ON SOME ASYMPTOTIC FORMULAS IN THE THEORY OF THE 
"FACTORISATIO NUMERORUM"

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Let $1 < a_1 \leq a_2 \leq \ldots$ be a sequence of integers. Denote by $f(n)$ the number of representations of $n$ as the product of the $a$'s, where two representations are considered equal only if they contain the same factors in the same order. As far as I know the first papers written on the subject are those of L. Kalmár,\(^1\) who proved by using the methods of analytic number theory that if $a_k = k + 1$ then

$$F(n) = \sum_{r=1}^{n} f(r) = -\frac{n^\rho}{\zeta(\rho)} \left[1 + o(1)\right],$$

$\rho$ is defined as the unique positive root of $\zeta(\rho) = 2$. He also gives estimates for the error term.

Another paper on this subject is that of E. Hille.\(^2\) He obtains among others the following results: Let $p_1 < p_2 < \ldots$ be a sequence of primes and $a_1 < a_2 < \ldots$ the sequence of integers composed of these primes, then

$$F(n) = cn^{\rho}[1 + o(1)],$$

where $\sum \frac{1}{a_i^{\rho}} = 1, \rho > 0$. Hille uses the theorem of Wiener and Ikehara.

In the present paper we assume that $\sum \frac{1}{a_i^{1+\epsilon}}$ converges for every $\epsilon$ and that the $a$'s are not all powers of $a_1$, then we prove that

$$F(n) = cn^{\rho}[1 + o(1)],$$

where $\sum \frac{1}{a_i^{\rho}} = 1, \rho > 0$. The proof will be elementary.

First we need 2 Lemmas.

LEMMA 1

$$F(n) = \sum_k F\left[\frac{n}{a_k}\right] + 1.\(^3\)$$

PROOF. Follows immediately by considering those products in which $a_k$ is the first factor, and summing for $a_k$.

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\(^3\) The use of this identity was suggested to me by L. Kalmár.
Lemma 2.

\[ 0 < \lim_{n \to \infty} \frac{F(n)}{n^p} \leq \lim_{n \to \infty} \frac{F(n)}{n^p} < \infty. \]

Proof. Put \( F(n) = c_n n^p \). We have from (4)

\[ c_n n^p < \max_{i \leq \frac{n}{2}} c_i \sum_{j=1}^{n \overline{}} \frac{a_j}{a_k} + 1, \]

hence

\[ c_n < \max_{i \leq \frac{n}{2}} c_i + \frac{1}{n^p}. \]

Thus by induction

\[ c_n < 1 + \sum_{m=1}^{\infty} \frac{1}{2^{m^p}} < \infty, \]

which proves the first half of (5).

The proof of the second half of (5) will be slightly more complicated. Put \( F(n) = c'_n (n + 1)^p \). It suffices to prove that \( \lim c'_n > 0 \). From \( \left[ \frac{n}{a_k} \right] \geq n + 1 \) we obtain by (4)

\[ c'_n (n + 1)^p > \min_{i \leq \frac{n}{2}} c'_i \sum_{j=1}^{n \overline{}} \frac{(n + 1)^p}{a_k} = \min_{i \leq \frac{n}{2}} c'_i (n + 1)^p \left( 1 - \sum_{a_k > n} \frac{1}{a_k} \right). \]

Thus

\[ c'_n > \min_{i \leq \frac{n}{2}} c'_i \left( 1 - \sum_{a_k > n} \frac{1}{a_k} \right). \]

Hence by induction

\[ c'_n > \prod_{m=1}^{\infty} \left( 1 - \sum_{a_k > m} \frac{1}{a_k} \right). \]

The product on the right side (if extended to infinity) converges since

\[ \sum_{m=1}^{\infty} \sum_{a_k > m} \frac{1}{a_k} \leq \sum_{a_k} \log a_k < \sum_{a_k} \frac{1}{a_k}, \]

converges. This proves \( \lim c'_n > 0 \), and completes the proof of Lemma 2.

Now we can prove our theorem. Suppose that (3) does not hold, denote

\[ 0 < c = \lim_{n \to \infty} \frac{F(n)}{n^p} = \lim_{n \to \infty} \frac{F(n)}{(n + 1)^p} < \lim_{n \to \infty} \frac{F(n)}{n^p} = \lim_{n \to \infty} \frac{F(n)}{(n + 1)^p} = C < \infty. \]
Let \( m \) be sufficiently large and such that \( F(m) > (C - \delta)(m + 1)^k \). Clearly a fixed \( k \) exists (depending only on \( c \) and \( C \)) such that for every \( x \) satisfying \( m \leq x \leq m(1 + k) \)

\[
F(x) > \frac{C + c}{2}.
\]

Now let \( a_i \) be the least \( a \) which is not a power of \( a_1 \). Consider any \( x \) satisfying \( m_0 \leq x \leq m(1 + k) \). We have by (4), (6), (7) and \[
\left\lfloor \frac{x}{a_i} \right\rfloor + 1 \geq \frac{x + 1}{a_i}.
\]

Thus

\[
\frac{F(x)}{(x + 1)^k} > c + \frac{C - c}{2a_i} - o(1).
\]

