A well known conjecture of Catalan states that if \( f(n) \) is the sum of all divisors of \( n \) except \( n \), then the sequence of iterates of \( f(n) \) is either eventually periodic or ends at 1. It not only seems impossible to prove this, but it is also very difficult to verify.\(^1\)

Another conjecture of Poulet,\(^3\) which appears equally difficult to prove, has the doubtful merit that it is easy to verify. Let \( \sigma(n) \) be the sum of all divisors of \( n \), and let \( \phi(n) \) be Euler’s function. Then for any integer \( n \) the sequence

\[
\sigma(n) = n, \quad f_{2k+1}(n) = \sigma(f_{2k}(n)), \quad f_{2k}(n) = \phi(f_{2k-1}(n))
\]

is eventually periodic.

We have verified this conjecture to \( n = 10000 \) (extending Poulet’s verification) by using Glaisher’s tables.\(^5\) The checking was facilitated by the following observation: if the conjecture is to be checked for all \( n < x \), it is enough to find a member of the sequence other than the first which is less than \( x \).

The longest cycle found was in the sequence \( f_4(9216) \). It starts with \( f_4(9216) \), and is: 34560, 122640, 27648, 81800, 30976, 67963, 54432, 183456, 48384, 163520, 55296, 163800, 34560. However our method of checking does not show that this is the largest cycle up to 10000, and in fact Poulet found that \( f_4(1800) \) has the same length 12.

As a rule \( \phi(\sigma(n)) \) is less than \( n \). In fact, it can be shown that for every \( \epsilon > 0 \), \( \phi(\sigma(n)) < \epsilon n \), except for a set of density 0. The proof follows from the following two observations:

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(1) For a given prime \( p \), the set of all \( n \) such that \( \sigma(n) \equiv 0 \pmod{p} \) is of density 1.

The set of all integers not divisible by any prime \( q \) of the form \( pq - 1 \) is of density zero, since \( \sum q / q \) diverges. Hence the set of all integers divisible by a prime \( q > N \) of this type is of density 1. But the set of all integers divisible by \( q^2, q > N \), is of density less than \( \sum q > N 1 / q^2 = o(1) \). Therefore, if \( x \) is large, the number of \( n \) less than \( x \) such that \( \sigma(n) \equiv 0 \pmod{p} \) exceeds \( (1 - \epsilon)x \).

(2) Except for \( \epsilon x \) integers \( n \) less than \( x \), \( \sigma(n) < c(\epsilon)n \).

This follows from the fact that \( \sum n < \sigma(n) \sim n^2 \ln n / 12 \).

Choose \( p \) so that \( \prod q \leq n (1 - 1 / q) < \delta / c(\epsilon) \). Then, if \( x \) is sufficiently large, all except \( \eta x + \epsilon x \) integers \( n \) less than \( x \) have \( \sigma(n) < c(\epsilon)n \), \( \sigma(n) \equiv 0 \pmod{q} \) for all \( q \leq p \). But, with these exceptions, \( \phi[\sigma(n)] < \delta n \), which completes the proof, since \( \eta \) and \( \epsilon \) are arbitrary.

In much the same way it can be shown that for every \( c > 0 \), \( \sigma[\phi(n)] > cn \) except for a set of density zero.

Actually, much more can be shown. Except for a set of density zero, \( e^\gamma \phi[\sigma(n)] \log \log n \sim \sigma(n) \), and \( e^{-\gamma} \phi[\phi(n)] / \log \log n \sim \phi(n) \), where \( \gamma \) is Euler's constant. The proof is suppressed, but it might be noted that the reason for this result is that, for almost all \( n \), \( \sigma(n) \) and \( \sigma(n) \) are both divisible by all primes less than \( (\log \log n)^{1 - \epsilon} \), and by relatively few primes greater than \( (\log \log n)^{1 + \epsilon} \).

There exist numbers for which \( \phi[\sigma(n)] = n \). Up to 2500 these numbers are 1, 2, 8, 12, 128, 240, 720; while two further solutions are \( 2^{18} \) and \( 2^{24} \). Poulet gives many others; we do not know whether there are infinitely many solutions.

We state two further conjectures:

(a) Form the sequence \( \sigma(n), \sigma(\sigma(n)), \phi(\sigma(\sigma(n))), \sigma(\phi(\sigma(\sigma(n)))) \) in which the functions are successively applied in the order \( \sigma, \sigma, \phi, \sigma, \phi, \sigma, \phi, \cdots \). This sequence seems to tend to infinity if \( n \) is large enough.

(b) On the other hand, the sequence \( \phi(n), \phi(\phi(n)), \sigma(\phi(\phi(n))), \cdots \), in which the order is \( \phi, \phi, \sigma, \phi, \sigma, \phi, \sigma, \cdots \), seems to converge to 1, for all \( n \).

Obviously many more such conjectures can be formulated.

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