ON THE SMOOTHNESS PROPERTIES OF A FAMILY OF
BERNOULLI CONVOLUTIONS.*

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Let $L(u, a), -\infty < u < +\infty$ denote the Fourier-Stieltjes transform,

$$
\int_{-\infty}^{\infty} e^{iux}d\sigma(x),
$$
of a distribution function $\sigma(x), -\infty < x < +\infty$. Thus if $\beta(x)$ is the distribution function which is 0, $\frac{1}{2}, 1$ according as $x \leq -1, -1 < x \leq 1, 1 < x$, then $L(u, \beta) = \cos u$; and so, if $b$ is a positive constant, $\cos (u/b)$ is the transform of the distribution function $\beta(bx)$. Hence, if $a$ is a positive constant, the infinite convolution

$$
\sigma_a(x) = \beta(ax) * \beta(a^2x) * \beta(a^3x) * \ldots
$$
is convergent if and only if $a > 1$; its Fourier-Stieltjes transform being

$$
L(u, \sigma_a) = \prod_{n=1}^{\infty} \cos \left(\frac{u}{a^n}\right), \quad (a > 1).
$$

It is known that the distribution function $\sigma_a$ is continuous for every $a > 1$ and, in fact, is either absolutely continuous or purely singular, depending on the value of $a$. In this direction it is known that the set of points $x$ in the neighborhood of which $\sigma_a(x)$ is not constant is either the interval $x \leq a/(a - 1)$ or a nowhere dense perfect set of measure zero contained in this interval according as $1 < a \leq 2$ or $2 < a$. While this implies that $\sigma_a(x)$ is singular if $2 < a$ it does not imply that $\sigma_a(x)$ is absolutely continuous if $a < 2$. In fact it has recently been shown that there exist certain algebraic irrationalities $a < 2$ for which $L(u, \sigma_a)$ does not tend to zero with $1/u$ and so $\sigma_a$ cannot be absolutely continuous. (It was conjectured, loc. cit., that such values of $a$ are clustering at $a = 1 + \epsilon$ which would imply that they lie dense in the interval $1 < a < 2$). On the other hand it is known that those $a < 2$

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for which \( \sigma_a \) is absolutely continuous are certainly clustering at \( a = 1 + 0 \), since if \( a = 2^{1/m} \), where \( m \) is a positive integer, then \( \sigma_a \) has a continuous derivative of order \( m - 1 \).

The object of the present paper is to show that the successive smoothing of \( \sigma_a \) can be considered as the general case when \( a \to 1 + 0 \). In fact it will be shown that there exists, for every positive integer \( m \), a positive \( \eta(m) \) such that the set of those points \( a \) of the interval \( 1 < a < 1 + \eta(m) \) for which \( \sigma_a \) does not possess a continuous derivative of order \( m - 1 \) is a set of measure zero. To this end it is sufficient to prove that there exists, for every positive integer \( m \), a positive \( \delta(m) \) such that the set of those points \( a \) of the interval \( 1 < a < 1 + \delta(m) \) for which

\[
L(u, \sigma_a) = o\left(\left| u \right|^{-m}\right), \quad u \to \infty,
\]

does not hold is a set of measure zero.

Let \( c_1, c_2, \ldots, c_N \) be \( N \) positive integers which satisfy the following conditions:

(i) \( c_1 \leq 2 \);
(ii) \( c_i < c_{i+1} \),
(iii) \( c_{i+1} < 3c_i \),
(iv) there exists an \( \alpha \) such that \( 2^i < \alpha < 2 \) and \( |c_{i+1} - \alpha c_i| < 2 \),

\[
\text{(i) } (i = 1, 2, \ldots, N - 1); \quad \text{(ii) } (i = 1, 2, \ldots, N - 1); \quad \text{(iii) } (i = 1, 2, \ldots, N - 1).\]

**Lemma 1.** There exist two positive absolute constants \( \gamma_1, \gamma_2 \) such that if \( M \) is any fixed number \( > \gamma_2 \), there are less than \( \lceil M^{1/4} \rceil \) different sequences \( c_1, c_2, \ldots, c_N \) satisfying the requirements (i)-(iv), the inequality \( c_N \leq M \), and the condition that the number of those indices \( i \) \((i = 1, 2, \ldots, N)\) which satisfy \( |c_{i+1} - \alpha c_i| > \gamma_0 \) is less than \( \gamma_1 \log M \).

