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ON DIVERGENCE PROPERTIES OF THE LAGRANGE INTERPOLATION PARABOLAS

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Throughout the present paper, $-1 < x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} < 1$ denote the roots of the n-th Tchebicheff polynomial $T_n(x)$, and unless otherwise stated it is understood that the fundamental points of the Lagrange interpolation are the $x_i^{(n)}$.[1] It is well known that[2] there exists a continuous function whose interpolation parabolas diverge everywhere in $(-1, +1)$. In the present paper we prove that for $x_0 = \cos \frac{p}{q} \pi_1$, $p \equiv q \equiv 1 \pmod 2$, $(p_1, q) = 1$ there exists a continuous function $f(x)$ such that $L_n f(x_0) \to \infty$[3]. Turán and I[4] proved that this does not hold for any other point. In this direction Marcinkievicz[5] proved that if the fundamental points are the roots of $U_n(x) = T'_{n+1}(x)$ then for every continuous function $f(x)$ and every point x_0 there exists a sequence of integers $n_1 < n_2 < \cdots$ such that $L_{n_i}(f(x_0)) \to f(x_0)$. We remark that in the case of the Fourier series it is well known that there always exists a subsequence of the partial sums converging to $f(x_0)$. This fact may be of interest because there is often an analogous behaviour of the Lagrange interpolation parabola and the Fourier series.

First we prove some lemmas.

LEMMA 1.

$$x_i^{(m)} - x_j^{(n)} > \frac{1}{m^3}, \quad \textit{for } m \geqq n.$$

PROOF. Write

$$x_i^{(m)} = \cos \vartheta_i^{(m)}, \qquad \vartheta_i = \frac{2i-1}{2m}\pi.$$

Then we have

$$|x_i^{(m)} - x_j^{(n)}| > |\vartheta_i^{(m)} - \vartheta_j^{(n)}| \sin \frac{\pi}{2n} > \frac{\pi}{4n}\frac{\pi}{2mn} > \frac{1}{m^3} \text{ q.e.d.}$$

[1] For the employed notations see P. Erdös and P. Turán, Annals of Math., Vol. 38 (1937), p. 142–155. If there is no danger of confusion we will omit the upper index n.

[2] G. Grünwald, Annals of Math., Vol. 37 (1936), p. 908–918.

[3] $L_n(f(x))$ denotes the Lagrange interpolation parabola of $f(x)$.

[4] This result was stated in the Annals of Math., Vol. 38 (1937), p. 155 but there was a misprint.

[5] Acta Litt ac Scient. Szeged, Tom. 8, p. 127–130.

LEMMA 2. *Put* $x_0 = \cos \frac{p}{q}\pi$, $p \equiv q \equiv 1 \pmod 2$; *then constants* c_1 *and* c_2 *exist such that*

$$\min_{i=1,2,\cdots n} |x_0 - x_i^{(n)}| > \frac{c_1}{n}, \qquad |T_n(x_0)| > c_2.$$

PROOF.

$$|T_n(x_0)| = \cos\left(\frac{np}{q}\pi\right) \geqq \cos\left(\frac{\pi}{2} - \frac{\pi}{2q}\right) > c_2.$$

Put $x_j^{(n)} < x_0 < x_{j+1}^{(n)}$; then

$$\min_{i=1,2,\cdots n} |x_0 - x_i^{(n)}| > \frac{\pi}{2nq} \min\left(\sin\frac{2j-1}{2n}\pi, \sin\frac{2j+1}{2n}\pi\right) > \frac{c_1}{n}.$$

LEMMA 3.

$$\sum{}' |l_k^{(n)}(x_0)| < (\log n)^{\frac{1}{2}}$$

where $\sum'$ *indicates that the summation is extended only over the* $x_k^{(n)}$ *satisfying* $|x_k^{(n)} - x_0| > \frac{1}{(\log n)^{\frac{1}{2}}}$.

