[ h(t) = \frac{1}{\log 1/t}. \]

It seems likely that our method can be extended to prove the complete conjecture, but we have not yet succeeded in effecting this.

The object of this note is to prove the following theorem:

**Theorem.** If \( E \) is a linear closed set such that \( \lambda(E) \) is finite, then \( \tau(E) \) is zero.

**II.**

**Lemma 1.** If \( \{p_i\} \) and \( \{q_i\} (i = 1, 2, \ldots, k) \) denote two sets of positive numbers such that

\[ \sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i = 1, \]

then\(^\dagger\)

\[ \sum_{p_i > q_i} p_i \geq \frac{1}{2}. \]

For

\[ 1 = \sum_{i=1}^{k} p_i = \sum_{p_i > q_i} p_i + \sum_{p_i < q_i} p_i \leq \sum_{p_i > q_i} p_i + \frac{1}{2} \sum_{i=1}^{k} q_i, \]

and so

\[ \sum_{p_i > q_i} p_i \geq \frac{1}{2}. \]

**Lemma 2.** If \( E \) denotes a set of \( \kappa \) non-overlapping intervals \( \{I_n\} \) in \( (0, 1) \), where \( a_n \) is the length of \( I_n \) \( (n = 1, 2, \ldots, \kappa) \), and

\[ \sum_{n=1}^{\kappa} \frac{1}{\log 1/a_n} \leq \mu, \]

then

\[ \tau(E) \leq e^{-1/4\mu}. \]

Take any \( n \) points \( x_1, x_2, \ldots, x_n \) in \( E \). Let \( n_r \) be the number of these points which lie in \( I_r \) \( (r = 1, 2, \ldots, \kappa) \).

Write

\[ \frac{n_r}{n} = \frac{1}{q_r}, \quad \frac{1}{q_r} = \log \frac{1}{a_r} \sum_{i=1}^{\kappa} \frac{1}{a_i}. \]

\(^\dagger\) Conditions written beneath the symbols \( \sum, \Pi, \&c. \), mean that the operations are taken over those terms for which the condition is satisfied.
The conditions of Lemma 1 are satisfied, and so we deduce that
\[ \sum_{n_i \log 1/a_i > n/2} n_i \geq \frac{n}{2}. \]

But
\[ \prod_{i \neq j} |x_i - x_j| \leq \prod_{i=1}^{n} a_i^{n(n-1)}, \tag{1} \]

This approximation is obtained by replacing \( |x_i - x_j| \) by \( a_i \) if \( x_i \) and \( x_j \) both belong to \( I \), and by 1 otherwise. We now majorize the right-hand side of (1) by omitting those values of \( i \) for which
\[ n_i \log \frac{1}{a_i} \leq \frac{n}{2}. \]

Then
\[ \prod_{i \neq j} |x_i - x_j| \leq \exp \left( -\frac{n}{2} \sum_{n_i \log 1/a_i > n/2} (n_i-1) \right) \leq e^{-n/2(\log n - 1)} \]

Since this is true for any set of \( n \) points, it follows that
\[ d_n(E) \leq e^{-n(n-2)/4n(n-1)}. \tag{2} \]

We let \( n \) tend to infinity in (2) and the lemma follows.

**Corollary.** If \( E \) is a closed set such that \( \lambda(E) = 0 \), then \( \tau(E) = 0 \).

For given \( \epsilon > 0 \) we can, by the definition of \( \lambda(E) \), enclose \( E \) in a set of intervals \( \{I_n\} \) of respective lengths \( \{a_n\} \) such that
\[ \sum \frac{1}{\log 1/a_n} < \epsilon. \tag{3} \]

By the Heine-Borel theorem we can find a finite subset of \( \{I_n\} \) which also covers \( E \) and for which (3) is a fortiori satisfied. Finally we can modify these intervals so that they do not overlap while the condition (3) is still satisfied. Let \( \vartheta \) denote this final set of intervals. By (3) and Lemma 2,
\[ \tau(\vartheta) \leq e^{-1/4\epsilon}. \tag{4} \]

But, since \( E \subseteq \vartheta \), it is obvious from the definition of \( \tau \) that
\[ \tau(E) \leq \tau(\vartheta). \tag{5} \]

Since \( \epsilon \) is arbitrary, we deduce \( \tau(E) = 0 \).

