NOTE ON THE TRANSFINITE DIAMETER

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[*Extracted from the Journal of the London Mathematical Society, Vol.* 12, 1937.]

I.

We begin with some definitions. A function $h(X)$ is said to be a *measuring function*‡ if it has the following properties:—

(i) $h(X)$ is defined and is continuous in some range $0 \leqslant X \leqslant h_1$, and positive in $0 < X \leqslant h_1$;

(ii) $h''(X)$ exists and is less than or equal to 0 in $0 < X \leqslant h_1$;

(iii) $h(0) = 0$.

Now let h be a measuring function and E a linear set. Given any positive number ρ, we denote by $I(\rho, E)$ any set of intervals $\{I_k\}$ such that

(i) every point of E is interior to at least one of the I_k's, and

(ii) the length d_k of I_k is less than or equal to ρ $(k = 1, 2, \ldots)$.

Then we write $m_h^{(\rho)} E$ for the lower bound of $\sum_k h(d_k)$ for all possible sets $I(\rho, E)$.

† Received 5 December, 1936; read 10 December, 1936.

‡ *Cf.* Hausdorff, "Dimension und aüseres Mass", *Math. Annalen*, 79 (1918), 157–179.

Clearly, as ρ decreases $m_h^{(\rho)}(E)$ cannot decrease and so

$$\lim_{\rho \to 0} m_h^{(\rho)} E = m_h{}^* E \quad \text{exists.}$$

It is easily verified† that $m_h{}^*$ satisfies all the axioms of a Carathéodory measure function and so *measurability*, *measurable sets*, etc., can be introduced in the usual way. In this note we are especially interested in the function $h(x) = 1/\log(1/x)$ and, for this $h(x)$, we write $m_h E = \lambda E$.

Now suppose that there is some origin of coordinates on the line on which the set E lies. We denote by x_n indifferently a point and its distance from the origin, provided that no ambiguity can arise. Now we define

$$d_n(E) = \underset{x_1, x_2, \ldots, x_n \in E}{\text{upper bound}} \left\{ \prod_{i \neq j} |x_i - x_j| \right\}^{1/n(n-1)}.$$

Then it is known that $\lim_{n \to \infty} d_n$ exists; it is called the *transfinite diameter*‡ of E. We denote it by $\tau(E)$.

It is clear that, if $\bar{E}$ is the closure of E, $\tau(\bar{E}) = \tau(E)$. The following are the most important of the known relations between $\tau(E)$ and $m_h E$ for closed sets§.

(A) If $m_h E > 0$, where $\int_0 \frac{h(t)}{t}\, dt$ is finite, then $\tau(E) > 0$.

(B) If $\lambda E = 0$, then $\tau(E) = 0$.

(C) It has been conjectured‖ that if $\int_0 \frac{h(t)}{t}\, dt$ diverges and $m_h E$ is finite, not necessarily zero, then $\tau E = 0$. This result has not yet been proved generally, but Nevanlinna¶ has proved it for the special case where E is a "Cantor-set". In this note we prove it for a general closed set but

† *Cf.* Hausdorff, *op. cit.*

‡ *Cf.* M. Fekete, "Über den transfiniten Durchmesser ebener Punktmengen", *Math. Zeitschrift*, 32 (1930), 108–114.

§ For (A) see P. J. Myrberg, "Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannsche Fläche", *Acta Math.*, 61 (1933), 39–79. For (B) see J. W. Lindeberg, "Sur l'existence de fonctions d'une variable complexe et de fonctions holomorphes bornées", *Annales Acad. Scient. Fennicae*, 11 (1918), Nr. 6; *cf.* also P. J. Myrberg, "Bemerkung zur Theorie des transfiniten Durchmessers einer ebenen Punktmenge", *Annales Acad. Scient. Fennicae*, 33 (1930), Nr. 7.

‖ R. Nevanlinna, "Über die Kapazität der Cantorschen Punktmengen", *Monatshefte für Math. und Phys.*, 43 (1936), 435–447..

¶ R. Nevanlinna, *loc. cit.*

only for the special function

$$h(t) = \frac{1}{\log 1/t}.$$

It seems likely that our method can be extended to prove the complete conjecture, but we have not yet succeeded in effecting this.

