ON DIVERGENCE PROPERTIES OF THE LAGRANGE INTERPOLATION PARABOLAS

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Throughout the present paper, \(-1 < x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} < 1\) denote the roots of the \(n\)-th Tchebicheff polynomial \(T_n(x)\), and unless otherwise stated it is understood that the fundamental points of the Lagrange interpolation are the \(x_i^{(n)}\). It is well known that there exists a continuous function whose interpolation parabolas diverge everywhere in \((-1, 1)\). In the present paper we prove that for \(x_0 = \cos \frac{p \pi}{q}, p = q = 1 \text{ (mod 2)}\), \((p, q) = 1\) there exists a continuous function \(f(x)\) such that \(L_n f(x_0) \to \infty\). Turán and Erdős proved that this does not hold for any other point. In this direction Marcinkiewicz proved that if the fundamental points are the roots of \(U_n(x) = T_{n+1}(x)\) then for every continuous function \(f(x)\) and every point \(x_0\) there exists a sequence of integers \(n_1 < n_2 < \cdots\) such that \(L_{n_i} f(x_0) \to f(x_0)\). We remark that in the case of the Fourier series it is well known that there always exists a subsequence of the partial sums converging to \(f(x_0)\). This fact may be of interest because there is often an analogous behaviour of the Lagrange interpolation parabola and the Fourier series.

First we prove some lemmas.

Lemma 1.

\[ x_i^{(m)} - x_i^{(n)} > \frac{1}{m^2}, \text{ for } m \geq n. \]

Proof. Write

\[ x_i^{(m)} = \cos \theta_i^{(m)}, \quad \theta_i = \frac{2i - 1}{2m} \pi. \]

Then we have

\[ |x_i^{(m)} - x_i^{(n)}| > |	heta_i^{(m)} - \theta_i^{(n)}| \sin \frac{\pi}{2n} > \frac{\pi}{4n} \frac{\pi}{2m} > \frac{1}{m^2} \text{ q.e.d.} \]

1 For the employed notations see P. Erdős and P. Turán, Annals of Math., Vol. 38 (1937), p. 142-155. If there is no danger of confusion we will omit the upper index \(n\).
3 \(L_n f(x)\) denotes the Lagrange interpolation parabola of \(f(x)\).
4 This result was stated in the Annals of Math., Vol. 38 (1937), p. 155 but there was a misprint.
5 Acta Litt ae Scient. Szeged, Tom. 8, p. 127-130.
**Lemma 2.** Put \( x_0 = \cos \frac{p}{q} \pi, p \equiv q \equiv 1 \mod 2 \); then constants \( c_1 \) and \( c_2 \) exist such that

\[
\min_{i=1,2,\ldots,n} |x_0 - x_i^{(n)}| > \frac{c_1}{n}, \quad |T_n(x_0)| > c_2.
\]

**Proof.**

\[
|T_n(x_0)| = \cos \left( \frac{n p}{q} \pi \right) \geq \cos \left( \frac{\pi}{2} - \frac{\pi}{2q} \right) > c_2.
\]

Put \( x_i^{(n)} < x_0 < x_{i+1}^{(n)} \); then

\[
\min_{i=1,2,\ldots,n} |x_0 - x_i^{(n)}| > \frac{\pi}{2nq} \min \left( \sin \frac{2j - 1}{2n}, \sin \frac{2j + 1}{2n} \right) > \frac{c_1}{n}.
\]

**Lemma 3.**

\[
\sum' |l_k^{(n)}(x_0)| < (\log n)^3
\]

where \( \sum' \) indicates that the summation is extended only over the \( x_i^{(n)} \) satisfying \(|x_k^{(n)} - x_0| > \frac{1}{(\log n)^4}\). **Proof.**

\[
|l_k^{(n)}(x_0)| = \left| \frac{T_n(x_0)}{T'_n(x_k)(x_0 - x_k)} \right| < (\log n)^4
\]

since \( |T_n(x_0)| \leq 1 \) and \( T'_n(x_k) = \frac{n}{\sqrt{1 - x_k^2}} \geq n \), which proves the Lemma.

