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ON POLYNOMIALS WITH ONLY REAL ROOTS[1]

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We start from some well known elementary properties of the parabola of second order. Denote its axis by t, the vertex by C; in any point C_1 of the axis draw the perpendicular to it, meeting the parabola at A_1 and A_2 respectively. If the foot of the perpendiculars from A_1 and A_2 to the tangent at the vertex are denoted by B_1 and B_2 respectively and the point of intersection of the tangents at A_1 and A_2 by T (Fig. 1) then we have

$$\overline{TC_1} = 2\,\overline{CC_1}$$

and

$$F = \tfrac{2}{3}\Delta A_1A_2T = \tfrac{2}{3}\,\text{rect}\;A_1A_2B_2B_1,$$

where F denotes the area of the parabola above $\overline{A_1A_2}$.

It is evident that these properties do not hold for polynomials of degree $n > 2$. We shall now enquire if for $n > 2$ the ratio of the above distances and areas remains between certain limits.

Let $g(x)$ be a polynomial of degree n with $n \geqq 2$, p and q ($p < q$) two consecutive real roots and suppose $g(x)$ to be positive throughout the interval (p, q). Let $g(x)$ assume its maximum value in (p, q) at b. The tangents at p and q meet at a point T with t as ordinate. In general $t > 0$ and for $g'(p) = g'(q) = 0$ we assign to t the value 0. Throughout this paper $\triangle pqT$ will be referred to as the "*tangential triangle*"; its area is given by $\frac{1}{2}(q - p)t$. The part of the curve between p and q may be included in the "*tangential rectangle*" of area $(q - p)g(b)$, and $\int_p^q g(x)\,dx$ defines the "*area of the curve.*"

The ratio $t/g(b)$ and that of the *area of the tangential triangle to the area of the curve* which, in the case $n = 2$ take the values 2 and $3/2$ respectively, may in the general case, assume any value. They may be less then any number we may choose, however small; they may even equal 0, e.g. if p is a multiple root, i.e. if $g'(p) = 0$ and thus also $t = 0$. But they may also be larger than any number we may choose, however large; as, e.g. in the case $g(x) = 1 - x^{2n}$. Here $p = -1$, $q = +1$, $b = 0$, $g(0) = 1$, $t = 2n$, hence

$$t/g(0) = 2n \quad\text{and}\quad \frac{q-p}{2}\,t \Big/ \int_p^q g(x)\,dx > 2n/2 = n.$$

[1] In this paper we consider only polynomials with real coefficients.

This example at the same time illustrates that the ratio of *area of the curve to the area of the tangential rectangle* may approach unity as close as we like. But, as the example $g(x) = (1 - x^2)^n$ shows, the same ratio may assume values less then any preassigned number however small.

Thus we see that, for *any* polynomial the above ratios cannot be enclosed between definite limits (trivial ones disregarded). *But for polynomials with only real roots we proved the following theorems.*

THEOREM I. $t \leqq 2g(b)$.

THEOREM II. *The area of the curve is greater than or equal to* $\frac{2}{3}$ *the area of the tangential triangle*:

$$\int_p^q g(x)\,dx \geqq \frac{2}{3}\left(\frac{q-p}{2}\,t\right).$$

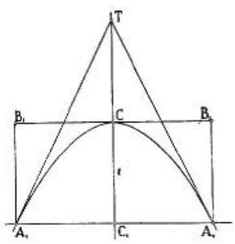

FIG. 1

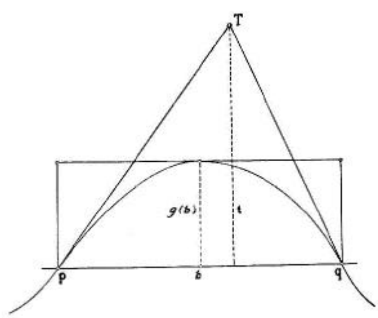

FIG. 2

THEOREM III. *The area of the curve is less than or equal to* $\frac{2}{3}$ *the area of the tangential rectangle*:

$$\int_p^q g(x)\,dx \leqq \tfrac{2}{3}[(q-p)g(b)].$$

In all cases equality holds only for $n = 2$.