The object of this note is to prove the following theorem:

THEOREM. *If E is a linear closed set such that $\lambda(E)$ is finite, then $\tau(E)$ is zero.*

II.

LEMMA 1. *If $\{p_i\}$ and $\{q_i\}$ $(i = 1, 2, \ldots, k)$ denote two sets of positive numbers such that*

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i = 1,$$

then†

$$\sum_{p_i > \frac{1}{2}q_i} p_i \geqslant \tfrac{1}{2}.$$

For

$$1 = \sum_{i=1}^{k} p_i$$

$$= \sum_{p_i > \frac{1}{2}q_i} p_i + \sum_{p_i \leqslant \frac{1}{2}q_i} p_i$$

$$\leqslant \sum_{p_i > \frac{1}{2}q_i} p_i + \tfrac{1}{2}\sum_{i=1}^{k} q_i$$

and so

$$\sum_{p_i > \frac{1}{2}q_i} p_i \geqslant \tfrac{1}{2}.$$

LEMMA 2. *If E denotes a set of κ non-overlapping intervals $\{I_n\}$ in $(0, 1)$, where a_n is the length of I_n $(n = 1, 2, \ldots, \kappa)$, and*

$$\sum_{n=1}^{\kappa} \frac{1}{\log 1/a_n} \leqslant \mu,$$

then

$$\tau(E) \leqslant e^{-1/4\mu}.$$

Take any n points $x_1, x_2, \ldots, x_n$ in E. Let n_r be the number of these points which lie in I_r $(r = 1, 2, \ldots, \kappa)$.

Write

$$p_i = \frac{n_i}{n}, \quad \frac{1}{q_i} = \log\frac{1}{a_i} \sum_{i=1}^{\kappa} \frac{1}{\log 1/a_i}.$$

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† Conditions written beneath the symbols Σ, Π, &c., mean that the operations are taken over those terms for which the condition is satisfied.

The conditions of Lemma 1 are satisfied, and so we deduce that

$$\sum_{n_i \log 1/a_i > n/2\mu} n_i \geqslant \tfrac{1}{2}n.$$

But
$$\prod_{i \neq j} |x_i - x_j| \leqslant \prod_{i=1}^{\kappa} a_i^{n_i(n_i-1)}. \tag{1}$$

This approximation is obtained by replacing $|x_i - x_j|$ by a_r if x_i and x_j both belong to I_r and by 1 otherwise. We now majorize the right-hand side of (1) by omitting those values of i for which

$$n_i \log \frac{1}{a_i} \leqslant \frac{n}{2\mu}.$$

Then
$$\prod_{i \neq j} |x_i - x_j| \leqslant \exp\left(-\frac{n}{2\mu} \sum_{n_i \log 1/a_i > n/2\mu} (n_i - 1)\right)$$
$$\leqslant e^{-n/2\mu(\frac{1}{2}n - \kappa)}.$$

Since this is true for any set of n points, it follows that

$$d_n(E) \leqslant e^{-n(n-2\kappa)/4\mu n(n-1)}. \tag{2}$$

We let n tend to infinity in (2) and the lemma follows.

COROLLARY. *If E is a closed set such that $\lambda(E) = 0$, then $\tau(E) = 0$.*

For given $\epsilon > 0$ we can, by the definition of $\lambda(E)$, enclose E in a set of intervals $\{I_n\}$ of respective lengths $\{a_n\}$ such that

$$\sum \frac{1}{\log 1/a_n} < \epsilon. \tag{3}$$

By the Heine-Borel theorem we can find a finite subset of $\{I_n\}$ which also covers E and for which (3) is *a fortiori* satisfied. Finally we can modify these intervals so that they do not overlap while the condition (3) is still satisfied. Let ϑ denote this final set of intervals. By (3) and Lemma 2,

$$\tau(\vartheta) \leqslant e^{-1/4\epsilon}. \tag{4}$$

But, since $E \subset \vartheta$, it is obvious from the definition of τ that

$$\tau(E) \leqslant \tau(\vartheta). \tag{5}$$

Since ϵ is arbitrary, we deduce $\tau(E) = 0$.

This corollary is the result quoted under (B) in § I and first established by Lindeberg†. Lindeberg's proof, however, depends on results in the theory of functions.

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† See foot-note 4 above.

III.

