ADDITIVE ARITHMETICAL FUNCTIONS AND STATISTICAL INDEPENDENCE.*

By Paul Erdös and Aurel Wintner.

Introduction. The object of this paper is to solve the problem[1] as to the existence of an asymptotic distribution function of an arbitrary (real) additive arithmetical function $f(n)$, and to embed this problem into the general theory of infinite convolutions.[2] This aim will be reached by an approach which subjects $f(n)$ to no a priori limitation whatever. Actually, the sufficient condition found *loc. cit.*[1], III for the existence of the asymptotic distribution function turns out to be a necessary condition as well.

The result to be obtained is to the effect that the statistical independence of the components which belong to the different prime numbers is such as to make the condition of convergence in relative measure not only sufficient but also necessary for the existence of an asymptotic distribution function.[3]

The crucial point to be proved is that, on the one hand, $f(n)$ cannot have an asymptotic distribution function distinct from the infinite convolution which is formally associated with $f(n)$; and that, on the other hand, the mere convergence of this formal convolution implies the existence of an asymptotic distribution function.

In particular, the theory of divergent infinite convolutions[4] will become applicable to every $f(n)$ which does not possess an asymptotic distribution function.

1. For a given function $g(n)$ defined for $n = 1, 2, \cdots$, let $M\{g(n)\}$ denote the limit of the ratio of $g(1) + g(2) + \cdots + g(n)$ and n, as $n \rightarrow \infty$, if this limit exists. If S is an increasing sequence of positive integers, let $S(n)$ denote its characteristic function, i. e., the function defined for $n = 1, 2, \cdots$ by placing $S(n) = 1$ or $S(n) = 0$ according as n is or is not

* Received March 23, 1939.

[1] Cf. P. Erdös, I, *Journal of the London Mathematical Society*, vol. 10 (1935), pp. 120-125; II, *ibid.*, vol. 12 (1937), pp. 7-11; III, *ibid.*, vol. 13 (1938), pp. 119-127. In III, further references are given. The earliest instance of the treatment of a non-trivial case, that of (the logarithm of) the ratio $\phi(n)/n$ belonging to Euler's ϕ-function, is due to I. Schur (cf. I. Schoenberg, *Mathematische Zeitschrift*, vol. 28 (1928), p. 194).

[2] Cf. B. Jessen and A. Wintner, *Transactions of the American Mathematical Society*, vol. 38 (1935), pp. 48-88.

[3] Cf. P. Hartman, E. R. van Kampen and A. Wintner, *American Journal of Mathematics*, vol. 61 (1939), pp. 476-486.

[4] Cf. E. R. van Kampen and A. Wintner, *American Journal of Mathematics*, vol. 59 (1937), pp. 635-654.

contained in the set S. If the average $M\{S(n)\}$ exists, it is called the density of S. Generally, let $\bar{M}\{S(n)\}$ denote the upper limit of the ratio of $S(1) + S(2) + \cdots + S(n)$ and n, as $n \rightarrow \infty$.

For a given real-valued function $f = f(n)$ and for any real number x, let S_x denote the set of those n at which $f(n) < x$. If there exists a monotone function $\sigma = \sigma(x)$, $-\infty < x < +\infty$, which satisfies the boundary conditions $\sigma(-\infty) = 0$, $\sigma(+\infty) = 1$ and is such that, at every continuity point x of σ, the limit $M\{S_x(n)\}$ exists and equals $\sigma(x)$, then f is said to have an asymptotic distribution function, σ.

2. For a given function $f(n)$ and for every positive integer k, define a function $f^{(k)}(n)$ by placing

$$f^{(k)}(n) = \begin{cases} 0, \text{ if } n \not\equiv 0 \pmod{p_k}, \\ f(p_k{}^l), \text{ if } p_k{}^l | n \text{ and } p_k{}^{l+1} \nmid n, \end{cases} \tag{1}$$

where p_k denotes the k-th prime number. Put

$$f_k(n) = \sum_{j=1}^{k} f^{(j)}(n). \tag{2}$$

It is clear that a function $f(n)$ is additive if and only if so is each of the functions $f_1(n), f_2(n), \cdots$ and one has

$$f(n) = \sum_{k=1}^{\infty} f^{(k)}(n), \tag{3}$$

(where $f(n) = f_k(n)$ for every $k > h_n$, if h_n is sufficiently large for a fixed n). In fact, the additivity of $f(n)$, i. e., the property that $f(n_1 n_2) = f(n_1) + f(n_2)$ whenever $(n_1, n_2) = 1$, is equivalent to the property that, for arbitrary positive integers k; $j_1, \cdots, j_k$,

$$f(\prod_{i=1}^{k} p_i{}^{j_i}) = \sum_{i=1}^{k} f(p_i{}^{j_i}), \text{ while } f(1) = 0. \tag{4}$$