Without loss of generality we may assume that \( x_0 > 0 \). Let \( x_i^{(n)} < x_0 < x_{i+1}^{(n)} \). Now we prove

**Lemma 4.** Suppose \( 0 < x_k^{(n)} < x_j^{(n)} \) (i.e., \( \frac{n}{2} < k < j \)); then

\[
|l_k^{(n)}(x_0)| > \frac{c_4}{j - k}.
\]

**Proof.** We have

\[
|l_k^{(n)}(x_0)| = \left| \frac{T_n(x_0)}{T'_n(x_k)(x_0 - x_k)} \right| \geq \frac{c_2 \sqrt{1 - x_k^2}}{n(x_k^2 - x_k)} > \frac{c_4}{n(x_{j+1} - x_k)},
\]

by Lemma 2. Now \( x_{j+1} - x_k < (j + 1 - k) \frac{\pi}{n} < \frac{c_5(j - k)}{n} \), which proves the Lemma.

**Lemma 5.**

\[
\sum_{(2k - 1, n) - 1} |l_k^{(n)}(x_0)| > c_6 \frac{\log n}{\log \log n}.
\]
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PROOF. By Lemma 4 we have

$$\sum_{(2k-1,n)=1}^{j-k} \left| l_k^{(n)}(x_0) \right| > \sum' \left| l_k^{(n)}(x_0) \right| > c_3 \sum'' \frac{1}{j-k}$$

where the two dashes indicate that the summation is extended only over those $k$ for which $(2k-1, n) = 1$ and $\frac{n}{2} < k < j$. It is clear that there are at least $c_7 n$ of the $x_k^{(n)}$ between 0 and $x_0$, thus

$$\sum'' \frac{1}{j-k} > \sum'' \frac{1}{j-k}$$

where the three dashes indicate that the summation is extended only over those $k$ which satisfy $(2k-1, n) = 1$ and $j - k < c_7 n$.

Denote by $v(n)$ the number of different odd prime factors of $n$. It is well known that $v(n) < c_8 \frac{\log n}{\log \log n}$. (This result is an immediate consequence of the prime number theorem, but can also be obtained in an elementary way.)

The number of integers $k$ satisfying $j - x < k < j$, $(2k-1, n) = 1$ equals by the sieve of Eratosthenes

$$x - \sum_{p \mid n} \left( \left\lfloor \frac{x}{p} \right\rfloor - \left\lfloor \frac{x}{pq} \right\rfloor - \cdots \right) > x \prod_{p \mid n} \left( 1 - \frac{1}{p} \right) - 2^{v(n)}$$

$$> c_9 \frac{x}{\log \log n} - 2^{v(n)} \frac{\log n}{\log \log n} > c_9 \frac{x}{\log \log n} \text{ for } x > \sqrt{n}, \quad (p \text{ odd})$$

since it is well known that $\prod_{p \mid n} \left( 1 - \frac{1}{p} \right) > \frac{c_{11}}{\log \log n}$. Thus

$$\sum'' \frac{1}{j-k} > \frac{c_{10}}{\log \log n} \sum_{r \mid \sqrt{n}} \frac{1}{r} > c_8 \frac{\log n}{\log \log n} \text{ q.e.d.}$$

THEOREM 1. There exists a continuous function $f(x)$ such that $L_n(f(x)) \to \infty$.

PROOF. Write

$$f(x) = \sum_{n \to x_0} f_n(x)$$

\[1 \text{ i.e. } \left| x_r^{(n)} - x_s^{(n)} \right| \leq \frac{x}{n}, \quad r, s = 1, 2, \ldots.\]

\[2 \left\lfloor \frac{x}{p} \right\rfloor \text{ denotes the number of the } k's \text{ in the interval } j - x < k < j \text{ for which } 2k-1 \text{ is divisible by } p. \quad \text{It is clear that } \left\lfloor \frac{x}{p} \right\rfloor \text{ differs from } \frac{x}{p} \text{ by less than } 1.\]

\( f_n(x) \) is defined as follows:
\[
  f_n(x_k^{(n)}) = \text{signum} \{ x_k^{(n)}(x_0) \} \quad \text{for} \quad (2k - 1, n) = 1,
\]
\[
  f_n \left( x_k^{(n)} \pm \frac{1}{2^m} \right) = 0,
\]
in the intervals \( (x_k^{(n)}, x_k^{(n)} + \frac{1}{2^m}) \) and \( (x_k^{(n)}, x_k^{(n)} - \frac{1}{2^m}) \), \( f_n(x) \) is linear and elsewhere \( f_n(x) = 0 \).