PROOF.

$$|l_k^{(n)}(x_0)| = \left|\frac{T_n(x_0)}{T_n'(x_k)(x_0 - x_k)}\right| < \frac{(\log n)^{\frac{1}{2}}}{n}$$

since $|T_n(x_0)| \leqq 1$ and $T_n'(x_k) = \frac{n}{\sqrt{(1 - x_k^2)}} \geqq n$, which proves the Lemma. Without loss of generality we may assume that $x_0 > 0$. Let $x_j^{(n)} < x_0 < x_{j+1}^{(n)}$. Now we prove

LEMMA 4. *Suppose* $0 < x_k^{(n)} < x_j^{(n)}$ $\left(\text{i.e., } \frac{n}{2} < k < j\right)$; *then*

$$|l_k^{(n)}(x_0)| > \frac{c_3}{j-k}.$$

PROOF. We have

$$|l_k^{(n)}(x_0)| = \left|\frac{T_n(x_0)}{T_n'(x_k)(x_0 - x_k)}\right| \geqq \frac{c_2\sqrt{(1 - x_k^2)}}{n(x_0 - x_k)} > \frac{c_4}{n(x_{j+1} - x_k)},$$

by Lemma 2. Now $x_{j+1} - x_k < (j + 1 - k)\frac{\pi}{n} < \frac{c_5(j-k)}{n}$, which proves the Lemma.

LEMMA 5.

$$\sum_{(2k-1,n)=1} |l_k^{(n)}(x_0)| > c_6 \frac{\log n}{\log \log n}.$$

PROOF. By Lemma 4 we have

$$\sum_{(2k-1,n)=1} |l_k^{(n)}(x_0)| > \sum'' |l_k^{(n)}(x_0)| > c_3 \sum'' \frac{1}{j-k}$$

where the two dashes indicate that the summation is extended only over those k for which $(2k - 1, n) = 1$ and $\frac{n}{2} < k < j$. It is clear[6] that there are at least $c_7 n$ of the $x_k^{(n)}$ between 0 and $x_j^{(n)}$, thus

$$\sum'' \frac{1}{j-k} > \sum''' \frac{1}{j-k}$$

where the three dashes indicate that the summation is extended only over those k which satisfy $(2k - 1, n) = 1$ and $j - k < c_7 n$.

Denote by $\nu(n)$ the number of different odd prime factors of n. It is well known that $\nu(n) < c_8 \frac{\log n}{\log\log n}$. (This result is an immediate consequence of the prime number theorem, but can also be obtained in an elementary way.) The number of integers k satisfying $j - x < k < j$, $(2k - 1, n) = 1$ equals by the sieve of Eratosthenes

$$x - \sum_{p|n}\left[\frac{x}{p}\right]' + \sum_{pq|n}\left[\frac{x}{pq}\right]' - \cdots{}^{7} \geqq x\prod_{p|n}\left(1 - \frac{1}{p}\right) - 2^{\nu(n)}$$

$$> c_9 \frac{x}{\log\log n} - 2^{c_8 \log n/\log\log n} > c_{10}\frac{x}{\log\log n} \text{ for } x > \sqrt{n}, \quad (p \text{ odd})$$

since it is well known that $\prod_{p|n}\left(1 - \frac{1}{p}\right) > \frac{c_{11}}{\log\log n}$.[8] Thus

$$\sum''' \frac{1}{j-k} > \frac{c_{10}}{\log\log n}\sum_{c_7 n > r > \sqrt{n}} \frac{1}{r} > c_6 \frac{\log n}{\log\log n} \text{ q.e.d.}$$

THEOREM 1. *There exists a continuous function $f(x)$ such that $L_n(f(x_0)) \to \infty$.*

PROOF. Write

$$f(x) = \sum_{n=n_0}^{\infty} \frac{f_n(x)}{\sqrt{\log n}}.$$

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[6] i.e. $|x_{r+1}^{(n)} - x_r^{(n)}| \leqq \frac{\pi}{n}$, $r = 1, 2, \cdots$.

[7] $\left[\frac{x}{p}\right]'$ denotes the number of the k's in the interval $j - x < k < j$ for which $2k - 1$ is divisible by p. It is clear that $\left[\frac{x}{p}\right]'$ differs from $\frac{x}{p}$ by less than 1.

[8] E. Landau, *Über den Verlauf der zahlentheoretischen Function.* Archiv der Math. und Phys., Ser. 3, Vol. 5, (1903), p. 86–91.