In what follows, $f(n)$ will denote an arbitrarily given real-valued additive function.

3. Consider first the particular case of those functions $f(n)$ which are bounded on the sequence of all prime numbers $p_1, p_2, \cdots$,

$$|f(p)| < \text{const.} \tag{5}$$

Then $f(n)$ cannot have an asymptotic distribution function unless the series

$$\sum \frac{f(p)^2}{p} \text{ is convergent.} \tag{6}$$

In fact, suppose, if possible, that $\sum f(p)^2/p = \infty$. Then, if ω denotes any real number, a recent theorem [5] states that the density (§ 1) of the sequence of those natural numbers m which satisfy the inequality

$$f(m) < \sum_{p<m} \frac{f(p)}{p} + \omega \cdot \left\{ 2 \sum_{p<m} \frac{f(p)^2}{p} \right\}^{\frac{1}{2}}$$

exists and is represented by the normal distribution function $\pi^{-\frac{1}{2}} \int_{-\infty}^{\omega} \exp - t^2 dt$.

This clearly implies that if a denotes any positive number, the density of the sequence of those natural numbers m which satisfy the inequality $|f(m)| < a$ is 0. But then $f(n)$ does not have an asymptotic distribution function, since the boundary conditions of § 1 are violated. This completes the proof.

4. The assumption (5) also implies that $f(n)$ cannot have an asymptotic distribution function unless the partial sums of the series $\sum \frac{f(p)}{p}$ form a bounded sequence of numbers.

In fact, if this sequence were not bounded, it would follow from (6) by a straightforward application of the method of Turán,[6] that there exists for every $\epsilon > 0$ a sufficiently large $C = C_\epsilon > 0$ in such a way that the number of those positive integers m which satisfy the condition

$$-C + \sum_{p<m} \frac{f(p)}{p} < f(m) < C + \sum_{p<m} \frac{f(p)}{p}$$

exceeds $(1-\epsilon)m$ for every sufficiently large m. But then the definitions of § 1 imply that $f(n)$ cannot have an asymptotic distribution function. This contradiction completes the proof.

5. Next, it will be shown that if (5) is satisfied, $f(n)$ cannot have an asymptotic distribution function unless the series

$$\sum \frac{f(p)}{p} \text{ is convergent.} \tag{7}$$

The proof of (7) will be based on the estimate

$$\frac{1}{n} \sum_{m=1}^{n} f(m)^2 < \text{Const.}, \tag{8}$$

which will be proved in § 6. Suppose that (8) has already been established.

It is clear from (8) that for every $\epsilon > 0$ and for every n one has $\sum |f(j)| < \epsilon n$, if the summation index j runs through those positive integers j which satisfy both inequalities $j < n$, $|f(j)| \geqq \epsilon^{-1} \cdot \text{Const.}$ It follows, there-

[5] P. Erdös and M. Kac, *Proceedings of the National Academy*, vol. 25 (1939), pp. 206-207.

[6] P. Turán, *Journal of the London Mathematical Society*, vol. 11 (1936), pp. 125-133.

fore, from the definitions of § 1 that $f(n)$ cannot have as asymptotic distribution function unless the limit $M\{f(n)\}$ exists. Since $f(n)$ is supposed to have an asymptotic distribution function, and since $M\{f(n)\}$ is defined as the limit of the ratio of $f(1) + \cdots + f(n)$ and n, it follows that if

$$\sum_{m=1}^{n} f(m) - n \sum_{p \leqq n} \frac{f(p)}{p} = o(n)$$

is true, then so is (7). But this $o(n)$-estimate is certainly true in case the additive function $f(n)$ is such that $f(p^k) = f(p)$ holds for every prime p and for every $k \geqq 2$, since in this case

$$\sum_{m=1}^{n} f(m) = \sum_{p \leqq n} \left[\frac{n}{p}\right] f(p) = n \sum_{p \leqq n} \frac{f(p)}{p} + o(n)$$

in virtue of (5). Finally, it is known[7] that the case of any $f(n)$ satisfying (5) may be reduced to the case in which not only (5) holds, but also $f(p^k) = f(p)$.