It is clear that the proof of Lemma 2 actually proves the following slightly stronger result:

LEMMA 3. *Let E be a finite set of non-overlapping intervals $\{I_r\}$ $(r = 1, 2, \ldots, \kappa)$ such that*

$$\sum_{r=1}^{\kappa} \frac{1}{\log 1/a_i} \leqslant \mu,$$

where a_r is the length of I_r $(1 \leqslant r \leqslant \kappa)$.

Let $x_1, x_2, \ldots, x_n$ be any n points in E.

Then
$$\prod_{i \neq j}' |x_i - x_j| \leqslant e^{-n^2/(4+\epsilon)\mu} \tag{6}$$

if $n \geqslant n_0 = n_0(\kappa, \epsilon)$, where Π' indicates that the product is to be taken only over those values of i, j for which x_i, x_j both belong to the same I_r.

We proceed to the proof of the theorem stated at the end of § I. Let E be a closed set in $(0, 1)$ such that $\lambda(E) = l$ is finite and positive. Suppose that $\tau(E) = t > 0$. Given $\rho > 0$, we can, as above, find a finite set $\{I_n\}$ of non-overlapping intervals such that

$$\Sigma \frac{1}{\log 1/a_r} < 2l, \tag{7}$$

where a_r is the length of I_r and $a_r \leqslant \rho$. Let $\alpha = \min(a_i)$ and

take
$$\rho_1 = \alpha^{4N}, \tag{8}$$

where
$$N = 64l \log 2/t. \tag{9}$$

Now cover E with a finite set of non-overlapping intervals, each of length less than or equal to ρ_1, such that (7) is again satisfied. Clearly we can suppose that all these intervals lie inside the original intervals $\{I_r\}$; we denote by I_{rs} $(s = 1, 2, \ldots, p_r)$ the set of the former which lie in I_r, and by a_{rs} the length of I_{rs}.

Now take any n points $x_1, x_2, \ldots, x_n$ of E, where n is sufficiently large.

By (7) and Lemma 3,

$$\Pi_1 |x_i - x_j| \leqslant e^{-n^2/10l}, \tag{10}$$

where Π_1 is taken over those pairs i, j for which x_i, x_j belong to the same interval I_{rs}.

Consider $\Pi_2 |x_i - x_j|$ taken over those pairs i, j for which x_i, x_j lie in the same I_r but not in the same I_{rs}. Clearly

$$\Pi_2 |x_i - x_j| \leqslant \prod_i a_i^{2 \sum_{p \neq q} n_{ip} n_{iq}}, \tag{11}$$

where n_{ip} is the number of points x in I_{ip}.

We replace $\Pi_2|x_i-x_j|$ by $\Pi_3|x_i-x_j|$, where Π_3 is taken over those x_i, x_j which do not lie in the same I_{rs} but lie in an I_r for which

$$a_r^{n_r} < e^{-n/4l}. \tag{12}$$

Then it is clear that

$$\prod_{i\neq j} |x_i-x_j| \leqslant \Pi_1|x_i-x_j| \, \Pi_3|x_i-x_j|. \tag{13}$$

Suppose that, for $i=1, 2, \ldots, i_0$,

$$n_{ir} \leqslant \tfrac{1}{2} n_i \quad (r=1, 2, \ldots, p_i) \tag{14}$$

and for each $i>i_0$ there exists r_i $(1 \leqslant r_i \leqslant p_i)$ such that

$$n_{ir_i} > \tfrac{1}{2} n_i. \tag{15}$$

There is clearly no loss of generality in this assumption since we can order the set $\{I_r\}$ as we please.

Then†

$$\begin{aligned}\left(\frac{t}{2}\right)^{n^2} &\leqslant \prod_{\substack{i>i_0 \\ a_i^{n_i}<e^{-n/4l}}} a_{ir_i}^{n_{ir_i}(n_{ir_i}-1)} \\ &\leqslant \prod_{\substack{i>i_0 \\ a_i^{n_i}<e^{-n/4l}}} a_i^{4N\, n_{ir_i}(n_{ir_i}-1)} \quad \text{by (8)}, \\ &< \prod_{\substack{i>i_0 \\ a_i^{n_i}<e^{-n/4l}}} a_i^{4N\, \frac{1}{2}n_i(\frac{1}{2}n_i-1)} \quad \text{by (15)}, \\ &< e^{-nN/4l \sum_{\substack{i>i_0 \\ a_i^{n_i}<e^{-n/4l}}} (n_i-2)}. \end{aligned} \tag{16}$$

