INTEGRAL DISTANCES

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In the present note we are going to prove the following result:

For any \( n \) we can find \( n \) points in the plane not all on a line such that their distances are all integral, but it is impossible to find infinitely many points with integral distances (not all on a line).\(^1\)

**Proof.** Consider the circle of diameter 1, \( x^2 + y^2 = 1/4 \). Let \( p_1, p_2, \ldots \) be the sequence of primes of the form \( 4k+1 \). It is well known that

\[
p_i^2 = a_i^2 + b_i^2, \quad a_i \neq 0, \quad b_i \neq 0,
\]

is solvable. Consider the point (on the circle \( x^2 + y^2 = 1/4 \)) whose distance from \((-1/2, 0)\) is \( b_i/p_i \). Denote this point by \((x_i, y_i)\). Consider the sequence of points \((-1/2, 0), (1/2, 0), (x_i, y_i), i = 1, 2, \ldots \). We shall show that any two distances are rational. Suppose this has been shown for all \( i < j \). We then prove that the distance from \((x_i, y_i)\) to \((x_j, y_j)\) is rational. Consider the 4 concyclic points \((-1/2, 0), (1/2, 0), (x_i, y_i), (x_j, y_j)\); 5 distances are clearly rational, and then by Ptolemy's theorem the distance from \((x_i, y_i)\) to \((x_j, y_j)\) is also rational. This completes the proof. Thus of course by enlarging the radius of the circle we can obtain \( n \) points with integral distances.

It is very likely that these points are dense in the circle \( x^2 + y^2 = 1/4 \), but this we cannot prove. It is easy to obtain a set which is dense on \( x^2 + y^2 = 1/4 \) such that all the distances are rational. Consider the

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\(^1\) Anning gave 24 points on a circle with integral distances. Amer. Math. Monthly vol. 22 (1915) p. 321. Recently several authors considered this question in the Mathematical Gazette.
point \( x_1 \) whose distance from \((-1/2, 0)\) is 3/5; the distance from \((0, 1/2)\) is of course 4/5. Denote \((-1/2, 0)\) by \(P_1\), \((1/2, 0)\) by \(P_2\), and let \(\alpha\) be the angle \(P_2P_1X_1\). \(\alpha\) is known to be an irrational multiple of \(\pi\). Let \(x_i\) be the point for which the angle \(P_1P_2X_i\) equals \(i\alpha\); the points \(X_i\) are known to be dense on the circle \(x^2 + y^2 = 1/2\), and all distances between \(x_i\) and \(x_j\) are rational because if \(\sin \alpha\) and \(\cos \alpha\) are rational, clearly \(\sin i\alpha\) and \(\cos i\alpha\) are also rational.

To give another configuration of \(n\) points with integral distances, let \(m^2\) be an odd number with \(d\) divisors, and put

\[
m^2 = x_i - y_i.
\]

This equation has clearly \(d\) solutions. Consider now the points

\[
(m, 0), \quad (0, y_i) \quad i = 1, 2, \ldots.
\]

It is immediate that all the distances are integral.

These configurations are all of very special nature. Several years ago Ulam asked whether it is possible to find a dense set in the plane such that all the distances are rational. We do not know the answer.

Now we prove that we cannot have infinitely many points \(P_1, P_2, \ldots\) in the plane not all on a line with all the distances \(P_iP_j\) being integral.

First we show that no line \(L\) can contain infinitely many points \(Q_1, Q_2, \ldots\). Let \(P\) be a point not on \(L, Q_i\) and \(Q_j\) two points very far away from \(P\) and very far from each other. Put \(d(PQ_i) = a,\ d(Q_1Q_i) = b,\ d(PQ_i) = c\). \((d(A, B)\) denotes the distance from \(A\) to \(B\).)

\[
(1) \quad c \leq a + b - 1.
\]

Let \(Q_iR\) be perpendicular to \(PQ_i\). We have

\[
a < d(PR) + (d(Q_iR))^2/d(PR), \quad b < d(Q_jR) + (d(Q_iR))^2/d(Q_jR).
\]

Thus from (1)

\[
(d(Q_iR))^2 \left(\frac{1}{d(PR)} + \frac{1}{d(Q_jR)}\right) > 1
\]

which is clearly false for \(a\) and \(b\) sufficiently large. \((d(Q_iR)\) is clearly less than the distance of \(P\) from \(L\). This completes the proof.

There clearly exists a direction \(P_iX\) such that in every angular neighborhood of \(P_iX\) there are infinitely many \(P_i\).

Let \(P_3\) be a point not on the line \(P_iX\).

Denote the angle \(XP_3P_2\) by \(\alpha\), \(0 < \alpha < \pi\). Evidently the \(P_i\) cannot form a bounded set. Let \(Q\) be one of the \(P_i\) sufficiently far away from
Let $P_1$, where the angle $QP_1X$ equals $\epsilon$ ($\epsilon$ sufficiently small). Denote $d(P_1, P_2) = a$, $d(P_1, Q) = b$, $d(P_2, Q) = c$. We evidently have
\[ c^2 = a^2 + b^2 - 2ab \cos (\alpha - \epsilon). \]
a, $b$, $c$ all are integers. From this we shall show that if $b$ and $c$ are sufficiently large, $\epsilon$ sufficiently small, then
\[ (2) \quad c = b - a \cos \alpha. \]

Put
\[ \epsilon = b - a \cos \alpha + \delta, \quad \delta > 0. \]

Then
\[ (b - a \cos \alpha + \delta)^2 = b^2 - 2ab \cos \alpha + a^2 \cos^2 \alpha + 2\delta (b - a \cos \alpha) + \delta^2 > a^2 + b^2 - 2ab \cos (\alpha - \epsilon) \]
if $b$ is sufficiently large and $\epsilon$ sufficiently small. Similarly we dispose of the case $\delta < 0$. Thus (2) is proved.

From (2) we have
\[ a^2 + b^2 - 2ab \cos (\alpha - \epsilon) = b^2 - 2ab \cos \alpha + a^2 \cos^2 \alpha \]
or
\[ \cos (\alpha - \epsilon) - \cos \alpha = \frac{a^2 \sin^2 \alpha}{2b}. \]

Thus we clearly obtain
\[ \epsilon < c_1/b. \]

Thus clearly all the points $Q_i$ have distance less than $c_1$ from the line $P_1X$. Let $Q_1, Q_2, Q_3$ be three such points not on a line, where $d(Q_i Q_j)$ are large. Let $Q_1 Q_3$ be the largest side of the triangle $Q_1Q_2Q_3$. Let $Q_2R$ be perpendicular to $Q_1Q_3$. We have as before
\[ d(Q_2 Q_3) \leq d(Q_1, Q_2) + d(Q_1 Q_3) - 1; \]
also
\[ d(Q_2 Q_3) - d(Q_2 R) < \epsilon, \quad d(Q_3 R) - d(Q_3 R) < \epsilon \]
an evident contradiction; this completes the proof.

By a similar argument we can show that we cannot have infinitely many points in $n$-dimensional space not all on a line, with all the distances being integral.

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