This completes the proof of (7) on the assumption (8).

6. In order to prove (8), notice first that

$$\sum_{m=1}^{n} f(m)^2 = \sum\sum \left[\frac{n}{pq}\right] f(p)f(q) + \sum_{p \leqq n} \left[\frac{n}{p}\right] f(p)^2,$$

where the summation indices p, q of the double sum run through all pairs of distinct primes. Hence, $\frac{1}{n} \sum_{m=1}^{n} f(m)^2$ cannot exceed

$$\left(\sum_{p < n^{\frac{1}{2}}} \frac{f(p)}{p}\right)^2 + 2 \sum_{n^{\frac{1}{2}} \leqq p \leqq n} \frac{f(p)}{p} \sum_{qp \leqq n} \frac{f(q)}{q} + \frac{1}{n} \sum_{qp \leqq n} | f(p)f(q) | + \sum_{p \leqq n} \frac{f(p)^2}{p}.$$

Since the partial sums of the series $\sum \frac{f(p)}{p}$ are bounded (§ 4), it follows that

$$\frac{1}{n} \sum_{m=1}^{n} f(m)^2 = O(1) + O\left(\sum_{n^{\frac{1}{2}} \leqq p \leqq n} \frac{| f(p) |}{p} O(1)\right) + \frac{1}{n} \sum_{qp \leqq n} | f(p)f(q) | + \sum_{p \leqq n} \frac{f(p)^2}{p}.$$

Consequently, from (5) and (6),

$$\frac{1}{n} \sum_{m=1}^{n} f(m)^2 = O(1) + O\left(\sum_{n^{\frac{1}{2}} \leqq p \leqq n} \frac{1}{p}\right) + \frac{1}{n} O\left(\sum_{qp \leqq n} 1\right) + O(1).$$

This clearly implies (8). Thus, the proof of (7) is now complete.

7. For any $y = f(n)$, define $y^+ = f^+(n)$ by placing

(9) $$y^+ = y \text{ or } y^+ = 1 \text{ according as } | y | < 1 \text{ or } | y | \geqq 1.$$

[7] P. Erdös, *loc. cit.*[1], I, p. 124.

Then an $f(n)$ which satisfies (5) cannot have an asymptotic distribution function unless the series

$$\Sigma \frac{f^*(p)}{p} \text{ and } \Sigma \frac{f^*(p)^2}{p} \text{ are convergent.} \tag{10}$$

In fact, the convergence of the second of the series (10) is clear from (6) and (9). On the other hand, (7) and (9) show that the first of the series (10) is certainly convergent if $\Sigma' \mid f(p) \mid / p < \infty$, where the summation Σ' ranges over those primes p at which $\mid f(p) \mid \geqq 1$. Hence, it is sufficient to assure that $\Sigma' f(p)^2/p < \infty$. But this is clear from (6).

8. It will now be easy to prove the following theorem:

(i). *An additive function $f(n)$ has an asymptotic distribution function if and only if* (10) *is satisfied.*

That (10) is a sufficient condition for the existence of the asymptotic distribution function of $f(n)$, has been proved by Erdös, loc. cit.[1], III. On the other hand the necessity of (10) is established by § 7 for those $f(n)$ which satisfy (5). Thus, (i) will be proved by showing that the result of § 7 holds without the restriction (5) also.