Hence

$$\begin{aligned}\sum_{\substack{i>i_0 \\ a_i^{n_i}<e^{-n/4l}}} (n_i-2) &< \frac{4ln}{N} \log \frac{2}{t} \\ &< \frac{n}{16}, \quad \text{by (9)}.\end{aligned}$$

Hence, if n is sufficiently large compared with the number of intervals,

$$\sum_{\substack{i>i_0 \\ a_i^{n_i}<e^{-n/4l}}} n_i < \frac{n}{15} \quad \text{(say)}. \tag{17}$$

† We may obviously assume that $\left(\frac{t}{2}\right)^{n^2} \leqslant \prod_{i\neq j} |x_i-x_j|$, taken over all i, j.

It follows that

$$\sum_{\substack{i=1 \\ a_i^{n_i}<e^{-n/4l}}}^{i_0} n_i \geqslant \sum_{a_i^{n_i}<e^{-n/4l}} n_i - \sum_{\substack{i>i_0 \\ a_i^{n_i}<e^{-n/4l}}} n_i$$

$$\geqslant \frac{n}{2} - \frac{n}{15} \quad \text{by (1) and (17)}$$

$$= \frac{13n}{30}. \tag{18}$$

For $i \leqslant i_0$ we have

$$2\Sigma\, n_{ip} n_{iq} = n_i^2 - \Sigma\, n_{ip}^2 > n_i^2 - \tfrac{1}{2} n_i \Sigma\, n_{ip} = \tfrac{1}{2} n_i^2.$$

Hence
$$\Pi_3 |x_i - x_j| \leqslant \prod_{\substack{i \leqslant i_0 \\ a_i^{n_i}<e^{-n/4l}}} a_i^{\frac{1}{2}n_i^2}$$

$$\leqslant e^{-n/8l \sum\limits_{\substack{i \leqslant i_0 \\ a_i^{n_i}<e^{-n/4l}}} ni}$$

$$\leqslant e^{13n^2/240l} \tag{19}$$

by (18), if n is sufficiently large.

Hence
$$\Pi |x_i - x_j| \leqslant e^{-(n^2/10l)-(n^2/20l)} \tag{20}$$

by (10) and (19).

Now take a finer covering of E, $\{I_{rst}\}$, deduced from $\{I_{rs}\}$ as the latter was deduced from $\{I_r\}$. Then, by (20), we have

$$\Pi_4 |x_i - x_j| \leqslant \bar{e}^{3n^2/20l}, \tag{21}$$

where Π_4 is taken over those pairs i, j for which x_i, x_j belong to the same interval I_{rs}. This follows by arguing with the sets $\{I_{rs}\}$ and $\{I_{rst}\}$ as we argued above with $\{I_r\}$ and $\{I_{rs}\}$.

Also, from (19),

$$\Pi_5 |x_i - x_j| \leqslant e^{-n^2/20l}, \tag{22}$$

where Π_5 is taken over those pairs i, j for which x_i, x_j lie in the same I_r but not in the same I_{rs}.,

From (21) and (22).

$$\Pi |x_i - x_j| \leqslant e^{-4n^2/20l}. \tag{23}$$

This process can obviously be continued indefinitely and we deduce that, if k is any positive integer, then, for all sufficiently large n, we have, for any set of points of E, $x_1, x_2, \ldots, x_n$,

$$\prod_{i \neq j} |x_i - x_j| \leqslant e^{-kn^2/20l}.$$

Hence, for every k,

$$\tau(E) \leqslant e^{-k/20l}, \tag{24}$$

and so

$$\tau(E) = 0.$$

This completes the theorem.

IV.

There are some remarks that seem relevant.

(*a*) We have stated the theorem for linear sets. It is obvious from the proof that the linearity of the sets is quite inessential and that the proof is valid for sets in Euclidean space of any number of dimensions. If the sets lie in R_n, then we have merely to replace intervals and their lengths in our proof by convex n-dimensional regions and their diameters respectively.

(*b*) The theorem, proved for closed sets in (0, 1), is obviously true for all bounded closed sets.

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Printed by C. F. Hodgson & Son, Ltd., Newton St., London, W.C.2.