Theorem I and its proof under 2 is due to G. Szekeres.[2]

[2] Oral communication.

Theorems II and III may be written

$$\frac{2}{3}\left(\frac{q-p}{2}t\right) \leqq \int_p^q g(x)\,dx \leqq \frac{2}{3}[(q-p)g(b)].$$

Hence

$$\frac{2}{3}\left(\frac{q-p}{2}t\right) \leqq \frac{2}{3}[(q-p)g(b)],$$

i.e. the area of the tangential triangle is less than or equal to the area of the tangential rectangle.

From this inequality we immediately obtain

$$t \leqq 2g(b)$$

i.e. *theorem I is a consequence of II and III.* Nevertheless we give Szekeres's proof as it leads to the result immediately and by very simple means.

In 3 we prove two lemmas required and in 4 and 5 our theorems II and III.

2. Proof of theorem I due to G. Szekeres

Let

$$g(x) = A(x - a_1)(x - a_2) \cdots (x - a_n) \qquad (n \geqq 2),$$

where for the real a_i's

$$a_1 \geqq a_2 \geqq \cdots \geqq a_{k-1} > a_k \geqq \cdots \geqq a_n . \qquad (2 \leqq k \leqq n).$$

Assume

$$p = a_k \quad \text{and} \quad q = a_{k-1} .$$

At a certain b_{k-1} in (a_k , a_{k-1}) $g'(x)$ vanishes and, if we suppose $g(x)$ to be positive in (a_k , a_{k-1}), the function assumes at b_{k-1} its maximum. Thus $g(x) > 0$ $(a_k < x < a_{k-1})$, $b = b_{k-1}$ and if we denote by T_{k-1} the point of intersection of the tangents at a_{k-1} and a_k, we have to prove

$$t_{k-1} \leqq 2g(b_{k-1}),$$

where t_{k-1} denotes the ordinate of T_{k-1}.

If one of the values $g'(a_{k-1})$ and $g'(a_k)$ equals 0 then $t_{k-1} = 0$ and our theorem is a trivial one. (The same holds for $g'(a_{k-1}) = g'(a_k) = 0$ for in this case, according to our definition, $t_{k-1} = 0$). For $n = 2$ we have, by simple computation $t_{k-1} = 2g(b_{k-1})$. Hence we may assume

$$g'(a_{k-1})g'(a_k) \neq 0 \quad \text{and} \quad n \geqq 3. \tag{1}$$

By easy computation

$$t_{k-1} = (a_{k-1} - a_k) \cdot \frac{|g'(a_{k-1})|\,|g'(a_k)|}{|g'(a_{k-1})| + |g'(a_k)|} = \frac{a_{k-1} - a_k}{2} \cdot \frac{2}{\dfrac{1}{|g'(a_{k-1})|} + \dfrac{1}{|g'(a_k)|}}$$

i.e.

$$t_{k-1} \leqq \frac{a_{k-1} - a_k}{2} (|g'(a_{k-1})| |g'(a_k)|)^{\frac{1}{2}} = s_{k-1}$$

since the harmonic mean of two positive numbers is not greater than their geometric mean.

Now we prove

$$s_{k-1} < 2g\left(\frac{a_{k-1} + a_k}{2}\right)$$

by which, properly said, we prove more than is contained in theorem I; for we prove that t_{k-1} is less that the double of the value assumed at the bisection of the interval (a_k, a_{k-1}).

Without loss of generality we may suppose

$$a_{k-1} = 1 \quad \text{and} \quad a_k = -1.$$

Hence, by (1), according to which -1 and $+1$ are simple roots, we have

$$(2) \qquad |a_i| > 1 \qquad (i = 1, 2, \cdots k-2, k+1, \cdots, n)$$

and

$$s_{k-1} = (|g'(-1)| |g'(1)|)^{\frac{1}{2}}, \qquad \frac{a_{k-1} + a_k}{2} = 0.$$

$$|g'(1)| = |A(1 - a_1)(1 - a_2) \cdots (1 - a_{k-2})(1 + 1)(1 - a_{k+1}) \cdots (1 - a_n)|$$
$$= 2|A(a_1 - 1)(a_2 - 1) \cdots (a_{k-2} - 1)(a_{k+1} - 1) \cdots (a_n - 1)|.$$