Suppose, therefore, that $f(n)$ has an asymptotic distribution function, and that there does not exist a constant which satisfies (5) for all primes. It may be assumed that there does not exist a number K with the property that the sum of the reciprocal values of those primes p at which $\mid f(p) \mid > K$ is a convergent series. In fact, if this series is convergent for some sufficiently large K, then the truth of the statement (10) may be obtained by a repetition of the corresponding considerations, applied loc. cit.[7]

9. Consequently, it may be assumed that there exists an infinite sequence of primes, say $p^{(1)}, p^{(2)}, \cdots$, in such a way that, on the one hand, the sum of the reciprocal values of all these $p^{(j)}$ is a divergent series, while the sum of the squares of these reciprocal values is less than $\frac{1}{4}$, say; and, on the other hand, one has $\mid f(p^{(j)}) \mid \rightarrow \infty$, as $j \rightarrow \infty$. Now, the existence of such a sequence $p^{(1)}, p^{(2)}, \cdots$ is incompatible with the existence of an asymptotic distribution function for $f(n)$.

In fact, if $f(n)$ has an asymptotic distribution function, then there exists, by § 1, a sufficiently large $\lambda > 0$ which has the property that the density of the set of those integers n which satisfy $\mid f(n) \mid \leqq \lambda$ exists and does not exceed $\frac{3}{4}$, say. Thus, it is clear from the choice of the sequence $p^{(1)}, p^{(2)}, \cdots$, that the density of the set of those integers n which satisfy the condition $\mid f(n) \mid \leqq \lambda$ and are not divisible by the square of any of the primes $p^{(1)}, p^{(2)}, \cdots$, cannot be less than $\frac{1}{2}$. Hence, on choosing the value of λ

fixed, one sees that the sum of the reciprocal values of those integers m which satisfy both inequalities $|f(m)| \leqq \lambda$, $m \leqq n$ and are not divisible by the square of any of the primes $p^{(1)}, p^{(2)}, \cdots$, must exceed $\frac{1}{2} \log n$ for every sufficiently large n. Now, it is readily seen from the elementary considerations which were used in the proof [8] of the sufficiency of the condition (10), that one can find two integers m, say m_1 and m_2, which satisfy the condition $|f(m)| \leqq \lambda$ and are such that, on the one hand, $m_2 = m_1 d$ for a suitable integer d and, on the other hand, $p^{(l)} \mid d$ for a suitable element $p^{(l)}$ of the sequence $p^{(1)}, p^{(2)}, \cdots$.

Clearly, this implies a contradiction whenever the prime $p^{(l)}$ satisfies the condition $f(p^{(l)}) > 2\lambda$. And this condition may be satisfied by a suitable choice of $p^{(l)}$. In fact, the value of λ has been fixed, while $|f(p^{(j)})| \to \infty$, as $j \to \infty$.

This completes the proof of (i).

10. Let $f(n)$ be an arbitrary additive function, which need not have an asymptotic distribution function. On choosing any fixed $k \geqq 1$, and denoting by $u_1, \cdots, u_k$ real variables each of which varies continuously from $-\infty$ to $+\infty$, one readily verifies from the definitions (1)-(2), that limits

$$M\{\exp i \sum_{j=1}^{k} u_j f^{(j)}(n)\} \quad \text{and} \quad M\{\exp i u_j f^{(j)}(n)\}; \quad j = 1, \cdots, k,$$

where $M\{g(n)\} = \lim (g(1) + \cdots + g(n))/n$, exist uniformly in every fixed $(u_1, \cdots, u_k)$-sphere $u_1^2 + \cdots + u_k^2 < \text{const.}$, and that

$$M\{\exp i \sum_{j=1}^{k} u_j f^{(j)}(n)\} = \prod_{j=1}^{k} M\{\exp i u_j f^{(j)}(n)\}. \tag{11}$$

This means [9] that *the k functions $f^{(1)}(n), \cdots, f^{(k)}(n)$ are statistically independent*, and implies,[10] therefore, not only that each of the additive functions (1) and (2) possess an asymptotic distribution function, say $\sigma^{(k)}(x)$ and $\sigma_k(x)$, but also that

$$L(u; \sigma_k) = \prod_{j=1}^{k} L(u; \sigma^{(j)}) \text{ for } -\infty < u < +\infty, \tag{12}$$

$L(u; \rho)$ denoting the Fourier-Stieltjes transform

$$L(u; \rho) = \int_{-\infty}^{+\infty} e^{iux} d\rho(x). \tag{13}$$

In fact, if $g(n)$ has the asymptotic distribution function $\rho(x)$, then

---

[8] P. Erdös, *loc. cit.* [1], III, Proof of Lemma 3.