$f_n(x)$ is defined as follows:

$$f_n(x_k^{(n)}) = \text{signum } l_k^{(n)}(x_0) \quad \text{for} \quad (2k-1, n) = 1,$$

$$f_n\left(x_k^{(n)} \pm \frac{1}{2^{2^n}}\right) = 0,$$

in the intervals $\left(x_k^{(n)}, x_k^{(n)} + \frac{1}{2^{2^n}}\right)$ and $\left(x_k^{(n)}, x_k^{(n)} - \frac{1}{2^{2^n}}\right)$, $f_n(x)$ is linear and elsewhere $f_n(x) = 0$.

First we show that $f(x)$ is continuous. It suffices to show that

$$\sum_{n=n_0}^{\infty} \frac{f_n(x)}{\sqrt{\log n}}$$

is uniformly convergent, i.e. that

$$\sum_{n>n(\epsilon)} \frac{f_n(x)}{\sqrt{\log n}} < \epsilon.$$

If for a certain y, $f_n(y)$ and $f_m(y)$, $m > n$ are both different from 0, we have for a certain k_1 and k_2

$$|x_{k_1}^{(n)} - y| < \frac{1}{2^{2^n}}, \qquad |x_{k_2}^{(m)} - y| < \frac{1}{2^{2^m}},$$

i.e.

$$|x_{k_1}^{(n)} - x_{k_2}^{(m)}| < \frac{2}{2^{2^n}}.$$

But by Lemma 1

$$|x_{k_1}^{(n)} - x_{k_2}^{(m)}| > \frac{1}{m^3}$$

hence $2m^3 > 2^{2^n}$, i.e. $m > n^2$ for $n > 3$. Thus

$$\sum_{n>n(\epsilon)} \frac{f_n(x)}{\sqrt{\log n}} < \sum_{r>r(\epsilon)} \frac{1}{\sqrt{\log 2^{2^r}}} < \epsilon.$$

Put

$$\varphi_1(x) = \sum_{r=n_0}^{n-1} \frac{f_r(x)}{\sqrt{\log r}}, \qquad \varphi_2(x) = \sum_{r>n} \frac{f_r(x)}{\sqrt{\log r}}.$$

Then

$$L_n(f(x_0)) = L_n(\varphi_1(x)) + L_n\left(\frac{f_n(x)}{\sqrt{\log n}}\right) + L_n(\varphi_2(x)).$$

First we show that $L_n(\varphi_2(x)) = 0$. It will evidently suffice to show that for every k, $\varphi_2(x_k^{(n)}) = 0$ or that for $r > n$, $f_r(x_k^{(n)}) = 0$. If for a certain $r > n, f_r(x_k^{(n)}) \neq 0$ we have for a certain l

$$| x_k^{(n)} - x_l^{(r)} | < \frac{1}{2^{2^r}},$$

which does not hold for by Lemma 1 for $2^{2^r} > r^3$.

Next we estimate $L_n(\varphi_1(x))$. If for a certain $x_k^{(n)}$, $f_r(x_k^{(n)}) \neq 0$ then for a certain l

$$| x_k^{(n)} - x_l^{(r)} | < \frac{1}{2^{2^r}}$$

which by Lemma 1 means that

$$2^{2^r} < n^3 \qquad \text{or} \qquad r < 2 \log \log n \text{ for } n > n_0 .$$

Thus if for a certain $x_k^{(n)}$, $\varphi_1(x_k^{(n)}) \neq 0$ then by Lemma 2

$$| x_k^{(n)} - x_0 | > \min_{i=1,2,\cdots r} | x_i^{(r)} - x_0 | - \frac{1}{2^{2^r}} > \frac{c_1}{r} - \frac{1}{2^{2^r}} > \frac{1}{(\log n)^{\frac{1}{2}}} \quad \text{for} \quad r > n_0.$$

Thus by Lemma 3

$$L_n(\varphi_1(x_0)) < c_{12} \sum_{|x_k - x_0| > (\log n)^{-\frac{1}{2}}} | l_k^{(n)}(x_0) | < c_{12}(\log n)^{\frac{1}{2}}$$

Now by Lemma 5

$$L_n(f_n(x_0)) = \sum_{(2k-1, n)=1} | l_k^{(n)}(x_0) | > c_6 \frac{\log n}{\log \log n}$$

since for $(2k - 1, n) \neq 1$ $f_n(x_0) = 0$. Thus finally

$$L_n(f(x_0)) > c_6 \frac{(\log n)^{\frac{1}{2}}}{\log \log n} - c_{12} (\log n)^{\frac{1}{2}} \to \infty .$$

Similarly we could prove that a continuous $f(x)$ exists such that $L_n(f(x_0))$ converges to any given value.