$$|g'(-1)| = 2|A(a_1 + 1)(a_2 + 1) \cdots (a_{k-2} + 1)(a_{k+1} + 1) \cdots (a_n + 1)|.$$

i.e. by (1) and (2)

$$s_{k-1} = (4A^2(a_1^2 - 1)(a_2^2 - 1) \cdots (a_{k-2}^2 - 1)(a_{k+1}^2 - 1) \cdots (a_n^2 - 1))^{\frac{1}{2}}$$
$$< 2(A^2 a_1^2 a_2^2 \cdots a_{k-2}^2 a_{k+1}^2 \cdots a_n^2)^{\frac{1}{2}} = 2|A a_1 a_2 \cdots a_{k-2} a_{k+1} \cdots a_n| = 2g(0).$$

Q.e.d.

3

Suppose that the roots $c_1, c_2, \cdots, c_\nu$ of the polynomial of degree ν ($\nu \geqq 3$)

$$h(x) = C(x - c_1)(x - c_2)(x - c_3) \cdots (x - c_\nu)$$

are all real; $c_1 > c_2 > c_3 \geqq \cdots \geqq c_\nu$, i.e. the two greatest roots are simple. Further let $h(x)$ be negative in (c_2, c_1) and positive in (c_3, c_2) i.e. $C > 0$. Then we prove

LEMMA 1. *If* $c_2 - c_3 \leqq c_1 - c_2$, *then* $\int_{c_3}^{c_1} h(x)\,dx \leqq 0$ and

LEMMA 2. *If*

$$\frac{1}{c_2 - c_3} + \frac{1}{c_2 - c_4} + \cdots + \frac{1}{c_2 - c_\nu} \leqq \frac{1}{c_1 - c_2},$$

then

$$\int_{c_3}^{c_1} h(x)\,dx \geqq 0.$$

In both lemmas equality holds only for $\nu = 3$ *with* $c_2 - c_3 = c_1 - c_2$.

Without loss of generality we may assume

$$C = 1, \qquad c_1 = 1, \qquad c_2 = 0.$$

Put $\gamma_i = -c_i$ for $3 \leqq i \leqq \nu$. Then

$$h(x) = (x - 1)x(x + \gamma_3)(x + \gamma_4) \cdots (x + \gamma_\nu)$$

and

$$0 < \gamma_3 \leqq \gamma_4 \leqq \cdots \leqq \gamma_\nu. \tag{3}$$

PROOF OF LEMMA 1.

We have to prove that for $\gamma_3 \leqq 1$, $\int_{-\gamma_3}^{1} h(x)\,dx \leqq 0$. It will be sufficient to prove that

$$|h(-x)| \leqq |h(x)|. \qquad (0 < x < \gamma_3(\leqq 1)).$$

By (3) and $\gamma_3 \leqq 1$, for $0 < x < \gamma_3$ we have

$$|h(x)| = (1 - x)x(x + \gamma_3)(x + \gamma_4) \cdots (x + \gamma_\nu)$$
$$\geqq (1 - x)x(x + \gamma_3)(-x + \gamma_4) \cdots (-x + \gamma_\nu)$$

(equality only for $\nu = 3$).

From $0 < x < \gamma_3 \leqq 1$ it is easy to verify that

$$(1 - x)(x + \gamma_3) \geqq (1 + x)(-x + \gamma_3)$$

(equality only for $\gamma_3 = 1$). Hence

$$|h(x)| \geqq (1 + x)x(-x + \gamma_3)(-x + \gamma_4) \cdots (-x + \gamma_\nu) = |h(-x)| \qquad (0 < x < \gamma_3),$$

which establishes lemma 1.

By the argument it is immediately clear that equality holds only for $\nu = 3$ and $\gamma_3 = 1$.

