Let $i \leq k \leq l$ be any integers. A theorem of Ramsey states that there exists a function $f(i, k, l)$ such that if $n \geq f(i, k, l)$, and if we select, from each combination of order $k$ of $n$ elements, a combination of order $i$, then there exists a combination of order $l$ all of whose combinations of order $i$ have been selected.

All the proofs give very bad estimates for $f(i, k, l)$. If $i = 2$ the theorem of Ramsey can be formulated as theorem about graphs: Let $n \geq \varphi(k, l)$, and consider any graph having $n$ points; then either the number of independent points is $\geq k$ or the graph contains a complete graph of order $l$.

Szekeres' proof gives $\varphi(k, l) \leq \left( k + l - \frac{2}{l} \right)$. This is probably very far from the best possible value. We do not even know whether or not $\lim \varphi(3, l) < \infty$ is true. Perhaps even the following stronger result holds: There exists an integer $c$ (independent of $n$) such that, given a graph without a triangle, we can number its vertices with the integers $1, 2, \ldots, c$, in such a way that no two vertices numbered with the same integer are connected. It is easy to see that $c \geq 4$.

Ramsey also proved that if $G$ is an infinite graph, then either $G$
contains an infinite set of independent points or \( G \) contains an infinite complete graph.

If the number of vertices of \( G \) is not countable, Duschnik, Miller and I proved the following theorem 4: Let the power of the points of \( G \) be \( m \); then either \( G \) contains an infinite complete graph, or \( G \) contains a set of \( m \) independent points. We can also state this theorem as follows: If we split the complete graph of \( m \) points into two subgraphs \( G_1 \) and \( G_2 \), then if \( G_1 \) does not contain an infinite complete graph, \( G_2 \) contains a set of \( m \) independent points.

In the present note we prove the following results:

**Theorem I:** Let \( a \) and \( b \) be infinite cardinals such that \( b > a^a \). If we split the complete graph of power \( b \) into a sum of \( a \) subgraphs at least one of them contains a complete graph of power \( > a \).

In particular: If \( b > c \) (the power of the continuum) and we split the complete graph of power \( b \) into a countable sum of subgraphs; at least one subgraph contains a non denumerable complete graph.

Theorem I is best possible. As a matter of fact, if \( b = a^a = 2^a \) we can split the complete graph of power \( b \) into the sum of subgraphs, such that no one of them contains a triangle. For the sake of simplicity we show this only in the case \( b = c = 2^{2^a} \). We write

\[
G = \sum_{k=1}^{\infty} G_k
\]

where \( G \) is a graph connecting every two points of the interval \((0, 1)\), and the edges of \( G_k \) connect two points \( x \) and \( y \) if \( \frac{1}{2^{k-1}} > y - x = \frac{1}{2^k} \).

Clearly none of the \( G_k \)'s contains any triangles.

Let us now assume that the generalized continuum hypothesis is true, i.e. \( 2^x = \aleph_{x+1} \). Let \( m = \aleph_{x+2} \), and let \( G \) be the complete graph containing \( m \) points, then we prove

**Theorem II:** Put \( G = G_1 + G_2 \); if \( G_1 \) does not contain a complete graph of power \( m \), then \( G_2 \) contains a complete graph of power \( \aleph_{x+1} \). From theorem I it would only follow that either \( G_1 \) or \( G_2 \) contains a complete graph of power \( \aleph_{x+1} \). By using results of paper of Sierpinski 5 it is not difficult to find a decomposition \( G = G_1 + G_2 \) such that

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neither $G_1$ nor $G_2$ contains a complete graph of power $\alpha$, which shows that theorem II can not be improved. (We have to assume that $\alpha$ is accessible).

Tukey and I have shown by using a result of Sierpinski $^6$ that the complete graph of power $\aleph_\beta$ can be decomposed into the countable sum of trees. Without assuming the continuum hypothesis we can not decide whether this also holds for the complete graph of power $\aleph_\alpha$.