THEOREM 2. *If* $x_0 \neq \cos \frac{p}{q} \pi$, $p \equiv q \equiv 1 \pmod 2$ *then there exists for every continuous* $f(x)$ *a sequence of integers* $n_1 < n_2 < \cdots$ *such that* $L_{n_i}(f(x_0)) \to f(x_0)$.

PROOF. First we prove that there exists a sequence if integers $n_1 < n_2 < \cdots$ such that $| T_{n_k}(x_0) | < \frac{c_{13}}{n}$. We need the following

LEMMA 6. *If* $x_0 \neq \frac{p}{q}$, $p \equiv q \equiv 1 \pmod 2$, *then the inequality*

$$\left| x_0 - \frac{2r - 1}{2n_k} \right| < \frac{c_{14}}{n_k^2}$$

has an infinite number of solutions.

PROOF. If x_0 is rational it is of the form $\frac{2r-1}{2n_k}$, thus the Lemma is trivial. Hence we may suppose that x_0 is irrational. It is well known that the equation $\left|x_0 - \frac{a}{b}\right| < \frac{1}{b^2}$ has an infinite number of solutions. If infinitely many of the b's are even the Lemma is proved, if not consider the least positive solution of

$$2a\,d - bf = 1.$$

Obviously $f \equiv 1 \pmod 2$ and $d < b$ thus

$$\left|x_0 - \frac{f}{2d}\right| \leqq \frac{1}{b^2} + \frac{1}{2bd} < \frac{c_{14}}{d^2}$$

which proves the Lemma.

If $\left|x_0 - \frac{2r-1}{2n_k}\right| < \frac{c_{14}}{n_k^2}$ we have

$$T_{n_k}(x_0) < \cos\left(\frac{\pi}{2} - \frac{c_{14}}{n_k}\right) < \frac{c_{13}}{n}.$$

Consider now a sequence of integers $n_1 < n_2 < \cdots$ with $\left|x_0 - \frac{2r-1}{n_k}\right| < \frac{c_{14}}{n_k^2}$. We are going to prove that $L_{n_k}(f(x_0)) \to f(x_0)$.

For $k \neq r$ we have

$$|l_k(x_0)| = \left|\frac{T_{n_k}(x_0)}{T'_{n_k}(x_k)(x_k - x_0)}\right| < \left|\frac{c_{13}}{n^2(x_k - x_0)}\right|.$$

Thus

$$\sum_{k\neq r} |l_k(x_0)| < \frac{c_{13}}{n^2} \sum_{k\neq r} \frac{1}{|x_k - x_0|} = o(1),^{9}$$

hence from

$$\sum_{k=1}^{n} l_k(x) \equiv 1$$

[9] We have

$$\sum_{k\neq r} \frac{1}{x_k - x_0} = \sum\nolimits'_{|x_k - x_0| \leqq (\log n)^{-1}} \frac{1}{|x_k - x_0|}$$
$$+ \sum\nolimits'_{|x_k - x_0| > (\log n)^{-1}} \frac{1}{|x_k - x_0|} < n \log n + cn \log n = o(n^2).$$

(The dash indicates that $k = r$ is omitted.)

it follows that

$$l_r(x_0) = 1 - o(1).$$

Thus

$$L_{n_k}(f(x_0)) = f(x_r)l_r(x_0) + \sum_{k \neq r} f(x_k)l_k(x_0) = (f(x_0) + \epsilon)[1 - o(1)] + o(1) \rightarrow f(x_0),$$

which proves Theorem 2.

On the other hand we can prove that for every x in $(-1, +1)$ there exists a continuous $f(x)$ such that

$$\lim_{n=\infty} \frac{\sum_{m \leqq n} L_m(f(x_0))}{n} = \infty \cdot$$

The proof is very similar to that of Theorem 1.

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