PROOF OF LEMMA 2. We have to prove that if

$$\frac{1}{\gamma_3} + \frac{1}{\gamma_4} + \cdots + \frac{1}{\gamma_{\nu-1}} + \frac{1}{\gamma_\nu} \leqq 1, \tag{4}$$

then

$$\int_{-\gamma_3}^{1} h(x)\,dx \geqq 0.$$

By (4) $\gamma_3 \geqq 1$; thus it will be sufficient to prove that

$$|\,h(x)\,| \leqq |\,h(-x)\,|. \qquad (0 < x < 1).$$

For $\nu = 3$ we have

$$h(x) = (x-1)x(x+\gamma_3) = (x-1)x(x+1)\,\frac{x+\gamma_3}{x+1}$$

$$= (x-1)x(x+1)\left(1 + \frac{\gamma_3 - 1}{x+1}\right).$$

If $\gamma_3 = 1$ then $|\,h(x)\,| \equiv |\,h(-x)\,|$ hence $\int_{-1}^{+1} h(x)\,dx = 0$. If $\gamma_3 > 1$ then

$$1 + \frac{\gamma_3 - 1}{-x+1} > 1 + \frac{\gamma_3 - 1}{x+1} \qquad (0 < x < 1),$$

hence $|\,h(x)\,| < |\,h(-x)\,|$ $(0 < x < 1)$.

Thus the theorem is true for $\nu = 3$ and we shall prove that if it holds for $\nu - 1$ it also holds for ν $(\nu \geqq 4)$.

Let us consider the number γ for which

$$\frac{1}{\gamma} = \frac{1}{\gamma_{\nu-1}} + \frac{1}{\gamma_\nu}. \tag{5}$$

By (4)

$$\frac{1}{\gamma_3} + \frac{1}{\gamma_4} + \cdots + \frac{1}{\gamma_{\nu-2}} + \frac{1}{\gamma} \leqq 1 \tag{6}$$

and thus

$$1 < \gamma < \gamma_{\nu-1} \leqq \gamma_\nu\,. \tag{7}$$

In the polynomial of degree $\nu - 1$

$$h^*(x) = h(x)\,\frac{x+\gamma}{(x+\gamma_{\nu-1})(x+\gamma_\nu)}$$

the coefficient of $x^{\nu-1}$ equals 1, each root is real and by (6) they satisfy (4), hence according to the hypothesis

$$|\,h^*(x)\,| \leqq |\,h^*(-x)\,|. \qquad (0 < x < 1).$$

By (5) and (7) we easily verify

$$\frac{x+\gamma}{(x+\gamma_{\nu-1})(x+\gamma_\nu)} > \frac{-x+\gamma}{(-x+\gamma_{\nu-1})(-x+\gamma_\nu)} \qquad (0 < x < 1),$$

hence

$$|h(x)| = |h^*(x)| \frac{(x+\gamma_{\nu-1})(x+\gamma_\nu)}{x+\gamma}$$
$$< |h^*(-x)| \frac{(-x+\gamma_{\nu-1})(-x+\gamma_\nu)}{-x+\gamma} = |h(-x)|. \quad (0<x<1),$$

which proves lemma 2.

4

Applying the notations of 2 theorem II may be written

$$\int_{a_k}^{a_{k-1}} g(x)\,dx \geqq \frac{2}{3}\left(\frac{a_{k-1}-a_k}{2}\,t_{k-1}\right).$$

If $g'(a_k)$ or $g'(a_{k-1})$ vanishes, $t_{k-1} = 0$ and the theorem becomes trivial, hence we suppose $g'(a_k)g'(a_{k-1}) \neq 0$.

In this case, according to 2

$$t_{k-1} = \frac{a_{k-1}-a_k}{2}\,\frac{2}{\dfrac{1}{|g'(a_{k-1})|}+\dfrac{1}{|g'(a_k)|}} \leqq \frac{a_{k-1}-a_k}{2}\,(|g'(a_{k-1})|\,|g'(a_k)|)^{\frac{1}{2}} = s_{k-1}.$$

Now we prove that

$$\int_{a_k}^{a_{k-1}} g(x)\,dx \geqq \frac{2}{3}\left(\frac{a_{k-1}-a_k}{2}\,s_{k-1}\right) = \frac{2}{3}\left[\left(\frac{a_{k-1}-a_k}{2}\right)^2 (|g'(a_{k-1})|\,|g'(a_k)|)^{\frac{1}{2}}\right].$$