**Proof of theorem I.** Let $G$ be the complete graph of power $\beta$; write

$$G = \sum \alpha G_{\alpha}, \quad \alpha < \Omega_\alpha,$$

where $\Omega_\alpha$ denotes the least ordinal corresponding to the power $\alpha$.

Let $p$ be any point of $G$. We split the remaining points of $G$ into a classes $Q_{\alpha}, \alpha_1 < \Omega_\alpha$, by the rule: a point $q$ is in $Q_{\alpha}$ if the line $pq$ is in $G_{\alpha}$. Take now an arbitrary point $p_{\alpha_1} \in Q_{\alpha_1}, (\alpha_1 = 1, 2, \ldots, \alpha_1 < \Omega_\alpha)$ and split the remaining points of $Q_{\alpha_1}$ into classes $Q_{\alpha_1, \alpha_2}, \alpha_2 < \Omega_\alpha$, by the rule: $q$ belongs to $Q_{\alpha_1, \alpha_2}$ if the line $p_{\alpha_1}q$ belongs to $G_{\alpha_2}$. Next we take an arbitrary point $p_{\alpha_2} \in Q_{\alpha_1, \alpha_2}$ and split the remaining points of $Q_{\alpha_1, \alpha_2}$ into classes $Q_{\alpha_1, \alpha_2, \alpha_3}, \alpha_3 < \Omega_\alpha$ etc. If $\alpha$ is not a limit ordinal we define the classes $Q_{\alpha_1, \alpha_2, \ldots, \alpha_k}$ in the obvious way from the classes $Q_{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}}, \alpha_k < \Omega_\alpha$. If $\alpha$ is a limit ordinal, we define the classes $Q_{\alpha_1, \alpha_2, \ldots, \alpha_i} (i < \alpha)$ as $\Pi Q_{\alpha_1, \alpha_2, \ldots, \alpha_i}$. Our construction can stop only if for some $\alpha$ all the classes $Q_{\alpha_1, \alpha_2, \ldots, \alpha_i}$ become empty; in other words if all the points of $G$ become $p_{\alpha_1, \alpha_2, \ldots, \alpha_i}$'s $(i < \alpha)$. Denote now by $\alpha^+$ the smallest power $> \alpha$, and by $\Omega_\alpha$, the smallest ordinal belonging to $\alpha^+$. We shall prove that not all the sets $Q_{\alpha_1, \alpha_2, \ldots, \alpha_i} (i < \Omega_{\alpha^+})$ can be empty. Clearly the power of the points $p_{\alpha_1, \alpha_2, \ldots, \alpha_i} (i < \Omega_{\alpha^+})$ does not exceed $\alpha^+ \cdot \alpha^\alpha = \alpha^\alpha (i.e. \alpha^\alpha > \alpha^+)$. But the power of the points of $G$ is by assumption $> \alpha^\alpha$; thus not all the points of $G$ are $p_{\alpha_1, \alpha_2, \ldots, \alpha_i}$'s $(i < \Omega_{\alpha^+})$. Let $\alpha$ be such a point, and consider the sets $Q_{\alpha_1, \alpha_2, \ldots, \alpha_i} (i < \Omega_{\alpha^+})$ with $r \subseteq Q_{\alpha_1, \alpha_2, \ldots, \alpha_i}$. Clearly $r = \Pi Q_{\alpha_1, \alpha_2, \ldots, \alpha_i} (i < \Omega_{\alpha^+})$ is non empty. If $\alpha$ is not a limit ordinal, $x_i$ runs at most through $\alpha$ values $(x_i < \Omega_\alpha)$ thus there must be an index $j$ $(j < \Omega_\alpha)$ which occurs in $Q_{\alpha_1, \alpha_2, \ldots, \alpha^+_j}$ times. Clearly $G_j$ contains a complete graph of power $\alpha^+$. For let $j = x_i = x_i = \ldots x_i = \ldots$, and consider the points $p_{x_1, x_2, \ldots, x_i}$. It is clear from our construction that the complete graph determined by these points is in $G_j$, this completes the proof of theorem I.