**Proof.** Suppose that \( |c_{i+1} - \alpha c_i| \leq \gamma_0 \) and \( |c_{i+2} - \alpha c_{i+1}| \leq \gamma_0 \) for a fixed \( i \). Then

\[
\left| \frac{c_{i+1}}{c_i} - \alpha \right| < \frac{1}{10c_i},
\]

hence

\[
\left| \frac{c^2_{i+1}}{c_i} - \alpha c_{i+1} \right| < \frac{c_{i+1}}{10c_i} < \frac{3}{10}
\]

by (iii). Consequently, since \( |c_{i+2} - \alpha c_{i+1}| < \gamma_0 \) by assumption,

\[
\left| \frac{c^2_{i+1}}{c_i} - c_{i+2} \right| < \frac{3}{10} + \frac{1}{10} < \frac{1}{2}
\]

and so \( c_{i+2} \) is uniquely determined as the nearest integer\(^5\) to \( c^2_{i+1}/c_i \).

Consequently if \( i_1, i_2, \ldots, i_t \) denote all those among the \( N \) indices \( i \) which satisfy the inequality \( |c_{i+1} - ac_i| > \frac{1}{20} \) then all indices \( i \) which are not of the form \( i_r + 1 \) or \( i_r + 2 \) for some \( r = 1, 2, \ldots, t \), are such that \( c_i \) is uniquely determined by \( c_{i-1} \) and \( c_{i-2} \). On the other hand, even if \( j \) is of the form \( i_r + 1 \) or \( i_r + 2 \), so that \( c_j \) is not uniquely determined by \( c_{j-1} \) and \( c_{j-2} \), then there are, by (iv), (or (i)), at most 4 choices for \( c_j \) after \( c_{j-1} \) has been determined. Hence there are at most \( 4^t \) different sequences \( c_1, c_2, \ldots, c_N \) which have a given set of exceptional indices \( i_1, i_2, \ldots, i_t \).

Finally (ii) and (iv) together with the assumption \( a_N \leq M \) clearly imply that \( N < 5 \log M \) for sufficiently large \( M \), say for \( M > \gamma_2 \). Since the number of exceptional indices \( i_1, i_2, \ldots, i_t \) is less than \( \gamma_1 \log M \), by the hypothesis of Lemma 1, it is seen that the number of distinct possible choices for a set of exceptional indices cannot exceed

\[
\left( \left\lfloor \frac{5 \log M}{\gamma_1} \right\rfloor \right) + \cdots + \left( \left\lfloor \frac{5 \log M}{\gamma_1 \log M} \right\rfloor \right)
\]

and is therefore less than \( M^{1/8} \) if \( \gamma_1 \) is chosen sufficiently small. Since it was shown above that there are at most \( 4^t \) sequences \( c_1, c_2, \ldots, c_N \) with a given set of exceptional indices, it follows that the number of distinct sequences \( c_1, c_2, \ldots, c_N \) which satisfy the requirements of Lemma 1 for a fixed \( M > \gamma_2 \) is less than

\[
M^{1/8} \cdot 4^t < M^{1/8} \cdot 4^{5 \gamma_1 \log M} < M^{1/8}
\]

if \( \gamma_1 \) is sufficiently small. This completes the proof of Lemma 1.