[9] P. Hartman, E. R. van Kampen and A. Wintner, *loc. cit.* [3], § 8.

[10] *Ibid.*, § 7.

$$M\{\exp iug(n)\} = L(u;\rho); \tag{14}$$

so that (12) follows from (11) by choosing all the u_j equal $(=u)$. And (12) means that the asymptotic distribution function $\sigma_k(x)$ of $f_k(n)$ is the convolution

$$\sigma_k = \sigma^{(1)} * \sigma^{(2)} * \cdots * \sigma^{(k)} \tag{15}$$

of the asymptotic distribution functions $\sigma^{(j)}(x)$ of the terms $f^{(j)}(x)$ of $f_k(x)$, i. e., of the sum (2) which belongs to the given $f(n)$ in virtue of (1). It is understood that the convolution, $\alpha * \beta$, of two distribution functions α, β is defined as the distribution function

$$\alpha(x) * \beta(x) = \int_{-\infty}^{+\infty} \alpha(x-y)\,d\beta(y)\,;\ \text{so that}\ L(u;\alpha*\beta) = L(u;\alpha)L(u;\beta).$$

It is readily verified that the asymptotic distribution function $\sigma^{(k)}(x)$ of the additive function (1) is the step function which has at $x = f(p^m)$ the jump $p^{-m}(1-p^{-1})$, where p denotes the k-th prime number and $m = 0, 1, \cdots$. It is understood that the values $f(p^m)$ belonging to the same $p = p_k$ need not be distinct for distinct m. Thus it is clear from (13) that

$$L(u;\sigma^{(k)}) = (1-p^{-1}) \sum_{m=0}^{\infty} p^{-m} \exp iuf(p^m), \text{ where } p = p_k. \tag{16}$$

10 bis. It should be mentioned that the italicized result of § 10 becomes wrong if statistical independence is meant in the sense of Kac and Steinhaus,[11] instead of the sense defined by Hartman, van Kampen and Wintner.[3] In fact, one can readily choose a sequence of numbers $f^{(1)}(2), f^{(1)}(2^2), f^{(1)}(2^3), \cdots$ in such a way that for the corresponding function $f^{(1)}(n)$, determined by (1), the limit $M\{S_x(n)\}$ mentioned at the end of § 1 does not exist at $x = 0$, say.

11. Let the additive function $f(n)$, the positive integer k and an $\epsilon > 0$ be arbitrarily given. Let S_k^ϵ denote the set of those positive integers n at which the function (2) satisfies the inequality $|f(n) - f_k(n)| > \epsilon$. Suppose that, in the notations introduced in § 1, one has $\bar{M}\{S_k^\epsilon(n)\} \to 0$, as $k \to \infty$. Suppose finally that this limit relation holds for every fixed $\epsilon > 0$. Then the functions $f_1(n), f_2(n), \cdots$ are said to tend to $f(n)$ in relative measure.

The following theorem will now be proved:

(ii). *An additive function $f(n)$ has an asymptotic distribution if and only if the function* (2) *tends, as $k \to \infty$, to $f(n)$ in relative measure, in which*

[11] M. Kac and H. Steinhaus, *Studia Mathematica*, vol. 7 (1937), pp. 1-15.

case the asymptotic distribution function of $f_k(n)$ *tends, as* $k \to \infty$, *to the asymptotic distribution function of* $f(n)$.

In fact, if $f_k(n)$ tends to $f(n)$ in relative measure, then $f(n)$ has an asymptotic distribution function which is the limit of the asymptotic distribution function of $f_k(n)$. This follows from a general theorem,[12] since every $f_k(n)$ has, by § 10, an asymptotic distribution function. Conversely, if $f(n)$ has an asymptotic distribution function, then (i), § 8, shows that (10) is satisfied. Since (10) is known [13] to be sufficient for the convergence of $f_k(n)$ to $f(n)$ in relative measure, the proof of (ii) is complete.