For $n = 2$ we have equality i.e. our statement holds for $n = 2$. We now suppose that it holds for numbers less than n $(n > 2)$ and prove that it also holds for n.

a/ If $2 < k < n$ then a_{k-1} and a_k are at least double roots of the polynomial

$$h(x) = \frac{g(x)}{(x-a_1)(x-a_n)}\,(x-a_{k-1})(x-a_k)$$

the degree of which is n, hence

$$h'(a_{k-1}) = h'(a_k) = 0. \tag{8}$$

On the other hand it is easy to see that

$$0 < h(x) < g(x). \qquad (a_k < x < a_{k-1}). \tag{9}$$

In the polynomials $h(x)$ and $g(x)$, x^n has the same coefficient, hence the degree of the polynomial

$$g_1(x) = g(x) - h(x)(\not\equiv 0)$$

does not exceed $n - 1$.

The values $a_2, a_3, \cdots, a_{n-1}$ are roots of $g(x)$ as well as of $h(x)$ thus they are at the same time roots of $g_1(x)$ hence, its coefficients being all real, all roots of $g_1(x)$ must be real too.

By (9), throughout (a_k, a_{k-1}) $g_1(x) > 0$ thus $g_1(x)$ satisfies the conditions of theorem II hence according to the hypothesis

$$\int_{a_k}^{a_{k-1}} g_1(x)\,dx \geqq \frac{2}{3}\left[\left(\frac{a_{k-1}-a_k}{2}\right)^2 (|g_1'(a_{k-1})|\,|g_1'(a_k)|)^{\frac{1}{2}}\right].$$

But by (8)

$$g_1'(a_{k-1}) = g'(a_{k-1}), \qquad g_1'(a_k) = g'(a_k),$$

hence, as by (9) $\int_{a_k}^{a_{k-1}} h(x)\,dx > 0$, we have

$$\int_{a_k}^{a_{k-1}} g(x)\,dx = \int_{a_k}^{a_{k-1}} g_1(x)\,dx + \int_{a_k}^{a_{k-1}} h(x)\,dx > \int_{a_k}^{a_{k-1}} g_1(x)\,dx$$

$$\geqq \frac{2}{3}\left[\left(\frac{a_{k-1}-a_k}{2}\right)^2 (|g'(a_{k-1})|\,|g'(a_k)|)^{\frac{1}{2}}\right] = \frac{2}{3}\left[\frac{a_{k-1}-a_k}{2}\,s_{k-1}\right].$$

b/ Of the cases $k = 2$ and $k = n$ it will be sufficient to deal with one of them for this one applied to the polynomial $g(-x)$ settles the other. Then let $k = 2$.

Without any loss of generality we may assume $a_1 = 1$, $a_2 = -1$. In order to satisfy $g(x) > 0$ throughout (a_2, a_1) we must suppose A to be negative. Let $A = -1$ and denote $\alpha_i = -a_i$ $(i = 3, 4, \cdots, n)$. Then

$$g(x) = -(x-1)(x+1)(x+\alpha_3)\cdots(x+\alpha_n),$$

$$1 \leqq \alpha_3 \leqq \alpha_4 \leqq \cdots \leqq \alpha_n$$

and

$$\frac{a_{k-1}-a_k}{2}\,s_k = (|g'(1)|\,|g'(-1)|)^{\frac{1}{2}}.$$

But as, by the hypothesis, $g'(a_1)g'(a_2) = g'(1)g'(-1) \neq 0$ we have

$$(10) \qquad 1 < \alpha_3 \leqq \alpha_4 \leqq \cdots \leqq \alpha_n .$$

Now consider the polynomial of degree $n - 1$

$$g_1(x) = \lambda\frac{g(x)}{x+\alpha_n} = -\lambda(x-1)(x+1)(x+\alpha_3)\cdots(x+\alpha_{n-1}) \qquad (\lambda > 0).$$

$g_1(x)$ is positive throughout $(-1, +1)$ and all its roots are real.