If \( a, \lambda \) are positive numbers let \( A_k = A_k(a, \lambda) \) and \( e_k = e_k(a, \lambda) \) be defined, for \( k = 1, 2, \ldots \), by placing

\[
\lambda a^k = A_k + e_k, \quad A_k \text{ integer, } -\frac{1}{2} < e_k \leq \frac{1}{2}.
\]

**Lemma 2.** There exists an absolute constant \( \gamma_2 \), which shall be chosen to be \( > \gamma_2 \), such that if \( M \) has a fixed value greater than \( \gamma_2 \), then the measure of the set \( \Gamma \) of those values \( a \) in the interval

\[
2^k < a < 2
\]

for which there exists in the interval

\[
1 < \lambda < 2
\]

\( a \lambda = \lambda(a) \) such that the inequalities

\[
\lambda a^k < M^2 \quad \text{and} \quad |e_k(a, \lambda)| > \frac{1}{2} a^k
\]

hold for at most \( \frac{1}{2} \gamma_1 \log M \) distinct values of \( k \), is less than \( M^{-1} \). It is under-
stood that \( e_k = e_k(a, \lambda) \) is defined as in (3), and that \( \gamma_1, \gamma_2 \) are the absolute constants occurring in Lemma 1.

Proof. Suppose, if possible, that Lemma 2 is false. Then there exist at least \([M^{1/4}]\) values of \( a \) in (4), say
\[
a_j, \quad (j = 1, 2, \ldots, [M^{1/4}]),
\]
which are in \( \Gamma \) and which are separated by \([M^{1/4}] - 1\) intervals each of which has a length not less than \( M^{-3/4} \); so that
\[
|a_j - a_k| \geq M^{-3/4}.
\]
Since \( a_j \) is in \( \Gamma \), there exists a \( \lambda = \lambda(a_j) \) in (5) such that
\[
e_k(a_j, \lambda(a_j)) < \frac{1}{2} \gamma_0
\]
holds for all but \( \frac{1}{2} \gamma_1 \log M \) values of \( k \) satisfying
\[
a_j^k \lambda(a_j) < M,
\]
where, according to (3)
\[
ea_k^j \lambda(a_j) = A_k(a_j, \lambda(a_j)) + e_k(a_j, \lambda(a_j)) = A_k^{(j)} + e_k^{(j)}, \text{ say}.
\]

It will be shown that

(I) The finite sequence of integers \( A_k^{(j)} \) belonging to a fixed \( j \) \((= 1, 2, \ldots, [M^{1/4}] )\) satisfies the hypotheses of Lemma 1 if this sequence of integers is identified with the sequence of integers \( c_1, c_2, \ldots, c_N \) occurring there; and that

(II) The sequences \( A_k^{(j)} \) corresponding to different values of \( j \) are distinct. Since there are \([M^{1/4}] \) such sequences this will contradict Lemma 1 and so complete the proof of Lemma 2.

In order to prove (I) notice first that (i), (ii), (iii) are obviously satisfied for \( c_1 = A_k^{(j)} \). Furthermore, by (8)
\[
A_k^{(j)} + e_k^{(j)} = a_j(A_k^{(j)} + e_k^{(j)}),
\]
and so, by (3) and (4)
\[
|A_k^{(j)} - a_j A_k^{(j)}| = |a_j e_k^{(j)} - e_k^{(j)}| < 2;
\]
so that (iv) is also satisfied, with \( a = a_j \). The hypothesis (6.1) assures that the assumption \( c_N \leq M \) of Lemma 1 is satisfied. In order to verify the remaining assumption of Lemma 1 recall that there are at most \( \frac{1}{2} \gamma_1 \log M \) values of \( k \) satisfying (6.1), (6.2). Thus there are at most \( \gamma_1 \log M \) values of \( k \) such that (6.1), (6.2) are satisfied either for \( k = i \) or for \( k = i + 1 \). But if \( i \) has a value distinct from one of these \( \gamma_1 \log M \) values, so that
\[
|e_i^{(j)}| < \frac{1}{2} \gamma_0 \text{ and } e_{i+1}^{(j)} < \frac{1}{2} \gamma_0,
\]
then, by (4),
Thus there are at most $\gamma_1 \log M$ indices $i$ for which
\[ |A_{\Delta_1}^{(\nu)} - a_i A_{\Delta_1}^{(\nu)}| < \frac{1}{2} \] 
This completes the proof of (1).

