ON A FAMILY OF SYMMETRIC BERNOULLI CONVOLUTIONS.*

By Paul Erdős.

1. For any fixed real number $a$ in the interval $0 < a < 1$, let $\lambda = \lambda(x; a)$, $-\infty < x < +\infty$, denote the distribution function which is defined as the convolution of the infinitely many symmetric Bernoulli distribution functions $\beta(ax^n, -\infty < x < +\infty$, where $n = 0, 1, 2, \ldots$, and $\beta(x)$ denotes the function which is $0$, $1$ or $1$ according as $x < -1$, $1 < x < 1$ or $x > 1$. In other words, $\lambda(x; a)$ is the distribution function whose Fourier-Stieltjes transform is the infinite product

$$L(u; a) = \prod_{n=0}^{\infty} \cos (a^nu); \quad -\infty < u < +\infty.$$ 

It is known that if a value of $a$ is not such as to make the (monotone) function $\lambda(x; a)$ of $x$ absolutely continuous for $-\infty < x < +\infty$, then $\lambda(x; a)$ is purely singular, that is to say such as to have neither a discontinuous nor an absolutely continuous component in its Lebesgue decomposition. It is also known that the set of those points of the $x$-axis at which the non-decreasing function $\lambda(x; a)$ is increasing either is the interval $-(1-a)^{-1} \leq x \leq (1-a)^{-1}$ or a nowhere dense perfect zero set contained in this interval, according as $a \leq \frac{1}{2}$ or $a < \frac{1}{2}$. While this clearly implies that $\lambda(x; a)$ is purely singular if $a < \frac{1}{2}$, it does not imply that $\lambda(x; a)$ is absolutely continuous if $a \geq \frac{1}{2}$. On the other hand, it is known that if $a$ has any of the values $\frac{1}{2}, (\frac{1}{2})^{1/2}, (\frac{1}{2})^{1/3}, \ldots$, then $\lambda(x; a)$ is absolutely continuous, and that $\lambda(x; (\frac{1}{2})^{1/k})$ acquires derivatives of arbitrary high order as $k \to \infty$, i.e., as $a = (\frac{1}{2})^{1/k} \to 1$.

2. Thus, one might be inclined to expect that the "smoothness" of $\lambda(x; a)$ for $-\infty < x < +\infty$ cannot decrease when $a$ is increasing, and that $\lambda(x; a)$, being absolutely continuous if $a = \frac{1}{2}$, is absolutely continuous if $\frac{1}{2} < a < 1$, and not only if $a = (\frac{1}{2})^{1/k}$. However, it turns out that such is

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* Received May 12, 1938.
not the case. For instance, it will be shown that if $a$ has the “Fibonacci” value $\frac{1}{3}(5^{1/2} - 1)$, a value which lies between $\frac{1}{3}$ and $(\frac{1}{3})^{1/2}$, then $\lambda(x; a)$ is not absolutely continuous and is therefore purely singular (though nowhere constant on its range $|x| \leq (1 - x)^{-1}$). A corresponding $a$-value between $(\frac{1}{3})^{1/2}$ and $(\frac{1}{3})^{1/3}$ is, for instance, the positive root of the cubic equation $a^3 + a^2 - 1 = 0$. That $\lambda(x; a)$ is singular for these algebraic irrationalities $a$, will be proved by showing that the necessary condition $L(u; a) \to 0$, $u \to \pm \infty$, of the Riemann-Lebesgue lemma is not satisfied at these particular $a$-values.

Let $a$ be a real algebraic integer which satisfies the inequality $a > 1$ and is such that, if $m$ denotes the degree of $a$, and $x_j$, where $j = 2, \ldots, m$, are the conjugates of $a$, then $|x_j| < 1$ for all $j$. Since $a^n + x_1^n + \cdots + x_m^n$ is a rational integer for $n = 0, 1, 2, \ldots$, it is clear that there exists a positive number $\theta < 1$ which has the property that the distance between $a^n$ and the nearest integer to $a^n$ is less than $\theta^n$ for every $n$.