First we show that \( f(x) \) is continuous. It suffices to show that
\[
  \sum_{n=n_0}^{\infty} \frac{f_n(x)}{\sqrt{\log n}}
\]
is uniformly convergent, i.e. that
\[
  \sum_{n=n_0}^{\infty} \frac{f_n(x)}{\sqrt{\log n}} < \epsilon.
\]
If for a certain \( y \), \( f_m(y) \) and \( f_m(y), m > n \) are both different from 0, we have for a certain \( k_1 \) and \( k_2 \)
\[
  |x_k^{(n)} - y| < \frac{1}{2^{2n}}, \quad |x_k^{(m)} - y| < \frac{1}{2^{2n}},
\]
i.e.
\[
  |x_k^{(n)} - x_k^{(m)}| < \frac{2}{2^{2n}}.
\]
But by Lemma 1
\[
  |x_k^{(n)} - x_k^{(m)}| > \frac{1}{m^3}
\]
hence \( 2m^3 > 2^{2n} \), i.e. \( m > n^2 \) for \( n > 3 \). Thus
\[
  \sum_{n=n_0}^{\infty} \frac{f_n(x)}{\sqrt{\log n}} < \sum_{r=r_0}^{\infty} \frac{1}{\sqrt{\log 2^{2r}}} < \epsilon.
\]
Put
\[
  \varphi_1(x) = \sum_{r=r_0}^{n-1} \frac{f_r(x)}{\sqrt{\log r}}, \quad \varphi_2(x) = \sum_{r=r_0}^{\infty} \frac{f_r(x)}{\sqrt{\log r}}.
\]
Then
\[
  L_n(f(x_0)) = L_n(\varphi_1(x)) + L_n \left( \frac{f_n(x)}{\sqrt{\log n}} \right) + L_n(\varphi_2(x)).
\]
First we show that \( L_n(\varphi_2(x)) = 0 \). It will evidently suffice to show that for every \( k \), \( \varphi_2(x_k^{(n)}) = 0 \) or that for \( r > n \), \( f_r(x_k^{(n)}) = 0 \). If for a certain \( r > n \), \( f_r(x_k^{(n)}) \neq 0 \) we have for a certain \( l \)

\[
|x_k^{(n)} - x_l^{(r)}| < \frac{1}{2^r},
\]

which does not hold for by Lemma 1 for \( 2^r > r^3 \).

Next we estimate \( L_n(\varphi_1(x)) \). If for a certain \( x_k^{(n)} \), \( f_r(x_k^{(n)}) \neq 0 \) then for a certain \( l \)

\[
|x_k^{(n)} - x_l^{(r)}| < \frac{1}{2^r},
\]

which by Lemma 1 means that

\[
2^r < n^3 \quad \text{or} \quad r < 2 \log \log n \quad \text{for} \quad n > n_0.
\]

Thus if for a certain \( x_k^{(n)} \), \( \varphi_1(x_k^{(n)}) \neq 0 \) then by Lemma 2

\[
|x_k^{(n)} - x_0| > \min_{i=1,2,...,r} |x_i^{(r)} - x_0| - \frac{1}{2^r} > \frac{c_1}{r} - \frac{1}{2^r} > \frac{1}{(\log n)^i} \quad \text{for} \quad r > n_0.
\]

Thus by Lemma 3

\[
L_n(\varphi_1(x_0)) < c_{12} \sum l_k^{(n)}(x_0) | < c_{12}(\log n)^i
\]

Now by Lemma 5

\[
L_n(f_n(x_0)) = \sum_{|x_k - x_0| > (\log n)^{-i}} |l_k^{(n)}(x_0)| > c_{14} \frac{\log n}{\log \log n}
\]

since for \( (2k - 1, n) \neq 1 \), \( f_n(x_0) = 0 \). Thus finally

\[
L_n(f(x_0)) > c_8 \left( \frac{\log n}{\log \log n} \right)^i - c_{12} (\log n)^i \to \infty.
\]

Similarly we could prove that a continuous \( f(x) \) exists such that \( L_n(f(x_0)) \) converges to any given value.