In order to prove (II), suppose, if possible, that (II) is false. Then there exists a pair of distinct indices $j$ and $k$ such that
\[ A_{\Delta_1}^{(\nu)} = A_{\Delta_1}^{(k)} \]
for all $i = 1, 2, \cdots, N$. Thus, by (3),
\[ |a_k \lambda(a_k) - a_j \lambda(a_j)| < 2 \]
holds, for all $l$ such that $a_k \lambda(a_k) \leq M$. In particular (9) holds if $l$ is an index for which
\[ \frac{1}{4} M > a_k > \frac{1}{10} M. \]
Now it may be assumed that $a_k > a_j$ so that, by (7), $a_k \geq a_j + M^{-3/4}$. Then
\[ a_k \lambda(a_k) \geq a_k \lambda(a_k)(a_j + M^{-3/4}) \]
and so, by (9),
\[ a_k \lambda(a_k) \geq (a_j \lambda(a_j) - 2)(a_j + M^{-3/4}) = a_j \lambda(a_j) + a_j \lambda(a_j)M^{-3/4} - 2(a_j + M^{-3/4}). \]
Hence, by (5) and (10),
\[ a_k \lambda(a_k) \geq a_j \lambda(a_j) + \frac{1}{10} M^{1/4} - 2 - 2(a_j + M^{-3/4}) \]
if $M$ is sufficiently large, say $M > \gamma_3$. Thus
\[ |a_k \lambda(a_k) - a_j \lambda(a_j)| \geq 3. \]
This contradicts (9) (since by (10) $a_k \lambda(a_k) < M$) where one could write $l + 1$ for $l$. This contradiction proves (II).

The proof of Lemma 2 is now complete.

**Lemma 3.** There exists, on the interval (4) a zero set $Z$ which has the following property: if $a$ is a point of (4) not contained in $Z$ then there is a positive $\beta = \beta(a)$ such that if $M$ is any fixed number larger than $\beta$ and if $\lambda$ is any number in (5), then there are at least $\frac{1}{2} \gamma_1 \log M$ values of $k$ which satisfy both conditions (6.1), (6.2).

**Proof.** For any positive integer $h$ let $\Gamma_h$ denote the set of points $a$ on the interval (4) such that (6.1), (6.2) hold (for some $\lambda = \lambda(a)$ in (5)) for less than $\frac{1}{2} \gamma_1 \log M$ values of $k$ if $M = 2^h$. Then, by Lemma 2,
\[ \text{meas } \Gamma_h < 2^{-3h} \text{ if } 2^h > \gamma_3. \]
Thus if \( \Gamma_\mu \) denotes for any fixed \( \mu > \gamma_3 \) the \( a \)-set

\[
\Gamma = \Gamma_\mu = \sum_{\alpha > \mu} \Gamma_\alpha \quad \text{then meas} \Gamma_\mu < \frac{1}{4} \gamma_\mu^{-3}.
\]

It is clear from the definition of \( \Gamma \), that if \( a \) is not in \( \Gamma_\mu \) and if \( M > \mu \), then, even if \( M \) is not of the form \( 2^h \) for some \( h \), there are still at least \( \frac{1}{4} \gamma_1 \log M \) values of \( k \) satisfying (6.1), (6.2) for any value of \( \lambda \) in (5). Thus if \( a \) is not in \( \Gamma_\mu \) then there is a \( \beta = \beta(a) \) satisfying the requirements of Lemma 3; in fact one can choose \( \beta = \mu \). Then the set of points \( a \) in (4) such that there does not exist a \( \beta = \beta(a) \) satisfying the requirements of Lemma 3 is contained in \( \Gamma_\mu \) for every positive \( \mu \). Thus by (11), \( Z \) is a zero set. This completes the proof of Lemma 3.

**Lemma 4.** For every \( q > 0 \) there exists a \( p = p(q) > 1 \) and a zero set \( Z = Z_q \) of \( a \)-values contained in the interval

\[
1 < a < p(q)
\]

with the following properties: if \( a \) is a point of (12) not contained in \( Z_q \) then there exists an \( a = a(a) > 0 \) such that if \( M \) is any fixed number greater than \( a \), and if \( \lambda \) is any point of the interval (5), then there are at least \( q \log M \) values of \( k \) satisfying (6.1), (6.2).