Now we choose λ so that

$$(11) \qquad (|g'(+1)|\,|g'(-1)|)^{\frac{1}{2}} = (|g_1'(+1)|\,|g_1'(-1)|)^{\frac{1}{2}}.$$

Now

$$g'(+1) = -(1+1)(1+\alpha_3)\cdots(1+\alpha_{n-1})(1+\alpha_n),$$

$$g_1'(+1) = -\lambda(1+1)(1+\alpha_3)\cdots(1+\alpha_{n-1}).$$

Hence

$$|g_1'(+1)| = \lambda \frac{|g'(+1)|}{\alpha_n + 1}.$$

Similarly

$$|g_1'(-1)| = \lambda \frac{|g'(-1)|}{\alpha_n - 1},$$

hence

$$|g_1'(+1)||g_1'(-1)| = \frac{\lambda^2}{\alpha_n^2 - 1} |g'(+1)||g'(-1)|.$$

(11) will be satisfied if ($\alpha_n > 1$!) $\lambda^2 = \alpha_n^2 - 1 (> 0)$ i.e. if

$$\lambda = +(\alpha_n^2 - 1)^{\frac{1}{2}}.$$

In this case

$$\alpha_n - 1 < \lambda < \alpha_n . \tag{12}$$

Now form the polynomial $h(x) = g_1(x) - g(x)$. Its degree is n, its $n - 1$ roots: $+1, -1, -\alpha_3, -\alpha_4, \cdots, -\alpha_{n-1}$ and consequently also the n^{th} one are all real. We prove that the n^{th} root must lie in $(-1, 0)$.

By (12)

$$h(0) = g_1(0) - g(0) = \lambda \frac{g(0)}{\alpha_n} - g(0) < 0 \tag{13}$$

and, as $g'(-1) > 0$,

$$h'(-1) = g_1'(-1) - g'(-1) = \frac{\lambda}{\alpha_n - 1} g'(-1) - g'(-1)$$

$$= g'(-1)\left(\frac{\lambda}{\alpha_n - 1} - 1\right) > 0.$$

By this and by $h(-1) = 0$, it follows that somewhere in $(-1, 0)$ $h(x)$ must vanish. Denote this root by $-\alpha$ $(0 < \alpha < 1)$.

$h(x)$ satisfies the requirements of lemma 1 as 1 and $-\alpha$ are simple roots of $h(x)$, $1 - (-\alpha) > (-\alpha) - (-1)$ and as, by (13), throughout $(-1, -\alpha)$ $h(x) > 0$ whereas throughout $(-\alpha, 1)$ $h(x) < 0$. Hence by lemma 1.

$$\int_{-1}^{+1} h(x)\, dx < 0.$$

Consequently

$$\int_{-1}^{+1} g(x)\, dx = \int_{-1}^{+1} g_1(x)\, dx - \int_{-1}^{+1} h(x)\, dx > \int_{-1}^{+1} g_1(x)\, dx.$$

But $g_1(x)$ is of degree $n - 1$. Thus by the hypothesis

$$\int_{-1}^{+1} g_1(x)\,dx \geqq \tfrac{2}{3}(|g_1'(+1)|\,|g_1'(-1)|)^{\frac{1}{2}},$$

which with (11) gives

$$\int_{-1}^{+1} g(x)\,dx > \tfrac{2}{3}(|g'(+1)|\,|g'(-1)|)^{\frac{1}{2}},$$

which proves theorem II.

5

By applying the notations introduced in 2, we write theorem III in the form

$$\int_{a_k}^{a_{k-1}} g(x)\,dx \leqq \tfrac{2}{3}(a_{k-1} - a_k)g(b_{k-1}).$$

We prove it by induction. Theorem III holds for $n = 2$ (equality). We shall prove that if it is true for numbers less than n $(n > 2)$ then it also holds for n.

a/ If $2 < k < n$ then a_k and a_{k-1} are at least simple roots of the polynomial of degree n

$$h(x) = \frac{g(x)}{(x - a_1)(x - a_n)}(x - b_{k-1})^2$$

whereas b_{k-1} is exactly a double root. Thus $h(b_{k-1}) = h'(b_{k-1}) = 0$. In the interior of the intervals (a_k, a_{k-1}), with the exception of b_{k-1}, $h(x)$ is less than 0.