$^6$ W. Sierpinski, *ibid.*
Proof of theorem II. We state theorem II as follows: Let $G$ be a graph containing $\aleph_{x+2}$ points. Then if each set of independent points has power $< \aleph_{x+2}$, our graph contains a complete graph of power $\aleph_{x+1}$.

Let $p_1, p_2, \ldots, p_n$ be a complete set of independent points $(x_1 < \Omega \aleph_{x+1})$. Clearly every other point of $G$ is connected with at least one of the $p$'s. The point $q$ of $G$ will belong to class $Q_{p_i}$ if $p_i$ is the $p$ with smallest index with which $q$ is connected. In each $Q_{p_i}$, consider now a maximal system of independent points. Thus we obtain the points $p_{1i}, p_{2i}, \ldots, p_{ni}, \ldots$ and we split the remaining points of $Q_{p_i}$ into classes as before; the point $q \in Q_{p_i}$ belongs to $Q_{p_{ai}}$, if $p_{ai}$ is the point of lowest index with which $q$ is connected. We can continue this process as in the proof of theorem I. We claim that this process can not stop in $\aleph_{x+1}$ steps, in other words, the sets $Q_{p_{ai}}, Q_{p_{bi}}, \ldots, \subseteq Q_{p_{ci}}, \ldots, j < \Omega \aleph_{x+1}$, cannot all be empty. For if these sets were all empty, all points of $G$ would be $p_{ai}, p_{bi}, \ldots, p_{ci}$ for some $j < \Omega \aleph_{x+1}$. But the number of these points does not exceed $\aleph_{x+1} \aleph_{x+1} = \aleph_{x+1}$, by the generalized hypothesis of the continuum.

Consider, then, a sequence of sets $Q_{p_{1i}}, Q_{p_{2i}}, Q_{p_{3i}}, \ldots, Q_{p_{ni}}, \ldots, j < \Omega \aleph_{x+1}$ whose intersection is non empty. Clearly our graph contains the complete graph determined by the points $p_{1i}, p_{2i}, \ldots, p_{ni}, j < \Omega x + 1$ and this completes the proof of theorem II.

I do not know whether theorem II remains true if the power of the points of $G$ is $\aleph_{x+1}$, where $\aleph_x$ is a limit cardinal.

If the power of the points of $G$ is a limit cardinal e.g. $\aleph_x$, the theorem is certainly false. Let $M$ be the set of points of $G$ and write $M = \sum_{i=1}^{\infty} M_i$ where the power of $M$ is $\aleph_x$. We define $G$ as follows: Two points of $G$ are connected if and only if they belong to the same $M_i$. Then clearly $G$ does not contain a complete graph of power $M$, and every system of independent points is countable.

In general, let $m$ be a limit cardinal, which is the sum of $\aleph_k$ sets of power $< m$, but not the sum of fewer than $\aleph_k$ such sets. Then we can construct a graph $G$ the power of whose points is $m$, such that $G$ does not contain a complete graph of power $m$, and every set of independent points has power $< \aleph_k$. On the other hand, perhaps the following result holds: If such a graph $G$ does not contain a complete graph of power $m$, then it contains a set of independent points of power $\aleph_{k-1}$.

Let $A$ be a set of power $m$, and let $n < m$. To every point $x \in A$, we
correspond a subset \( f(x) \) of \( A \) such that \( x \in f(x) \), and the power of \( f(x) \) is \( < n \). A subset \( B \) of \( A \) is called independent if \( B - f(B) \) is empty. If we assume the generalized continuum hypothesis we can prove that there always exists an independent set of power \( m \). This result has been proved previously, without using the continuum hypothesis, in the cases: (I) \( m \) is not a limit cardinal; (II) \( m \) is a countable sum of smaller cardinals.