This corollary is the result quoted under (B) in §1 and first established by Lindeberg†. Lindeberg’s proof, however, depends on results in the theory of functions.

† See foot-note 4 above.
III.

It is clear that the proof of Lemma 2 actually proves the following slightly stronger result:

**Lemma 3.** Let $E$ be a finite set of non-overlapping intervals $\{I_r\} (r = 1, 2, \ldots, \kappa)$ such that

$$\sum_{r=1}^{\kappa} \frac{1}{\log 1/a_r} \leq \mu,$$

where $a_r$ is the length of $I_r$ ($1 \leq r \leq \kappa$).

Let $x_1, x_2, \ldots, x_n$ be any $n$ points in $E$.

Then

$$\prod_{i \neq j} |x_i - x_j| \leq e^{-n^2/(4+\epsilon)^2},$$

if $n > n_0 = n_0(\kappa, \epsilon)$, where $\Pi'$ indicates that the product is to be taken only over those values of $i, j$ for which $x_i, x_j$ both belong to the same $I_r$.

We proceed to the proof of the theorem stated at the end of § I. Let $E$ be a closed set in $(0, 1)$ such that $\lambda(E) = 1$ is finite and positive. Suppose that $\tau(E) = t > 0$. Given $\rho > 0$, we can, as above, find a finite set $\{I_{sr}\}$ of non-overlapping intervals such that

$$\sum_{r} \frac{1}{\log 1/a_r} < 2t,$$

where $a_r$ is the length of $I_r$ and $a_r \leq \rho$. Let $a = \min (a_r)$ and take

$$\rho_1 = a^{4N},$$

where

$$N = 64t \log 2/t.$$

Now cover $E$ with a finite set of non-overlapping intervals, each of length less than or equal to $\rho_1$, such that (7) is again satisfied. Clearly we can suppose that all these intervals lie inside the original intervals $\{I_{sr}\}$; we denote by $I_{rs}$ ($s = 1, 2, \ldots, p_r$) the set of the former which lie in $I_r$, and by $a_{rs}$ the length of $I_{rs}$.

Now take any $n$ points $x_1, x_2, \ldots, x_n$ of $E$, where $n$ is sufficiently large.

By (7) and Lemma 3,

$$\prod_{i \neq j} |x_i - x_j| \leq e^{-n^2/10},$$

where $\Pi_{1}$ is taken over those pairs $i, j$ for which $x_i, x_j$ belong to the same interval $I_{rs}$.

Consider $\Pi_2 |x_i - x_j|$ taken over those pairs $i, j$ for which $x_i, x_j$ lie in the same $I_r$ but not in the same $I_{rs}$. Clearly

$$\Pi_2 |x_i - x_j| \leq \Pi q_i^2 \sum_{i \neq j} n_i n_j,$$

where $n_{ip}$ is the number of points $x$ in $I_{ip}$. 
We replace $\prod_{i,j} |x_i - x_j|$ by $\prod_i |x_i - x_j| \prod_j |x_i - x_j|$, where $\prod_j$ is taken over those $x_i, x_j$ which do not lie in the same $I_{x_i}$ but lie in an $I_r$ for which
\[ a_r^{\alpha} < e^{-n/u}. \] (12)

Then it is clear that
\[ \prod_{i \neq j} |x_i - x_j| \leq \prod_i |x_i - x_j| \prod_j |x_i - x_j|. \] (13)

Suppose that, for $i = 1, 2, \ldots, i_0$,
\[ n_i < a_i^{\alpha N} (r = 1, 2, \ldots, p) \] (14)
and for each $i > i_0$ there exists $r_i (1 \leq r_i \leq p)$ such that
\[ n_i > a_i^{\alpha N} (\text{say}). \] (15)

There is clearly no loss of generality in this assumption since we can order the set $\{I_r\}$ as we please.