**Theorem 2.** If \( x_0 \neq \cos \frac{p}{q} \pi \), \( p = q = 1 \pmod{2} \) then there exists for every continuous \( f(x) \) a sequence of integers \( n_1 < n_2 < \cdots \) such that \( L_n(f(x_0)) \to f(x_0) \).

**Proof.** First we prove that there exists a sequence if integers \( n_1 < n_2 < \cdots \) such that \( |T_{n_k}(x_0)| < \frac{c_{13}}{n_k} \). We need the following

**Lemma 6.** If \( x_0 \neq \frac{p}{q} \), \( p = q = 1 \pmod{2} \), then the inequality

\[
\left| x_0 - \frac{2r - 1}{2n_k} \right| < \frac{c_{14}}{n_k^2}
\]

has an infinite number of solutions.
Proof. If \( x_0 \) is rational it is of the form \( \frac{2r-1}{2nk} \), thus the Lemma is trivial. Hence we may suppose that \( x_0 \) is irrational. It is well known that the equation \( |x_0 - \frac{a}{b}| < \frac{1}{b^2} \) has an infinite number of solutions. If infinitely many of the \( b \)'s are even the Lemma is proved, if not consider the least positive solution of

\[
2ad - bf = 1.
\]

Obviously \( f = 1 \mod 2 \) and \( d < b \) thus

\[
|x_0 - \frac{f}{2d}| \leq \frac{1}{b^2} + \frac{1}{2bd} < \frac{c_{14}}{d^2}
\]

which proves the Lemma.

If \( |x_0 - \frac{2r-1}{2nk}| < \frac{c_{14}}{n^2} \) we have

\[
T_{nk}(x_0) < \cos \left( \frac{\pi}{2} - \frac{c_{14}}{n^2} \right) < \frac{c_{14}}{n^2}.
\]

Consider now a sequence of integers \( n_1 < n_2 < \ldots \) with \( |x_0 - \frac{2r-1}{n_k}| < \frac{c_{14}}{n_k^2} \). We are going to prove that \( L_{nk}(f(x_0)) \to f(x_0) \).

For \( k \neq r \) we have

\[
|l_k(x_0)| \leq \left| \frac{T_{nk}(x_0)}{T'(n_k)(x_k - x_0)} \right| < \left| \frac{c_{13}}{n^2(x_k - x_0)} \right|.
\]

Thus

\[
\sum_{k \neq r} |l_k(x_0)| < \frac{c_{13}}{n^2} \sum_{k \neq r} \frac{1}{|x_k - x_0|} = o(1),
\]

hence from

\[
\sum_{h=1}^n l_h(x) \equiv 1 \tag{1}
\]

We have

\[
\sum_{k \neq r} \frac{1}{x_k - x_0} = \sum' \frac{1}{|x_k - x_0|} \left( \sum_{|x_k - x_0| \leq (\log n)^{-1}} \frac{1}{x_k - x_0} \right)
\]

\[
+ \sum' \frac{1}{|x_k - x_0|} \left( \sum_{|x_k - x_0| > (\log n)^{-1}} \frac{1}{x_k - x_0} \right) < n \log n + cn \log n = o(n^2).
\]

(The dash indicates that \( k = r \) is omitted.)
it follows that

\[ l_r(x_0) = 1 - o(1). \]

Thus

\[ L_n(f(x_0)) = f(x_r)l_r(x_0) + \sum_{k \neq r} f(x_k)l_k(x_0) = (f(x_0) + \epsilon)[1 - o(1)] + o(1) \to f(x_0), \]

which proves Theorem 2.

On the other hand we can prove that for every \( x \) in \((-1, +1)\) there exists a continuous \( f(x) \) such that

\[ \lim_{n \to \infty} \frac{\sum_{m \leq n} L_m(f(x_0))}{n} = \infty. \]

The proof is very similar to that of Theorem 1.

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