Now choose $a = \frac{1}{a}$. Then, since $L(u; a) = \prod_{n=0}^{\infty} \cos(a^n u)$, one has, for every positive integer $k$,

$$L(\pi x^k; a) = C \prod_{n=1}^{k} \cos(a^n \pi), \quad \text{where } C = \prod_{n=2}^{\infty} \cos(a^n \pi)$$

is a non-vanishing constant, since every $\cos(a^n \pi) \neq 0$. Consequently by the above definition of the positive number $\theta < 1$,

$$|L(\pi x^k; a)| = |C| \prod_{n=1}^{k} |\cos(a^n \pi)| \geq C' \prod_{n=1}^{\infty} |\cos(\theta^n \pi)|,$$

where $C' = |C| \prod_{n=1}^{\infty} |\cos(\theta^n \pi)|$ and the product $\prod_{n=1}^{\infty}$ runs through those values of $n$ for which $\theta^n < \frac{1}{2}$. Hence, for every $k$,

$$|L(\pi x^k; a)| \geq C' \prod_{n=1}^{\infty} |\cos(\theta^n \pi)| = \text{const.} > 0,$$

if $\theta$ is chosen to be distinct from $\frac{1}{2}$. Since $a^k \to \infty$ as $k \to \infty$, it follows that $L(u; a)$ does not tend to 0 as $u \to \infty$, and so the distribution function $\lambda(x; a)$ is singular, for any $a = 1/x$ of the type described above.

It seems to be likely that these $a$ are clustering at $a = 1$ (this would imply that these $a$ lie everywhere dense between $a = 0$ and $a = 1$).

3. Needless to say, $L(u; a) \to 0$, $u \to \infty$, only is a necessary condition in order that $\lambda(x; a)$ be absolutely continuous. In fact, it is known\footnote{R. Kershner, American Journal of Mathematics, vol. 58 (1936), pp. 450-452.} that
if $a$ has any rational value which is not the reciprocal value of an integer, then there exists a positive $\gamma = \gamma(a)$ such that $L(u; a) = O(|\log u|^{-\gamma})$ as $u \to \infty$, whether the positive number $a(<1)$ is or is not greater than $\frac{1}{2}$. (It is easy to see that if $a = \frac{1}{3}, \frac{1}{4}, \ldots$, then $L(u; a) \to 0$ does not hold; while $L(u; a) = (\sin u)/u$ if $a = \frac{1}{2}$.) Actually, it may be shown that, whether $a > \frac{1}{2}$ or $a < \frac{1}{2}$, the Fourier-Stieltjes transform $L(u; a)$ tends, as $u \to \infty$, to 0, not only when $a$ is any rational number distinct from $\frac{1}{3}, \frac{1}{4}, \ldots$, but also for all irrational values of $a$ which do not belong to a certain enumerable set.

In order to prove this, notice first that all values $a$ between 0 and 1 which do not belong to a certain enumerable set are known to possess the following property: There does not exist any number $b > 0$ in such a way that if $\epsilon_k$ denotes, for fixed $a$ and fixed $b$, the distance between $b\alpha^n$ and the nearest integer to $b\alpha^n$, then $\epsilon_k < \frac{1}{2} (a^n + 1)^{-\epsilon_b}$ for every sufficiently large $n$. Let $a$ be chosen small enough not to possess this property.

Suppose, if possible, that $L(u; a)$ does not tend to 0 as $u \to \infty$, i.e., that there exists a sequence $u_1, \ldots, u_j, \ldots$ for which one has $u_j \to \infty$ as $j \to \infty$, while $|L(u_j; a)| > c$ holds for a sufficiently small positive $c = c(a)$ which is independent of $j$. Clearly, one can choose these $u_j$ in such a way that the sequence $\{b_j\}$ defined by $b_j = u_j a^k$ tends, as $j \to \infty$, to a limit, say $b$, where $k = k_j$ denotes the unique positive integer satisfying $a < u_j a^k \leq 1$; so that $k_j \to \infty$ as $j \to \infty$, and $a \leq b \leq 1$. But $|L(u_j; a)| > c$ may be written in the form

$$|L(b_j a^k; a)| = \left| \prod_{n=0}^{\infty} \cos (b_j a^n) \right| > c > 0$$

for every $j$ and $k = k_j$. Since $k_j \to \infty$ and $b_j \to b$ as $j \to \infty$, it follows by an obvious adaptation of the inequalities applied in § 2, that $b$ has the property excluded above by the choice of $a$. This contradiction completes the proof of the fact that $L(u; a) \to 0$, $u \to \infty$, holds for any of the $a$-values under consideration.

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