12. On comparing the definition (16) of $\sigma^{(k)}(x)$ with a general criterion [14] for the convergence of an arbitrary infinite convolution, one readily sees that (10) is precisely the condition for the convergence of the infinite convolution

$$\sigma = \sigma^{(1)} * \sigma^{(2)} * \cdots \tag{17}$$

But the k-th approximating convolution of (17) is the asymptotic distribution function (15) of $f_k(n)$. Hence, (i) and (ii) imply the following theorem:

(iii). *An additive function* $f(n)$ *has an asymptotic distribution if and only if the infinite convolution* (17) *is convergent. Furthermore, if* $f(n)$ *has at all an asymptotic distribution function, the latter is necessarily* (17).

13. As pointed out at the end of § 10, the distribution function $\sigma^{(k)}(x)$ is a step function. Hence, on combining (iii) with a general result [15] on infinite convolutions of step functions, one obtains the following theorem:

(iv). *A distribution function which is the asymptotic distribution function of an additive function* $f(n)$ *either is a step function or it is everywhere continuous, and in the latter case it is either purely singular or absolutely continuous.*

It is a rather intricate problem, at least in general, to distinguish between the cases of absolutely continuous and purely singular behavior.[16] On the other hand, the case of mere continuity is completely characterized by the following theorem: [17]

---

[12] B. Jessen and A. Wintner, *loc. cit.* [2], Theorem 24.

[13] P. Erdös, *loc. cit.* [1], III, Lemma 1.

[14] B. Jessen and A. Wintner, *loc. cit.* [2], Theorem 34.

[15] B. Jessen and A. Wintner, *loc. cit.* [2], Theorem 35.

[16] There are two extreme cases in which such a distinction is not too difficult; cf. P. Erdös, *American Journal of Mathematics*, vol. 61 (1939), pp. 722-725.

[17] If use is made of (i), the criterion (18) also follows as a consequence of the results obtained by Erdös, *loc. cit.* [1], III, by direct considerations.

(v). *A distribution function which is the asymptotic distribution function of an additive function $f(n)$ has no discontinuities if and only if*

$$\sum_{f(p)\neq 0} \frac{1}{p} \text{ is divergent.} \tag{18}$$

In fact, a general theorem of P. Lévy [18] states that a convergent infinite convolution $\rho^{(1)} * \rho^{(2)} * \cdots$ of distribution functions $\rho^{(k)}$ is continuous at every x if and only if the infinite product $\Pi d_k = 0$, where d_k denotes the largest jump $(0 \leqq d_k \leqq 1)$ of $\rho^{(k)}(x)$. On the other hand, (16) shows that the largest jump of $\sigma^{(k)}(x)$ is at most $1 - p^{-1} + p^{-2}$ if $f(p) \neq 0$, and at least $1 - p^{-2}$ if $f(p) = 0$, where $p = p_k$. Hence, (v) is implied by (iii).

14. By the spectrum of a distribution function $\rho = \rho(x)$, $-\infty < x < +\infty$, is meant the set of those x in the vicinity of which ρ is not constant, i. e., of those x for which one has $\rho(x - \epsilon) < \rho(x + \epsilon)$ whenever $\epsilon > 0$.

(vi). *If the additive function $f(n)$ has an asymptotic distribution function, the spectrum of the latter is identical with the closure set of the values attained by the function $f(n)$ for $n = 1, 2, \cdots$.*

In fact, (16) shows that (vi) is certainly true in the particular case $f(n) = f^{(k)}(n)$. Hence, on combining (iii) with a general theorem,[19] one sees from (3)-(4), where $f^{(k)}(1) = 0$, that (vi) is true for an arbitrary $f(n)$.

15. Further results may be obtained by combining (iii) with the theory of divergent infinite convolutions.[20]

THE INSTITUTE OF ADVANCED STUDY,
THE JOHNS HOPKINS UNIVERSITY.

[18] P. Lévy, *Studia Mathematica*, vol. 3 (1931), pp. 119-155, Théorème XIII. Another proof of Lévy's theorem has recently been communicated to one of us by Professor B. Jessen.

For a unified treatment of Lévy's theorem and of all of the convolution theorems used above, cf. a paper by E. R. van Kampen, which will appear in the *American Journal of Mathematics*.

[19] B. Jessen and A. Wintner, *loc. cit.* [2], Theorem 3.

[20] E. R. van Kampen and A. Wintner, *loc. cit.* [4], Theorems 4-9, and the investigations of P. Lévy, referred to there.