The coefficient of x^n in $h(x)$ and in $g(x)$ is the same, thus the degree of the polynomial $g_1(x) = g(x) - h(x)$ does not exceed $n - 1$. All roots of $g_1(x)$ are real, for it vanishes at $a_2, a_3, \cdots, a_{k-1}, a_k, \cdots a_{n-1}$ and thus the eventually resulting $(n-1)^{\text{th}}$ root cannot be complex. Hence, as $g_1'(b_{k-1}) = g'(b_{k-1}) - h'(b_{k-1}) = 0$ we have by hypothesis

$$\int_{a_k}^{a_{k-1}} g_1(x)\,dx \leqq \tfrac{2}{3}(a_{k-1} - a_k)g_1(b_{k-1}).$$

But as

$$g_1(b_{k-1}) = g(b_{k-1}) - h(b_{k-1}) = g(b_{k-1})$$

and since throughout (a_k, a_{k-1}) (with the exception of $x = b_{k-1}$, when equality holds) $g_1(x) > g(x)$ we may write

$$\int_{a_k}^{a_{k-1}} g_1(x)\,dx > \int_{a_k}^{a_{k-1}} g(x)\,dx,$$

hence

$$\int_{a_k}^{a_{k-1}} g(x)\,dx < \tfrac{2}{3}(a_{k-1} - a_k)g(b_{k-1}).$$

b/ Just as in 4 we have to consider only one of the cases $k = 2$ and $k = n$.

Let $k = 2$. Consider the polynomial of degree $n - 1$

$$g_1(x) = \lambda \frac{g(x)}{x - a_n}. \qquad (\lambda > 0).$$

We choose λ such as to make the maximal value of $g_1(x)$ in (a_2, a_1) (positive in this interval as well as that of $g(x)$) equal to that of $g(x)$. Such a λ exists. Hence, denoting the root of $g_1'(x)$ lying in (a_2, a_1) by b, we have

$$g_1(b) = g(b_1). \tag{14}$$

From the equations

$$\frac{g'(b_1)}{g(b_1)} = 0 \quad \text{and} \quad \frac{g_1'(b)}{g_1(b)} = 0$$

we obtain

$$\frac{1}{b_1 - a_2} + \frac{1}{b_1 - a_3} + \cdots + \frac{1}{b_1 - a_{n-1}} + \frac{1}{b_1 - a_n} = \frac{1}{a_1 - b_1}$$

and

$$\frac{1}{b - a_2} + \frac{1}{b - a_3} + \cdots + \frac{1}{b - a_{n-1}} = \frac{1}{a_1 - b}. \tag{15}$$

By which

$$b < b_1 .$$

The polynomial $h(x) = g_1(x) - g(x)$ is of degree n. Its $n - 1$ roots are $a_1, a_2, a_3, \cdots, a_{n-1}$. Furthermore, it must vanish in the interval (b, b_1) too, since by (14)

$$h(b) = g_1(b) - g(b) > 0 \quad \text{and} \quad h(b_1) = g_1(b_1) - g(b_1) < 0.$$

Denote the root of $h(x)$ lying in (b, b_1) by a. By (15)

$$\frac{1}{a - a_2} + \frac{1}{a - a_3} + \cdots + \frac{1}{a - a_{n-1}} < \frac{1}{a_1 - a},$$

hence, a_1 and a being simple roots, $h(x)$ satisfies all the conditions of lemma 2. Hence

$$\int_{a_2}^{a_1} h(x)\, dx > 0.$$

From this we obtain

$$\int_{a_2}^{a_1} g(x)\, dx = \int_{a_2}^{a_1} g_1(x)\, dx - \int_{a_2}^{a_1} h(x)\, dx < \int_{a_2}^{a_1} g_1(x)\, dx.$$

But according to the hypothesis, the degree of $g_1(x)$ being less than n,

$$\int_{a_2}^{a_1} g_1(x)\, dx \leqq \tfrac{2}{3}(a_1 - a_2) g_1(b),$$

hence by (14) we have

$$\int_{a_2}^{a_1} g(x)\, dx < \tfrac{2}{3}(a_1 - a_2)g(b_1),$$

and thus theorem III is completely proved.

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