**Proof.** Let \( a \) be a point in the interval \( 1 < a < 2^3 \) such that no integral power of \( a \) is a point of the zero set \( Z \) occurring in Lemma 3. Let \( p_1, p_2, \ldots, p_r \) be those prime numbers such that

\[
2^3 < a^{p_1} < a^{p_2} < \cdots < a^{p_r} < 2.
\]

Now if \( x \) is such that \( a^x = 2 \) then, by the elementary inequalities of Chebyshev, there are two absolute constants \( \gamma_4, \gamma_5 \) such that

\[
\gamma_4 \frac{x}{\log x} > r > \gamma_5 \frac{x}{\log x}.
\]

Since \( a^{p_j} (j = 1, 2, \ldots, r) \) is in the interval (4) and not a point of \( Z \), there are, by Lemma 3, for every \( \lambda \) in (5), at least \( \frac{1}{4} \gamma_1 \log M \) values of \( k \) satisfying

\[
|\lambda a^{p_j^k}| < M, \quad (14.1) \quad |\delta_0(a^{p_j}) \lambda| > \gamma_0
\]

provided \( M > \beta(a^{p_j}) \). Thus, if \( M > \max_{1 \leq i \leq r} \beta(a^{p_i}) \), there are at least \( \frac{1}{4} \gamma_1 \log M \) values of \( k \) satisfying (14.1), (14.2) for each \( i \) (\( i = 1, 2, \ldots, r \)). But there are at most \( \frac{x \log M}{p_i p_j \log 2} \) values of \( k \) such that

\[
(a^{p_i p_j})^k = (2^{p_i p_j /x})^k < \frac{1}{\lambda} M < M.
\]

Thus there are at least
values of \( k \) satisfying (6.1) and (6.2). Then by (13) the number of values \( k \) which satisfy (6.1) and (6.2) is not less than

\[
\frac{1}{2} r \gamma_1 \log M - \sum_{1 \leq \xi \leq M} \frac{\log M}{p_1(p, \log 2)}
\]

But this expression can be made greater than \( q \log M \) if \( x \) is chosen sufficiently large, i.e., if \( a \) is chosen sufficiently small, say \( a < \rho(q) \). This completes the proof of Lemma 4 since \( Z_q \) may be defined to be the zero set of points \( a \) in the interval (12), some integral power of which is a point of \( Z \).

**Theorem.** For every positive integer \( m \), there exists a positive \( \delta = \delta(m) \) such that the set of points \( a \) of the interval \( 1 < a < 1 + \delta(n) \) for which

\[
L(u, \sigma_a) = O(1/|u|^{-m}), \quad u \to \infty,
\]

does not hold, is a set of measure zero.

**Proof.** According to (1)

\[
L(u, \sigma_a) = \prod_{n=1}^{\infty} \cos (u/n^a), \quad (a > 1).
\]

Thus, if \( u \) is in the interval \( a^k < u \leq a^{k+1} \)

\[
L(u, \sigma_a) < \prod_{r=1}^{k} \cos (a^r(u/a^k)).
\]

Now let \( \lambda = u/a^k \) so that \( 1 < \lambda < 2 \). Then

\[
L(u, \sigma_a) < \prod_{r=1}^{k} |\cos (\lambda a^r)| = \prod_{\lambda a^r \leq u} |\cos (\lambda a^r)|.
\]

By Lemma 4, with \( M = u \), if \( a \) is chosen in the interval (12) and not in \( Z_q \) and if \( u > \alpha(a) \) there are at least \( q \log u \) factors in this last product which are less than \( \cos \pi/30 \) so that

\[
|L(u, \sigma_a)| < (\cos \pi/30)^{q \log u}, \quad u > \alpha(a).
\]

Since, according to Lemma 4, \( q > 0 \) can be chosen arbitrarily this completes the proof of the theorem.

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