Similarly we obtain that for the \( x \) satisfying \( a_i^a d_i m \leq x \leq a_i^a d_i m(1 + k) \)

\[
\frac{F(x)}{(x + 1)^k} > c + \delta_{a, \beta},
\]

where \( \delta_{a, \beta} \) depends only upon \( \alpha \) and \( \beta \). It is well known that the quotient of two consecutive integers of the form \( a_i^a d_i \) tends to 1. Thus there exists a sequence of integers \( A_1 < A_2 < \cdots < A_r \) of all the form \( a_i^a d_i \) and satisfying

\[
\frac{A_{i+1}}{A_i} < 1 + k, \quad i = 1, 2, \ldots, r - 1 \text{ and } A_r > a_1 A_1.
\]

Thus by (10) and since the intervals \([A_i m, A_i m(1 + k)]\) and \([A_{i+1} m, A_{i+1} m(1 + k)]\) overlap we have for \( A_i m \leq x \leq a_i A_i m \)

\[
\frac{F(x)}{(x + 1)^k} > c + \min \delta_{a, \beta} = c + \delta,
\]

for sufficiently large \( m \), where \( \delta \) is fixed and depends only on \( c \) and \( C \). Consider now the integers \( x \) satisfying \( a_i A_i m \leq x \leq a_i^a A_i m \) by (4), (6) and (11) we obtain as in (8) and (9)

\[
\frac{F(x)}{(x + 1)^k} > (c + \delta) \frac{1}{a_i^k} + c \sum_{a_i > a_i} \frac{1}{a_i^k} - o(1) = c + \delta \left( 1 - \sum \frac{1}{a_i^k} \right) - o(1).
\]

(i.e. \( \frac{x}{a_i} \) lies in \([A_i m, A_i m (1 + k)]\)). Similarly for the integers satisfying \( a_i^a A_i m \leq x \leq a_i^a A_i m \) we have

\[
\frac{F(x)}{(x + 1)^k} > \left[ c + \delta \left( 1 - \sum \frac{1}{a_i^k} \right) \right] \sum \frac{1}{a_i^k} + c \sum \frac{1}{a_i^k} + \delta \left( 1 - \sum \frac{1}{a_i^k} \right) - o(1).
\]
Finally we obtain for \( a_{i-1}^{k-1} A_1 m \leq x \leq a_i^k A_1 m \) \((k \text{ fixed}, \ m \text{ sufficiently large}) \)

\[
\frac{F(x)}{(x + 1)^\rho} > c + \delta \prod_{r = 1}^{k} \left( 1 - \sum_{a_i \geq a_{i+1}^+} \frac{1}{a_i^\rho} \right) - o(1).
\]

Denote

\[
\prod_{r = 1}^{\infty} \left( 1 - \sum_{a_i \geq a_{i+1}^+} \frac{1}{a_i^\rho} \right) = \eta.
\]

The product converges since \( \sum \frac{\log a_i}{a_i^\rho} \) converges. From (12) we have for \( A_1 m \leq x \leq a_i^k A_1 m \)

\[
\frac{F(x)}{(x + 1)^\rho} > c + \frac{\delta \eta}{2}.
\]

Now choose \( k \) so great that

\[
\prod_{r > k} \sum_{a_i \geq a_{i+1}^+} \frac{1}{a_i^\rho} > \frac{c + \frac{1}{2} \delta \eta}{c + \frac{1}{2} \delta \eta}.
\]

Then from (13) and (4) we have for \( A_1 a_1^m \leq x \leq A_1 a_1^{k+1} m \)

\[
F(x) > \sum_{a_i \geq a_{i+1}^+} F \left( \frac{x}{a_i} \right) > \left( c + \frac{\delta \eta}{2} \right) \sum_{a_i \geq a_{i+1}^+} \frac{(x + 1)^\rho}{a_i^\rho}.
\]

Similarly for any \( r \), in the interval \( A_1 a_i^m \leq x \leq A_1 a_i^{r+1} m \) we have by (14)

\[
\frac{F(x)}{(x + 1)^\rho} > \left( c + \frac{\delta \eta}{2} \right) \prod_{i > k} \sum_{a_i \geq a_i^+} \frac{(x + 1)^\rho}{a_i^\rho} > c + \frac{\delta \eta}{4}.
\]

Thus \( \lim \frac{F(x)}{(x + 1)^\rho} > c \). This contradicts (6) and completes the proof of our theorem.

It is easy to see that in our theorem, we can replace the assumption that \( \sum \frac{1}{a_i^\rho} \) converges by the following slightly more general one: There exists a \( k > 0 \) such that \( \sum \frac{1}{a_i^k} \) converges, and \( \sum \frac{\log a_i}{a_i^\rho} \) converges too.

Let \( a_k = k + 1 \). By using Lemma 2 we can prove that constants \( c_1 \) and \( c_2 \) exist, \( 0 < c_2 < c_1 < 1 \), such that for infinitely many \( n \)

\[
f(n) > \frac{n^\rho}{e^{(\log n)^{c_1}}}.
\]
and that for all $n > n_0$

$$f(n) < \frac{n^n}{e^{(\log n)^2}}.$$  

As I shall show in another paper the methods used here yield some asymptotic formulas in the theory of partitions.

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1. E. Hille proved that $f(n) > n^{r-1}$ for infinitely many $n$ (ibid).