Then
\[ \left( \frac{t}{2} \right)^{\alpha N} \leq \prod_{i > i_0} a_i^{\alpha N i (n_i - 1)} \] (17)
\[ \leq \prod_{i > i_0} a_i^{\alpha N i n_i (n_i - 1)} \] by (8),
\[ \leq \prod_{i > i_0} a_i^{\alpha N i n_i (n_i - 1)} \] by (15),
\[ \leq e^{-n N/4} \sum_{i > i_0} a_i^{\alpha N i n_i (n_i - 2)}. \] (16)

Hence
\[ \sum_{i > i_0} (n_i - 2) < 4n \] \[ \frac{\log \frac{2}{i}}{N} < \frac{n}{16}, \] by (9).

Hence, if $n$ is sufficiently large compared with the number of intervals,
\[ \sum_{i > i_0} n_i < \frac{n}{15} \] (say). (17)

† We may obviously assume that $\left( \frac{t}{2} \right)^{\alpha N} \leq \prod_{i,j} |x_i - x_j|$, taken over all $i, j$. 
It follows that
\[ \sum_{i=1}^{n_1} n_i \geq \sum_{a_i \in_{\mathcal{E}}} n_i - \sum_{a_i \in_{\mathcal{E}}} n_i \]
\[ \geq \frac{n}{2} - \frac{n}{15} \quad \text{by (1) and (17)} \]
\[ = \frac{13n}{30}. \quad (18) \]

For \( i \leq i_0 \) we have
\[ 2 \sum n_i p_i q_i = n_i^2 - \sum n_i p_i q_i > n_i^2 - \frac{1}{2} n_i \sum n_i p_i = \frac{1}{2} n_i^2. \]

Hence
\[ \Pi_2 |x_i - x_j| \leq \Pi \sum_{a_i \in_{\mathcal{E}}} n_i \]
\[ \leq e^{-n/2} \sum_{i \leq i_0} n_i \]
\[ \leq e^{13n^2/200}. \quad (19) \]

by (18), if \( n \) is sufficiently large.

Hence
\[ \Pi |x_i - x_j| \leq e^{-n^2/100} - n^2/200 \quad (20) \]

by (10) and (19).

Now take a finer covering of \( E \), \( \{I_{rs}\} \), deduced from \( \{I_{rs}\} \) as the latter was deduced from \( \{I_i\} \). Then, by (20), we have
\[ \Pi_5 |x_i - x_j| \leq e^{3n^2/200}, \quad (21) \]

where \( \Pi_5 \) is taken over those pairs \( i, j \) for which \( x_i, x_j \) belong to the same interval \( I_{rs} \). This follows by arguing with the sets \( \{I_{rs}\} \) and \( \{I_{rs}\} \) as we argued above with \( \{I_i\} \) and \( \{I_{rs}\} \).

Also, from (19),
\[ \Pi_5 |x_i - x_j| \leq e^{-n^2/200}, \quad (22) \]

where \( \Pi_5 \) is taken over those pairs \( i, j \) for which \( x_i, x_j \) lie in the same \( I_r \) but not in the same \( I_{rs} \).

From (21) and (22),
\[ \Pi |x_i - x_j| \leq e^{-4n^2/200}. \quad (23) \]

This process can obviously be continued indefinitely and we deduce that, if \( k \) is any positive integer, then, for all sufficiently large \( n \), we have, for any set of points of \( E \), \( x_1, x_2, \ldots, x_n \),
\[ \Pi_{i \neq j} |x_i - x_j| \leq e^{-kn^2/200}. \]
Hence, for every $k$,

$$\tau(E) \leqslant e^{-k/20},$$  \hspace{1cm} (24)

and so

$$\tau(E) = 0.$$

This completes the theorem.

IV.

There are some remarks that seem relevant.

(a) We have stated the theorem for linear sets. It is obvious from the proof that the linearity of the sets is quite inessential and that the proof is valid for sets in Euclidean space of any number of dimensions. If the sets lie in $R_n$, then we have merely to replace intervals and their lengths in our proof by convex $n$-dimensional regions and their diameters respectively.

(b) The theorem, proved for closed sets in $(0, 1)$, is obviously true for all bounded closed sets.

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